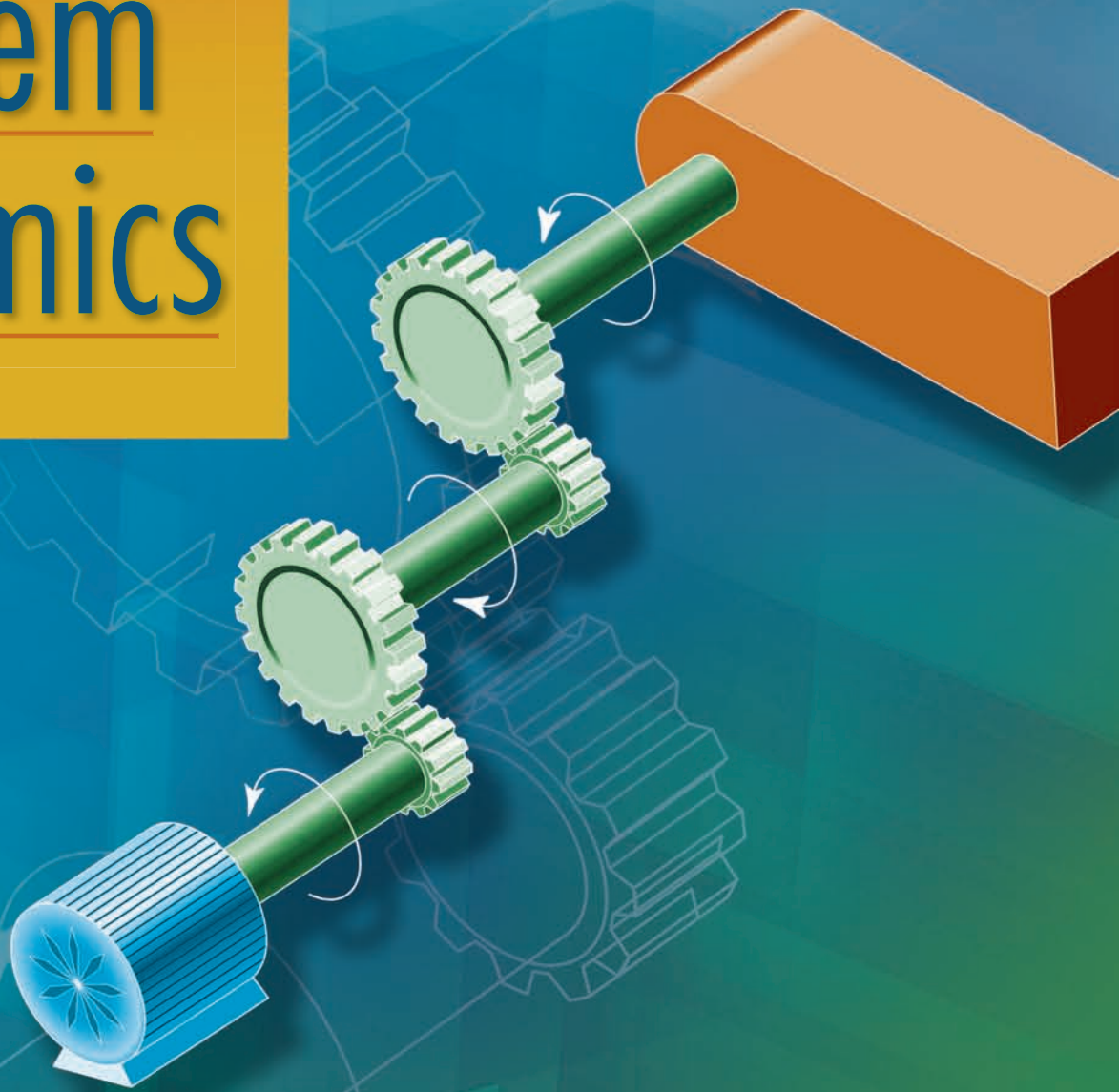Second Edition

System
Dynamics

William J. Palm III

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William J. Palm III
University of Rhode Island

Higher Education

To my wife, Mary Louise; and to my children, Aileene, Bill, and Andrew.

CONTENTS

PREFACE

System dynamics deals with mathematical modeling and analysis of devices and processes for the purpose of understanding their time-dependent behavior. While other subjects, such as Newtonian dynamics and electrical circuit theory, also deal with time-dependent behavior, system dynamics emphasizes methods for handling applications containing multiple types of components and processes such as electromechanical devices, electrohydraulic devices, and fluid-thermal processes. Because the goal of system dynamics is to understand the time-dependent behavior of a system of interconnected devices and processes as a whole, the modeling and analysis methods used in system dynamics must be properly selected to reveal how the connections between the system elements affect its overall behavior. Because systems of interconnected elements often require a control system to work properly, control system design is a major application area in system dynamics.

TEXT PHILOSOPHY

This text is an introduction to system dynamics and is suitable for such courses commonly found in engineering curricula. It is assumed that the student has a background in elementary differential and integral calculus and college physics (dynamics, mechanics of materials, thermodynamics, and electrical circuits). A previous course in differential equations is desirable but not necessary, as the required material on differential equations, as well as Laplace transforms and matrices, is developed in the text.

The decision to write a textbook often comes from the author's desire to improve on available texts. The decisions as to what topics to include and what approach to take emerge from the author's teaching experiences that give insight as to what is needed for students to master the subject. This text is based on the author's thirty-seven years of experience in teaching system dynamics.

This experience shows that typical students in a system dynamics course are not yet comfortable with applying the relevant concepts from earlier courses in dynamics and differential equations. Therefore, this text reviews and reinforces these important topics early on. Students often lack sufficient physical insight to relate the mathematical results to applications. The text therefore uses everyday illustrations of system dynamics to help students to understand the material and its relevance.

If laboratory sessions accompany the system dynamics course, many of the text's examples can be used as the basis for experiments. The text is also a suitable reference on hardware and on parameter estimation methods.

MATLAB® AND SIMULINK®[1]

MATLAB and Simulink are used to illustrate how modern computer tools can be applied in system dynamics.[2] MATLAB was chosen because it is the most widely used program in system dynamics courses and by practitioners in the field. Simulink,

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[1]MATLAB and Simulink are registered trademarks of The MathWorks, Inc.

[2]The programs in this text will work with the following software versions, or higher versions: Version 6 of MATLAB, Version 5 of Simulink, and Version 5 of the Control Systems Toolbox.

which is based on MATLAB and uses a diagram-based interface, is increasing in popularity because of its power and ease of use. In fact, students convinced the author to use Simulink after they discovered it on their own and learned how easy it is to use! It provides a useful and motivational tool.

It is, however, not necessary to cover MATLAB or Simulink in order to use the text, and it is shown how to do this later in the Preface.

CORE MATERIAL FOR SYSTEM DYNAMICS

This text has been designed to accommodate a variety of courses in system dynamics. The core material is in Chapters 1 through 6 and Chapters 8 and 9.

Chapter 1 introduces the basic terminology of system dynamics, covers commonly used functions, and reviews the two systems of units used in the text: British Engineering (FPS) units and SI units. These are the unit systems most commonly used in system dynamics applications. The examples and homework problems employ both sets of units so that the student will become comfortable with both. Chapter 1 also introduces methods for parameter estimation. These methods are particularly useful for obtaining spring constants and damping coefficients. The chapter then illustrates how MATLAB can be used for this purpose.

Chapter 2 covers rigid-body dynamics, including planar motion. Using the models developed in Chapter 2, Chapter 3 reviews solution methods for linear ordinary differential equations where either there is no forcing function (the homogeneous case) or where the forcing function is a constant. The chapter then develops the Laplace transform method for solving differential equations and applies it to equations having step, ramp, sine, impulse, and other types of forcing functions. It also introduces transfer function models.

Chapter 4 covers modeling of mechanical systems having stiffness and damping, and it applies the analytical methods developed in Chapter 3 to solve the models.

Chapter 5 develops the state-variable model, which is useful for certain analytical techniques as well as for numerical solutions. The optional sections of this chapter introduce Simulink, which is based on diagram descriptions, and apply the chapter's concepts using MATLAB.

Chapter 6 treats modeling of electric circuits, operational amplifiers, electromechanical devices, sensors, and electroacoustic devices. It also discusses how motor parameters can be obtained, and it shows how to analyze motor performance.

Chapters 8 and 9 cover analysis methods in the frequency domain and the time domain, respectively. Chapter 8 demonstrates the usefulness of the transfer function for understanding and analyzing a system's frequency response. It introduces Bode plots and shows how they are sketched and interpreted to obtain information about time constants, resonant frequencies, and bandwidth.

Chapter 9 integrates the modeling and analysis techniques of earlier chapters with an emphasis on understanding system behavior in the time domain, using step, ramp, and impulse functions primarily. The chapter covers step response specifications such as maximum overshoot, peak time, delay time, rise time, and settling time. Block diagram models are graphical representations of system structure. Chapter 9 introduces these models as preparation for Chapter 10, which deals with control systems.

ALTERNATIVE COURSES IN SYSTEM DYNAMICS

The choice of remaining topics depends partly on the desired course emphasis and partly on whether the course is a quarter or semester course.

Some courses omit fluid and thermal systems, which are covered in Chapter 7. This chapter can be skipped if necessary because only some examples in the remaining chapters, and not the theory and methods, depend on it. Part I of the chapter covers fluid systems. Part II covers thermal systems. These two parts are independent of each other. A background in fluid mechanics or heat transfer is not required to understand this chapter, but students should have had elementary thermodynamics before covering the material on pneumatic systems in Section 7.5.

Chapters 10, 11, and 12 deal with a major application of system dynamics, namely, control systems. Chapter 10 is an introduction to feedback control systems, including the PID control algorithm applied to first- and second-order plants. Chapter 11 deals with control systems in more depth and includes design methods based on the root locus plot and practical topics such as compensation, controller tuning, actuator saturation, reset wind-up, and state-variable feedback, with emphasis on motion control systems. Chapter 12 covers series compensation methods and design with the Bode plot.

Chapter 13 covers another major application area, vibrations. Important practical applications covered are vibration isolators, vibration absorbers, modes, and suspension system design.

At the author's institution, the system dynamics course is a junior course required for mechanical engineering majors. It covers Chapters 1 through 10, with some optional sections omitted. This optional material is then covered in a senior elective course in control systems, which also covers Chapters 11 and 12.

GLOSSARY AND APPENDICES

There is a glossary containing the definitions of important terms, four appendices, and an index. Appendices C and D are on the text website.

Appendix A is a collection of tables of MATLAB commands and functions, organized by category. The purpose of each command and function is briefly described in the tables.

Appendix B is a brief summary of the Fourier series, which is used to represent a periodic function as a series consisting of a constant plus a sum of sine terms and cosine terms. It provides the background for some applications of the material in Chapter 8.

Appendix C is a self-contained introduction to MATLAB, and it should be read first by anyone unfamiliar with MATLAB if they intend to cover the MATLAB and Simulink sections. It also provides a useful review for those students having prior experience with MATLAB.

Appendix D covers basic numerical methods, such as the Runge-Kutta algorithms, that form the basis for the differential equation solvers of MATLAB. It is not necessary to master this material to use the MATLAB solvers, but the appendix provides a background for the interested reader.

CHAPTER FORMAT

The format of each chapter follows the same pattern, which is

1. Chapter outline
2. Chapter objectives
3. Chapter sections
4. MATLAB sections (in most chapters)
5. Simulink section (in most chapters)

6. Chapter review
7. References
8. Problems

This structure has been designed partly to accommodate those courses that do not cover MATLAB and/or Simulink, by placing the optional MATLAB and Simulink material at the end of the chapter. Note that coverage of Simulink requires that the chapter's MATLAB sections also be covered. Because the chapter problems are arranged according to the chapter section whose concepts they illustrate, all problems requiring MATLAB and/or Simulink have been placed in separate, identifiable groups.

OPTIONAL TOPICS

In addition to the optional chapters (7, 10, 11, 12, and 13), some chapters have sections dealing with material other than MATLAB and Simulink that can be omitted without affecting understanding of the core material in subsequent chapters. All such optional material has been placed in sections near the end of the chapter. This optional material includes:

1. Function discovery, parameter estimation, and system identification techniques (Sections 1.4, 1.5, 8.5, and 9.4)
2. General theory of partial fraction expansion (Section 3.5)
3. Impulse response (Sections 3.6 and 4.6)
4. Motor performance (Section 6.5)
5. Sensors and electroacoustic devices (Section 6.6)

DISTINGUISHING FEATURES

The following are considered to be the major distinguishing features of the text.

1. **MATLAB.**   Stand-alone sections in most chapters provide concise summaries and illustrations of MATLAB features relevant to the chapter's topics.
2. **Simulink.**   Stand-alone sections in chapters 5 through 12 provide extensive Simulink coverage not found in most system dynamics texts.
3. **Parameter estimation.**   Coverage of function discovery, parameter estimation, and system identification techniques is given in Sections 1.4, 1.5, 8.5, and 9.4. Students are uneasy when they are given parameter values such as spring stiffness and damping coefficients in examples and homework problems, because they want to know how they will obtain such values in practice. These sections show how this is done.
4. **Motor performance evaluation.**   Section 6.5 discusses the effect of motor dynamics on practical considerations for motor and amplifier applications, such as motion profiles and the required peak and rated continuous current and torque, and maximum required voltage and motor speed. These considerations offer excellent examples of practical applications of system dynamics, but are not discussed in most system dynamics texts.
5. **System dynamics in everyday life.**   Commonly found illustrations of system dynamics are important for helping students to understand the material and its relevance. This text provides examples drawn from objects encountered in everyday life. These examples include a storm door closer, fluid flow from a

bottle, shock absorbers and suspension springs, motors, systems with gearing, chain drives, belt drives, a backhoe, a water tower, and cooling of liquid in a cup.

6. **Theme applications.** Two common applications provide themes for examples and problems throughout the text. These are motion control systems such as a conveyor system and a robot arm, and vehicle suspension systems.

WEBSITE

The publisher maintains a website for this text at www.mhhe.com/palm. An on-line instructors manual is available at this site. It contains solutions to the problems and other pedagogical aids, and is accessible to instructors who have adopted the text for their course. The site is also home to the text Appendices C & D.

ELECTRONIC TEXTBOOK OPTION

This text is offered through CourseSmart for both instructors and students. CourseSmart is an online resource where students can purchase access to this and other McGraw-Hill textbooks in a digital format. Through their browsers, students can access the complete text online for almost half the cost of a traditional text. Purchasing the eTextbook also allows students to take advantage of CourseSmart's web tools for learning, which include full text search, notes and highlighting, and email tools for sharing notes between classmates. To learn more about CourseSmart options, contact your sales representative or visit www.CourseSmart.com.

ACKNOWLEDGMENTS

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The University of Rhode Island provided an atmosphere that encourages teaching excellence, course development, and writing, and for that I am grateful.

Finally, I thank my wife, Mary Louise, and my children, Aileene, Bill, and Andrew, for their support, patience, and understanding.

William J. Palm III
Kingston, Rhode Island
August 2008

ABOUT THE AUTHOR

William J. Palm III is Professor of Mechanical Engineering and Applied Mechanics at the University of Rhode Island. In 1966 he received a B.S. from Loyola College in Baltimore, and in 1971 a Ph.D. in Mechanical Engineering and Astronautical Sciences from Northwestern University in Evanston, Illinois.

During his thirty-seven years as a faculty member, he has taught nineteen courses. One of these is a junior system dynamics course, which he developed. He has authored nine textbooks dealing with modeling and simulation, system dynamics, control systems, vibrations, and MATLAB. These include *Introduction to MATLAB 7 for Engineers* (McGraw-Hill, 2005) and *A Concise Introduction to MATLAB* (McGraw-Hill, 2008). He wrote a chapter on control systems in the *Mechanical Engineers' Handbook* third edition, (M. Kutz, ed., Wiley, 2006), and was a special contributor to the fifth editions of *Statics* and *Dynamics,* both by J. L. Meriam and L. G. Kraige (Wiley, 2002).

Professor Palm's research and industrial experience are in control systems, robotics, vibrations, and system modeling. He was the Director of the Robotics Research Center at the University of Rhode Island from 1985 to 1993, and is the co-holder of a patent for a robot hand. He served as Acting Department Chair from 2002 to 2003. His industrial experience is in automated manufacturing; modeling and simulation of naval systems, including underwater vehicles and tracking systems; and design of control systems for underwater vehicle engine test facilities.

Introduction

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Define the basic terminology of system dynamics.

2. Apply the basic steps used for engineering problem solving.

3. Apply the necessary steps for developing a computer solution.

4. Use units in both the FPS and the SI systems.

5. Develop linear models from given algebraic expressions.

6. Identify the algebraic form and obtain the coefficient values of a model, given a set of data.

7. Apply MATLAB to the methods of this chapter.

This chapter introduces the basic terminology of system dynamics, which includes the notions of *system, static* and *dynamic elements, input,* and *output*. Because we will use both the foot-pound-second (FPS) and the metric (SI) systems of units, the chapter introduces these two systems. Developing mathematical models of input-output relations is essential to the applications of system dynamics. Therefore, we begin our study by introducing some basic methods for developing algebraic models of static elements. We show how to use the methods of function identification and parameter estimation to develop models from data, and how to fit models to scattered data by using the least-squares method. We then show how to apply MATLAB for this purpose.

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[1]MATLAB is a registered trademark of The MathWorks, Inc.

Although Sections 1.4, 1.5, and 1.6 provide the foundation for understanding how to develop models of static elements, coverage of these sections is not required to understand the methods of the remaining chapters, because the appropriate models will be supplied in the examples and chapter problems and thus need not be derived. ∎

1.1 INTRODUCTION TO SYSTEM DYNAMICS

This text is an introduction to system dynamics. We presume that the reader has some background in calculus (specifically, differentiation and integration of functions of a single variable) and in physics (specifically, free body diagrams, Newton's laws of motion for a particle, and elementary dc electricity). In this section we establish some basic terminology and discuss the meaning of the topic "system dynamics," its methodology, and its applications.

SYSTEMS

The meaning of the term *system* has become somewhat vague because of overuse. The original meaning of the term is a combination of elements intended to act together to accomplish an objective. For example, a link in a bicycle chain is usually not considered to be a system. However, when it is used with other links to form a chain, it becomes part of a system. The objective for the chain is to transmit force. When the chain is combined with gears, wheels, crank, handlebars, and other elements, it becomes part of a larger system whose purpose is to transport a person.

The system designer must focus on how all the elements act together to achieve the system's intended purpose, keeping in mind other important factors such as safety, cost, and so forth. Thus, the system designer often cannot afford to spend time on the details of designing the system elements. For example, our bicycle designer might not have time to study the metallurgy involved with link design; that is the role of the chain designer. All the systems designer needs to know about the chain is its strength, its weight, and its cost, because these are the factors that influence its role in the system.

With this "systems point of view," we focus on how connections between the elements influence the overall behavior of the system. This means that sometimes we must accept a less-detailed description of the operation of the individual elements to achieve an overall understanding of the system's performance.

INPUT AND OUTPUT

Like the term "system," the meanings of *input* and *output* have become less precise. Nowadays, for example, a factory manager will call a meeting to seek "input," meaning opinions or data, from the employees, and the manager may refer to the products manufactured in the factory as its "output." However, in the system dynamics meaning of the terms, an *input* is a *cause;* an *output* is an *effect* due to the input. Thus, one input to the bicycle is the force applied to the pedal. One resulting output is the acceleration of the bike. Another input is the angle of the front wheel; the output is the direction of the bike's path of travel.

The behavior of a system element is specified by its *input-output relation,* which is a description of how the output is affected by the input. The input-output relation expresses the cause-and-effect behavior of the element. Such a description, which is represented graphically by the diagram in Figure 1.1.1, can be in the form of a table of numbers, a graph, or a mathematical relation. For example, a force f applied to a
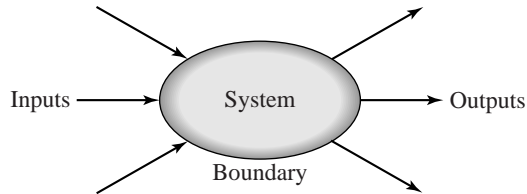
Figure 1.1.1 A system input-output diagram, showing the system boundary.

particle of mass m causes an acceleration a of the particle. The input-output or causal relation is, from Newton's second law, $a = f/m$. The input is f and the output is a.

The input-output relations for the elements in the system provide a means of specifying the connections between the elements. When connected together to form a system, the inputs to some elements will be the outputs from other elements.

The inputs and outputs of a system are determined by the selection of the system's boundary (see Figure 1.1.1). Any causes acting on the system from the world external to this boundary are considered to be system inputs. Similarly, a system's outputs are the outputs from any one or more of the system elements that act on the world outside the system boundary. If we take the bike to be the system, one system input would be the pedal force; another input is the force of gravity acting on the bike. The outputs may be taken to be the bike's position, velocity, and acceleration. Usually, our choices for system outputs are a subset of the possible outputs and are the variables in which we are interested. For example, a performance analysis of the bike would normally focus on the acceleration or velocity, but not on the bike's position.

Sometimes input-output relations are reversible, sometimes not. For example, we can apply a current as input to a resistor and consider the resulting voltage drop to be the output ($v = iR$). Or we can apply a voltage to produce a current through the resistor ($i = v/R$). However, acceleration is the cause of a change in velocity, but not vice versa. If we integrate acceleration a over time, we obtain velocity v; that is $v = \int a \, dt$. Whenever an output of an element is the time integral of the input and the direction of the cause-effect relation is not reversible, we say that the element exhibits *integral causality*. We will see that integral causality constitutes a basic form of causality for all physical systems.

Similar statements can be made about the relation between velocity and displacement. Integration of velocity produces displacement x: $x = \int v \, dt$. Velocity is the cause of displacement, but not vice versa.

Note that the mathematical relations describing integral causality can be reversed; for example, we may write $a = dv/dt$, but this does not mean that the cause-and-effect relation can be reversed.

STATIC AND DYNAMIC ELEMENTS

When the present value of an element's output depends only on the present value of its input, we say the element is a *static* element. For example, the current flowing through a resistor depends only on the present value of the applied voltage. The resistor is thus a static element. However, because no physical element can respond instantaneously, the concept of a static element is an approximation. It is widely used, however, because it results in a simpler mathematical representation; that is, an algebraic representation rather than one involving differential equations.

If an element's present output depends on past inputs, we say it is a *dynamic element*. For example, the present position of a bike depends on what its velocity has been from the start.

In popular usage, the terms static and dynamic distinguish situations in which no change occurs from those that are subject to changes over time. This usage conforms to the preceding definitions of these terms if the proper interpretation is made. A static element's output can change with time only if the input changes and will not change if the input is constant or absent. However, if the input is constant or removed from a dynamic element, its output can still change. For example, if we stop pedaling, the bike's displacement will continue to change because of its momentum, which is due to past inputs.

A *dynamic system* is one whose present output depends on past inputs. A *static system* is one whose output at any given time depends only on the input at that time. A static system contains all static elements. Any system that contains at least one dynamic element must be a dynamic system. *System dynamics,* then, is the study of systems that contain dynamic elements.

MODELING OF SYSTEMS

Table 1.1.1 contains a summary of the methodology that has been tried and tested by the engineering profession for many years. These steps describe a general problem-solving procedure. Simplifying the problem sufficiently and applying the appropriate fundamental principles is called *modeling,* and the resulting mathematical description is called a *mathematical model,* or just a *model*. When the modeling has been finished, we need to solve the mathematical model to obtain the required answer. If the model is highly detailed, we may need to solve it with a computer program.

The form of a mathematical model depends on its purpose. For example, design of electrical equipment requires more than a knowledge of electrical principles. An electric circuit can be damaged if its mounting board experiences vibration. In this case, its force-deflection properties must be modeled. In addition, resistors generate heat, and a thermal model is required to describe this process. Thus, we see that devices

Table 1.1.1 Steps in engineering problem solving.

1. Understand the purpose of the problem.
2. Collect the known information. Realize that some of it might turn out to be not needed.
3. Determine what information you must find.
4. Simplify the problem only enough to obtain the required information. State any assumptions you make.
5. Draw a sketch and label any necessary variables.
6. Determine what fundamental principles are applicable.
7. Think generally about your proposed solution approach and consider other approaches before proceeding with the details.
8. Label each step in the solution process.
9. If you use a program to solve the problem, hand check the results using a simple version of the problem. Checking the dimensions and units, and printing the results of intermediate steps in the calculation sequence can uncover mistakes.
10. Perform a "reality check" on your answer. Does it make sense? Estimate the range of the expected result and compare it with your answer. Do not state the answer with greater precision than is justified by any of the following:
    a. The precision of the given information.
    b. The simplifying assumptions.
    c. The requirements of the problem.

    Interpret the mathematics. If the mathematics produces multiple answers, do not discard some of them without considering what they mean. The mathematics might be trying to tell you something, and you might miss an opportunity to discover more about the problem.

can have many facets: thermal, mechanical, electrical, and so forth. No mathematical model can deal with all these facets. Even if it could, it would be too complex, and thus too cumbersome, to be useful.

For example, a map is a model of a geographic region. But if a single map contains all information pertaining to the roads, terrain elevation, geology, population density, and so on, it would be too cluttered to be useful. Instead, we select the particular type of map required for the purpose at hand. In the same way, we select or construct a mathematical model to suit the requirements of a particular study.

The examples in this text follow the steps in Table 1.1.1, although for compactness the steps are usually not numbered. In each example, following the example's title, there is a *problem statement* that summarizes the results of steps 1 through 5. Steps 6 through 10 are described in the *solution* part of the example. To save space, some steps, such as checking dimensions and units, are not always explicitly displayed. However, you are encouraged to perform these steps on your own.

CONTROL SYSTEMS

Often dynamic systems require a *control system* to perform properly. Thus, proper control system design is one of the most important objectives of system dynamics. Microprocessors have greatly expanded the applications for control systems. These new applications include robotics, mechatronics, micromachines, precision engineering, active vibration control, active noise cancellation, and adaptive optics. Recent technological advancements mean that many machines now operate at high speeds and high accelerations. It is therefore now more often necessary for engineers to pay more attention to the principles of system dynamics.

THEME APPLICATIONS

Two common applications of system dynamics are in (1) motion control systems and (2) vehicle dynamics. Therefore we will use these applications as major themes in many of our examples and problems.

Figure 1.1.2 shows a robot arm, whose motion must be properly controlled to move an object to a desired position and orientation. To do this, each of the several motors and drive trains in the arm must be adequately designed to handle the load, and the motor speeds and angular positions must be properly controlled. Figure 1.1.3 shows a typical motor and drive train for one arm joint. Knowledge of system dynamics is essential to design these subsystems and to control them properly.

Mobile robots are another motion control application, but motion control applications are not limited to robots. Figure 1.1.4 shows the mechanical drive for a conveyor system. The motor, the gears in the speed reducer, the chain, the sprockets, and the drive wheels all must be properly selected, and the motor must be properly controlled for the system to work well. In subsequent chapters we will develop models of these components and use them to design the system and analyze its performance.

Our second major theme application is vehicle dynamics. This topic has received renewed importance for reasons related to safety, energy efficiency, and passenger comfort. Of major interest under this topic is the design of vehicle suspension systems, whose elements include various types of springs and shock absorbers (Figure 1.1.5). *Active* suspension systems, whose characteristics can be changed under computer control, and vehicle-dynamics control systems are undergoing rapid development, and their design requires an understanding of system dynamics.

Figure 1.1.2 A robot arm.

Waist rotation

Shoulder rotation

Elbow rotation

Three wrist rotations
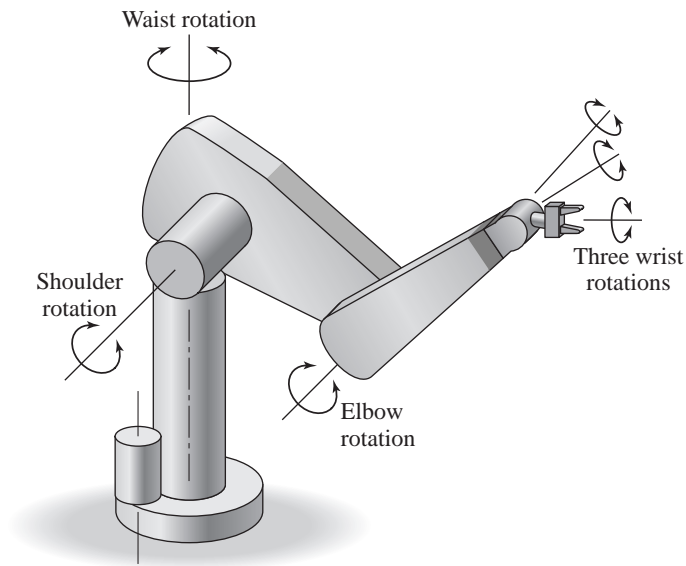
Figure 1.1.3 Mechanical drive for a robot arm joint.
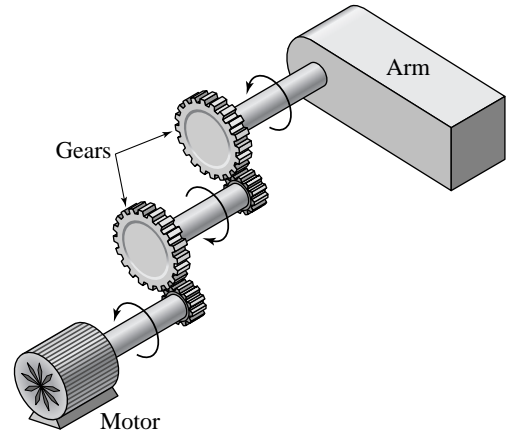
Arm

Gears

Motor

Figure 1.1.4 Mechanical drive for a conveyor system.

Load

Drive chains

Tachometer

Drive wheels

Sprocket 2

Drive shaft

Chain

Sprocket 1

Reducer

Motor

Figure 1.1.5 A vehicle suspension system.

Frame mount

Coil spring

Tie rod

Shock absorber

Frame mount

Steering arm

Wishbone
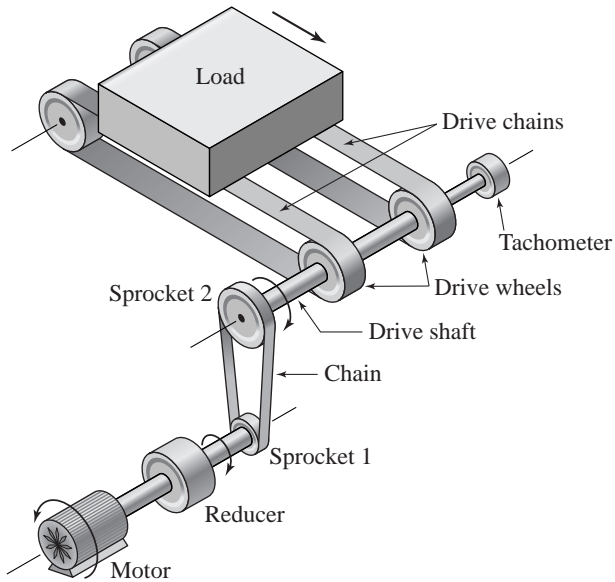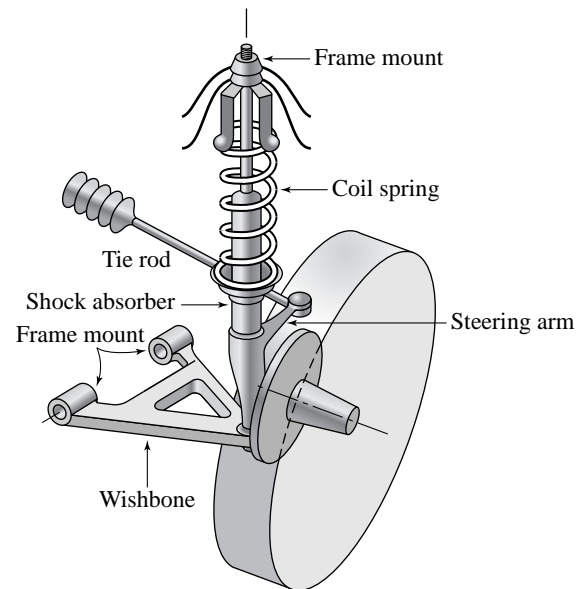
COMPUTER METHODS

The computer methods used in this text are based on MATLAB and Simulink.®1 If you are unfamiliar with MATLAB, Appendix C on the textbook website contains a thorough introduction to the program. No prior experience with Simulink is required;

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[1] Simulink is a registered trademark of The MathWorks, Inc.

Table 1.1.2 Steps for developing a computer solution.

1. State the problem concisely.
2. Specify the data to be used by the program. This is the "input."
3. Specify the information to be generated by the program. This is the "output."
4. Work through the solution steps by hand or with a calculator; use a simpler set of data if necessary.
5. Write and run the program.
6. Check the output of the program with your hand solution.
7. Run the program with your input data and perform a reality check on the output.
8. If you will use the program as a general tool in the future, test it by running it for a range of reasonable data values, and perform a reality check on the results. Document the program with comment statements, flow charts, pseudo-code, or whatever else is appropriate.

we will introduce the necessary methods as we need them. For the convenience of those who prefer to use a software package other than MATLAB or Simulink, we have placed all the MATLAB and Simulink material in optional sections at the end of each chapter. They can be skipped without affecting your understanding of the following chapters. If you use a program, such as MATLAB, to solve a problem, follow the steps shown in Table 1.1.2.

1.2 UNITS

In this book we use two systems of units, the *FPS* system and the metric SI. The common system of units in business and industry in English-speaking countries has been the foot-pound-second (FPS) system. This system is also known as the U.S. customary system or the British Engineering system. Much engineering work in the United States has been based on the FPS system, and some industries continue to use it. The metric Système International d'Unités (SI) nevertheless is becoming the worldwide standard. Until the changeover is complete, engineers in the United States will have to be familiar with both systems.

In our examples, we will use SI and FPS units in the hope that the student will become comfortable with both. Other systems are in use, such as the meter-kilogram-second (mks) and centimeter-gram-second (cgs) metric systems and the British system, in which the mass unit is a pound. We will not use these, because FPS and SI units are the most common in engineering applications. We now briefly summarize these two systems.

FPS UNITS

The FPS system is a *gravitational* system. This means that the primary variable is force, and the unit of mass is derived from Newton's second law. The *pound* is selected as the unit of force and the *foot* and *second* as units of length and time, respectively. From Newton's second law of motion, force equals mass times acceleration, or

$$f = ma \tag{1.2.1}$$

where f is the net force acting on the mass m and producing an acceleration a. Thus, the unit of mass must be

$$\text{mass} = \frac{\text{force}}{\text{acceleration}} = \frac{\text{pound}}{\text{foot/(second)}^2}$$

This mass unit is named the *slug*.

Table 1.2.1 SI and FPS units.

Quantity	Unit name and abbreviation	
	SI Unit	FPS Unit
Time	second (s)	second (sec)
Length	meter (m)	foot (ft)
Force	newton (N)	pound (lb)
Mass	kilogram (kg)	slug
Energy	joule (J)	foot-pound (ft-lb), Btu (= 778 ft-lb)
Power	watt (W)	ft-lb/sec, horsepower (hp)
Temperature	degrees Celsius (°C), degrees Kelvin (K)	degrees Fahrenheit (°F), degrees Rankine (°R)

Table 1.2.2 Unit conversion factors.

Length	1 m = 3.281 ft	1 ft = 0.3048 m
	1 mile = 5280 ft	1 km = 1000 m
Speed	1 ft/sec = 0.6818 mi/hr	1 mi/hr = 1.467 ft/sec
	1 m/s = 3.6 km/h	1 km/h = 0.2778 m/s
	1 km/hr = 0.6214 mi/hr	1 mi/hr = 1.609 km/h
Force	1 N = 0.2248 lb	1 lb = 4.4484 N
Mass	1 kg = 0.06852 slug	1 slug = 14.594 kg
Energy	1 J = 0.7376 ft-lb	1 ft-lb = 1.3557 J
Power	1 hp = 550 ft-lb/sec	1 hp = 745.7 W
	$1 \text{ W} = 1.341 \times 10^{-3}$ hp	
Temperature	$T°\text{C} = 5(T°\text{F} - 32)/9$	$T°\text{F} = 9T°\text{C}/5 + 32$

Through Newton's second law, the weight W of an object is related to the object mass m and the acceleration due to gravity, denoted by g, as follows: $W = mg$. At the surface of the earth, the standard value of g in FPS units is $g = 32.2$ ft/sec^2.

Energy has the dimensions of mechanical work; namely, force times displacement. Therefore, the unit of energy in this system is the *foot-pound* (ft-lb). Another energy unit in common use for historical reasons is the *British thermal unit* (Btu). The relationship between the two is given in Table 1.2.1. Power is the rate of change of energy with time, and a common unit is *horsepower*. Finally, temperature in the FPS system can be expressed in degrees *Fahrenheit* or in absolute units, degrees *Rankine*.

SI UNITS

The SI metric system is an *absolute* system, which means that the mass is chosen as the primary variable, and the force unit is derived from Newton's law. The *meter* and the *second* are selected as the length and time units, and the *kilogram* is chosen as the mass unit. The derived force unit is called the *newton*. In SI units the common energy unit is the newton-meter, also called the *joule,* while the power unit is the joule/second, or *watt*. Temperatures are measured in degrees Celsius, °C, and in absolute units, which are degrees *Kelvin,* K. The difference between the boiling and freezing temperatures of water is 100°C, with 0°C being the freezing point.

At the surface of the earth, the standard value of g in SI units is $g = 9.81$ m/s^2.

Table 1.2.2 gives the most commonly needed factors for converting between the FPS and the SI systems.

OSCILLATION UNITS

There are three commonly used units for frequency of oscillation. If time is measured in seconds, frequency can be specified as *radians/second* or as *hertz,* abbreviated Hz. One hertz is one cycle per second (cps). The relation between cycles per second f and radians per second ω is $2\pi f = \omega$. For sinusoidal oscillation, the *period P*, which is the time between peaks, is related to frequency by $P = 1/f = 2\pi/\omega$. The third way of specifying frequency is revolutions per minute (rpm). Because there are 2π radians per revolution, one rpm $= (2\pi/60)$ radians per second.

1.3 DEVELOPING LINEAR MODELS

A *linear* model of a static system element has the form $y = mx + b$, where x is the input and y is the output of the element. As we will see in Chapter 3, solution of dynamic models to predict system performance requires solution of differential equations. Differential equations based on linear models of the system elements are easier to solve than ones based on nonlinear models. Therefore, when developing models we try to obtain a linear model whenever possible. Sometimes the use of a linear model results in a loss of accuracy, and the engineer must weigh this disadvantage with advantages gained by using a linear model. In this section, we illustrate some ways to obtain linear models.

DEVELOPING LINEAR MODELS FROM DATA

If we are given data on the input-output characteristics of a system element, we can first plot the data to see whether a linear model is appropriate, and if so, we can extract a suitable model. Example 1.3.1 illustrates a common engineering problem—the estimation of the force-deflection characteristics of a cantilever support beam.

A Cantilever Beam Deflection Model | **EXAMPLE 1.3.1**

■ Problem
The deflection of a cantilever beam is the distance its end moves in response to a force applied at the end (Figure 1.3.1). The following table gives the measured deflection x that was produced in a particular beam by the given applied force f. Plot the data to see whether a linear relation exists between f and x.

Force f (lb)	0	100	200	300	400	500	600	700	800
Deflection x (in.)	0	0.15	0.23	0.35	0.37	0.5	0.57	0.68	0.77
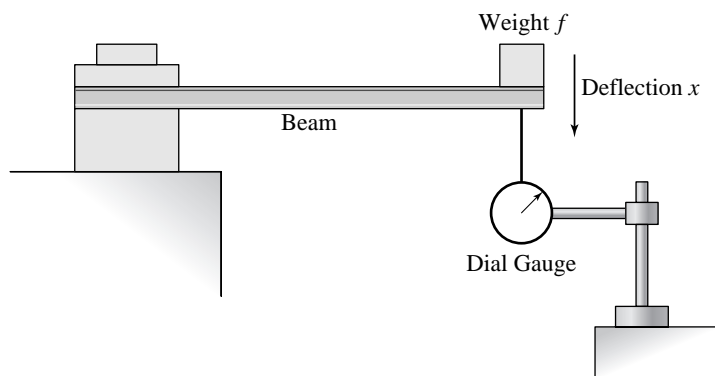
Figure 1.3.1 Measurement of beam deflection.

Weight f

Deflection x

Beam

Dial Gauge

Figure 1.3.2 Plot of beam deflection versus applied force.
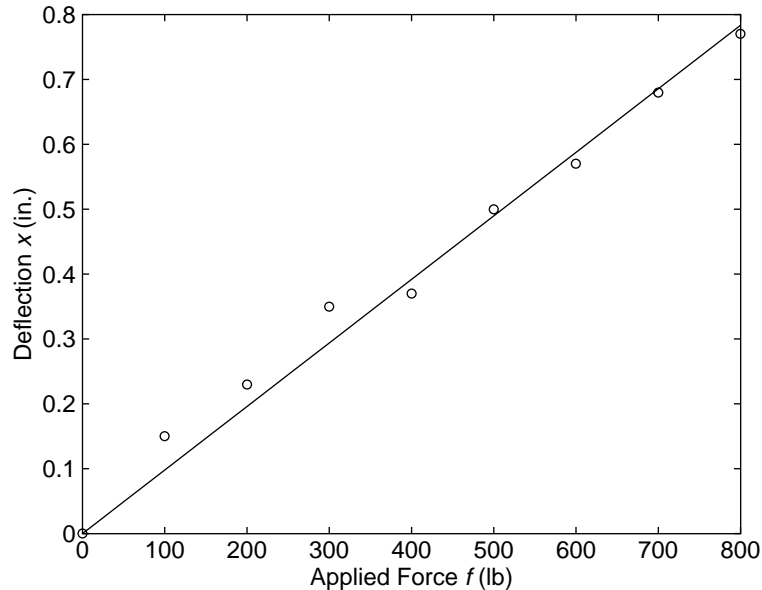
■ Solution

The plot is shown in Figure 1.3.2. Common sense tells us that there must be zero beam deflection if there is no applied force, so the curve describing the data must pass through the origin. The straight line shown was drawn by aligning a straightedge so that it passes through the origin and near most of the data points (note that this line is subjective; another person might draw a different line). The data lies close to a straight line, so we can use the linear function $x = af$ to describe the relation. The value of the constant a can be determined from the slope of the line, which is

$$a = \frac{0.78 - 0}{800 - 0} = 9.75 \times 10^{-4} \text{ in./lb}$$

As we will see in Chapter 4, this relation is usually written as $f = kx$, where k is the beam *stiffness*. Thus, $k = 1/a = 1025$ lb/in.

Once we have discovered a functional relation that describes the data, we can use it to make predictions for conditions that lie *within* the range of the original data. This process is called *interpolation*. For example, we can use the beam model to estimate the deflection when the applied force is 550 lb. We can be fairly confident of this prediction because we have data below and above 550 lb and we have seen that our model describes this data very well.

Extrapolation is the process of using the model to make predictions for conditions that lie *outside* the original data range. Extrapolation might be used in the beam application to predict how much force would be required to bend the beam 1.2 in. We must be careful when using extrapolation, because we usually have no reason to believe that the mathematical model is valid beyond the range of the original data. For example, if we continue to bend the beam, eventually the force is no longer proportional to the deflection, and it becomes much greater than that predicted by the linear model. Extrapolation has a use in making tentative predictions, which must be backed up later on by testing.

In some applications, the data contains so much scatter that it is difficult to identify an appropriate straight line. In such cases, we must resort to a more systematic and objective way of obtaining a model. This topic will be treated in Section 1.5.

LINEARIZATION

Not all element descriptions are in the form of data. Often we know the analytical form of the model, and if the model is nonlinear, we can obtain a linear model that is an accurate approximation over a limited range of the independent variable. Examples 1.3.2 and 1.3.3 illustrate this technique, which is called *linearization*.

Linearization of the Sine Function   **EXAMPLE 1.3.2**

■ Problem
We will see in Chapter 2 that the models of many mechanical systems involve the sine function $\sin\theta$, which is nonlinear. Obtain three linear approximations of $f(\theta) = \sin\theta$, one valid near $\theta = 0$, one near $\theta = \pi/3$ rad ($60°$), and one near $\theta = 2\pi/3$ rad ($120°$).

■ Solution
The essence of the linearization technique is to replace the plot of the nonlinear function with a straight line that passes through the reference point and has the same slope as the nonlinear function at that point. Figure 1.3.3 shows the sine function and the three straight lines obtained with this technique. Note that the slope of the sine function is its derivative, $d\sin\theta/d\theta = \cos\theta$, and thus the slope is not constant but varies with θ.

Consider the first reference point, $\theta = 0$. At this point the sine function has the value $\sin 0 = 0$, the slope is $\cos 0 = 1$, and thus the straight line passing through this point with a slope of 1 is $f(\theta) = \theta$. This is the linear approximation of $f(\theta) = \sin\theta$ valid near $\theta = 0$, line A in Figure 1.3.3. Thus we have derived the commonly seen small-angle approximation $\sin\theta \approx \theta$.
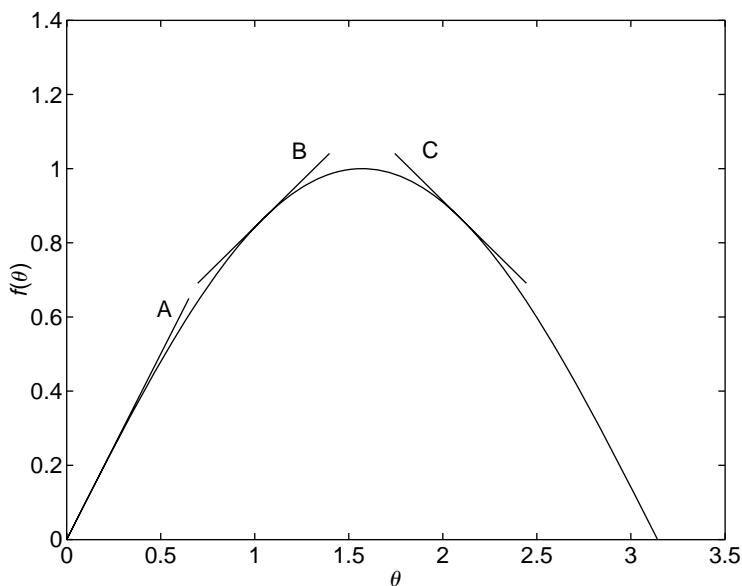
Figure 1.3.3 Three linearized models of the sine function.

Next consider the second reference point, $\theta = \pi/3$ rad. At this point the sine function has the value $\sin \pi/3 = 0.866$, the slope is $\cos \pi/3 = 0.5$, and thus the straight line passing through this point with a slope of 0.5 is $f(\theta) = 0.5(\theta - \pi/3) + 0.866$, line B in Figure 1.3.3. This is the linear approximation of $f(\theta) = \sin \theta$ valid near $\theta = \pi/3$.

Now consider the third reference point, $\theta = 2\pi/3$ rad. At this point the sine function has the value $\sin 2\pi/3 = 0.866$, the slope is $\cos 2\pi/3 = -0.5$, and thus the straight line passing through this point with a slope of -0.5 is $f(\theta) = -0.5(\theta - 2\pi/3) + 0.866$, line C in Figure 1.3.3. This is the linear approximation of $f(\theta) = \sin \theta$ valid near $\theta = 2\pi/3$.

In Example 1.3.2 we used a graphical approach to develop the linear approximation. The linear approximation can also be developed with an analytical approach based on the Taylor series. The Taylor series represents a function $f(\theta)$ in the vicinity of $\theta = \theta_r$ by the expansion

$$f(\theta) = f(\theta_r) + \left(\frac{df}{d\theta} \right)_{\theta=\theta_r} (\theta - \theta_r) + \frac{1}{2} \left(\frac{d^2 f}{d\theta^2} \right)_{\theta=\theta_r} (\theta - \theta_r)^2 + \cdots$$

$$+ \frac{1}{k!} \left(\frac{d^k f}{d\theta^k} \right)_{\theta=\theta_r} (\theta - \theta_r)^k + \cdots \tag{1.3.1}$$

Consider the nonlinear function $f(\theta)$, which is sketched in Figure 1.3.4. Let $[\theta_r, f(\theta_r)]$ denote the reference operating condition of the system. A model that is linear can be obtained by expanding $f(\theta)$ in a Taylor series near this point and truncating the series beyond the first-order term. If θ is "close enough" to θ_r, the terms involving $(\theta - \theta_r)^i$ for $i \geq 2$ are small compared to the first two terms in the series. The result is

$$f(\theta) = f(\theta_r) + \left(\frac{df}{d\theta} \right)_r (\theta - \theta_r) \tag{1.3.2}$$

Figure 1.3.4 Graphical interpretation of function linearization.
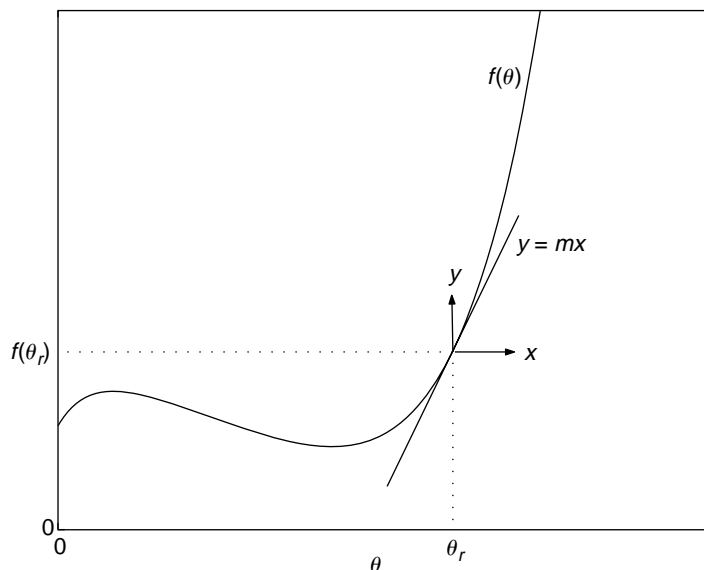
where the subscript r on the derivative means that it is evaluated at the reference point. This is a linear relation. To put it into a simpler form let m denote the slope at the reference point.

$$m = \left(\frac{df}{d\theta}\right)_r \tag{1.3.3}$$

Let y denote the difference between $f(\theta)$ and the reference value $f(\theta_r)$.

$$y = f(\theta) - f(\theta_r) \tag{1.3.4}$$

Let x denote the difference between θ and the reference value θ_r.

$$x = \theta - \theta_r \tag{1.3.5}$$

Then (1.3.2) becomes

$$y = mx \tag{1.3.6}$$

The geometric interpretation of this result is shown in Figure 1.3.4. We have replaced the original function $f(\theta)$ with a straight line passing through the point $[\theta_r, f(\theta_r)]$ and having a slope equal to the slope of $f(\theta)$ at the reference point. Using the (y, x) coordinates gives a zero intercept, and simplifies the relation.

| Linearization of a Square-Root Model | **EXAMPLE 1.3.3** |

■ Problem
We will see in Chapter 7 that the models of many fluid systems involve the square-root function $\sqrt{h}$, which is nonlinear. Obtain a linear approximation of $f(h) = \sqrt{h}$ valid near $h = 9$.

■ Solution
The truncated Taylor series for this function is

$$f(h) = f(h_r) + \left.\frac{d\left(\sqrt{h}\right)}{dh}\right|_r (h - h_r)$$

where $h_r = 9$. This gives the linear approximation

$$f(h) = \sqrt{9} + \left.\frac{1}{2}h^{-1/2}\right|_r (h - 9) = 3 + \frac{1}{6}(h - 9)$$

This equation gives the straight line shown in Figure 1.3.5.

Sometimes we need a linear model that is valid over so wide a range of the independent variable that a model obtained from the Taylor series is inaccurate or grossly incorrect. In such cases, we must settle for a linear function that gives a conservative estimate.

Figure 1.3.5 Linearization of the square-root function.
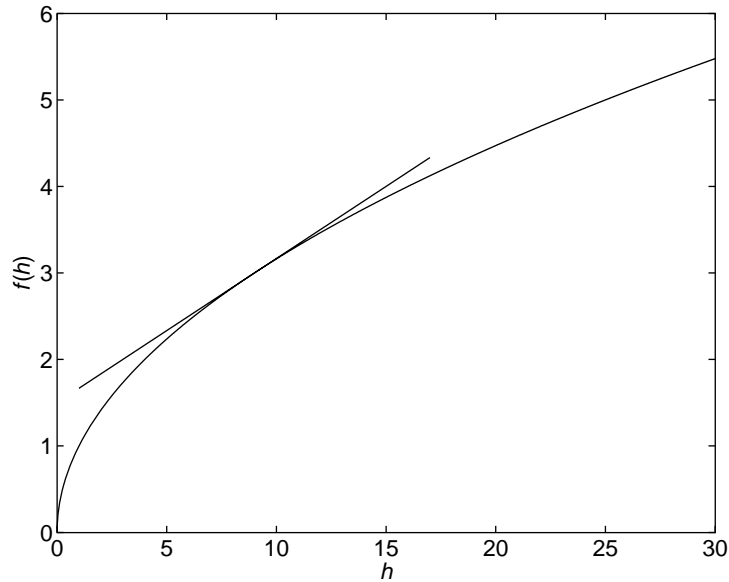
EXAMPLE 1.3.4

Modeling Fluid Drag

■ **Problem**

The drag force on an object moving through a liquid or a gas is a function of the velocity. A commonly used model of the drag force D on an object is

$$D = \frac{1}{2}\rho A C_D v^2 \tag{1}$$

where ρ is the mass density of the fluid, A is the object's cross-sectional area normal to the relative flow, v is the object's velocity relative to the fluid, and C_D is the *drag coefficient*, which is usually determined from wind-tunnel or water-channel tests on models. Curve A in Figure 1.3.6 is a plot of this equation for an Aerobee rocket 1.25 ft in diameter, with $C_D = 0.4$, moving through the lower atmosphere where $\rho = 0.0023$ slug/ft^3, for which equation (1) becomes

$$D = 0.00056v^2 \tag{2}$$

a. Obtain a linear approximation to this drag function valid near $v = 600$ ft/sec.
b. Obtain a linear approximation that gives a conservative (high) estimate of the drag force as a function of the velocity over the range $0 \le v \le 1000$ ft/sec.

■ **Solution**

a. The Taylor series approximation of equation (2) near $v = 600$ is

$$D = D|_{v=600} + \left.\frac{dD}{dv}\right|_{v=600} (v - 600) = 201.6 + 0.672(v - 600)$$

This straight line is labeled B in Figure 1.3.6. Note that it predicts that the drag force will be negative when the velocity is less than 300 ft/sec, a result that is obviously incorrect. This illustrates how we must be careful when using linear approximations.

b. The linear model that gives a conservative estimate of the drag force (that is, an estimate that is never less than the actual drag force) is the straight-line model that passes through the origin and the point at $v = 1000$. This is the equation $D = 0.56v$, shown by the straight line C in Figure 1.3.6.
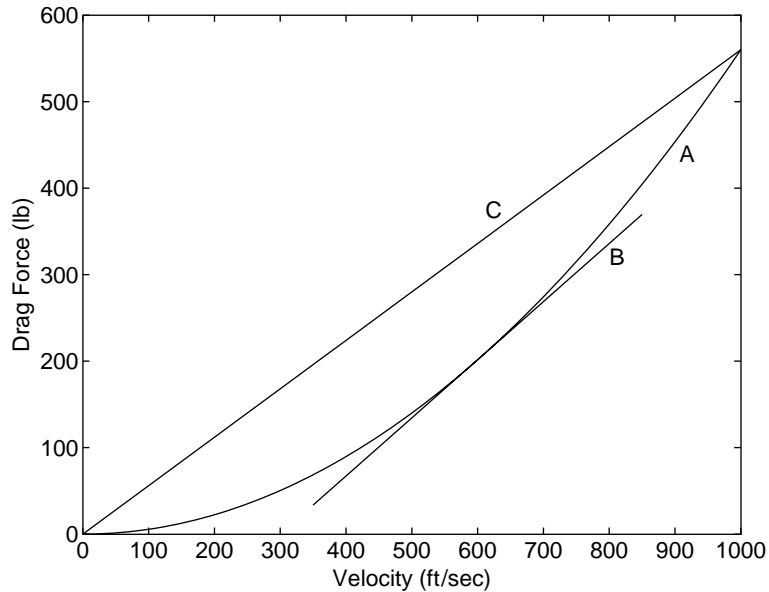
Figure 1.3.6 Models of fluid drag.

1.4 FUNCTION IDENTIFICATION AND PARAMETER ESTIMATION

Function identification, or function *discovery,* is the process of identifying or discovering a function that can describe a particular set of data. The term *curve fitting* is also used to describe the process of finding a curve, and the function generating the curve, to describe a given set of data. *Parameter estimation* is the process of obtaining values for the parameters, or coefficients, in the function that describes the data.

The following three function types can often describe physical phenomena.

1. The *linear* function $y(x) = mx + b$. Note that $y(0) = b$.
2. The *power* function $y(x) = bx^m$. Note that $y(0) = 0$ if $m \geq 0$, and $y(0) = \infty$ if $m < 0$.
3. The *exponential* function $y(x) = b(10)^{mx}$ or its equivalent form $y = be^{mx}$, where e is the base of the natural logarithm ($\ln e = 1$). Note that $y(0) = b$ for both forms.

For example, the linear function describes the voltage-current relation for a resistor ($v = iR$) and the velocity versus time relation for an object with constant acceleration a ($v = at + v_0$). The distance d traveled by a falling object versus time is described by a power function ($d = 0.5gt^2$). The temperature change ΔT of a cooling object can be described by an exponential function ($\Delta T = \Delta T_0 e^{-ct}$).

Each function gives a straight line when plotted using a specific set of axes:

1. The linear function $y = mx + b$ gives a straight line when plotted on rectilinear axes. Its slope is m and its y intercept is b.
2. The power function $y = bx^m$ gives a straight line when plotted on log-log axes.
3. The exponential function $y = b(10)^{mx}$ and its equivalent form, $y = be^{mx}$, give a straight line when plotted on semilog axes with a logarithmic y axis.

Figure 1.4.1 The power function $y = 2x^{-0.5}$ and the exponential function $y = 10(10^{-x})$, plotted on rectilinear, semi-log, and log-log axes, respectively.
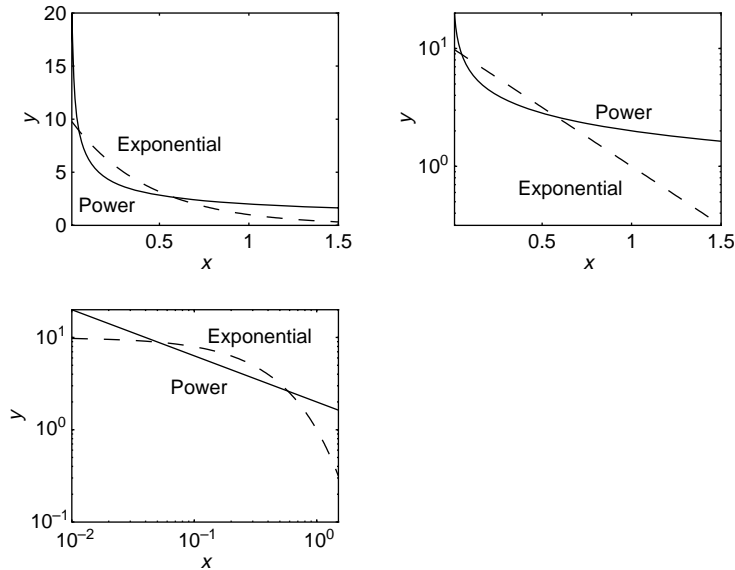
These properties of the power and exponential functions are illustrated in Figure 1.4.1, which shows the power function $y = 2x^{-0.5}$ and the exponential function $y = 10(10^{-x})$.

When we need to identify a function that describes a given set of data, we look for a set of axes (rectilinear, semi-log, or log-log) on which the data forms a straight line, because a straight line is the one most easily recognized by eye, and therefore we can easily tell if the function will fit the data well.

Using the following properties of base-ten logarithms, which are shared with natural logarithms, we have:

$$\log(ab) = \log a + \log b$$

$$\log(a^m) = m \log a$$

Take the logarithm of both sides of the power equation $y = bx^m$ to obtain

$$\log y = \log(bx^m) = \log b + m \log x$$

This has the form $Y = B + mX$ if we let $Y = \log y$, $X = \log x$, and $B = \log b$. Thus if we plot Y versus X on rectilinear scales, we will obtain a straight line whose slope is m and whose intercept is B. This is the same as plotting $\log y$ versus $\log x$ on rectilinear scales, so we will obtain a straight line whose slope is m and whose intercept is $\log b$. This process is equivalent to plotting y versus x on log-log axes. Thus, if the data can be described by the power function, it will form a straight line when plotted on log-log axes.

Taking the logarithm of both sides of the exponential equation $y = b(10)^{mx}$ we obtain:

$$\log y = \log[b(10)^{mx}] = \log b + mx \log 10 = \log b + mx$$

because $\log 10 = 1$. This has the form $Y = B + mx$ if we let $Y = \log y$ and $B = \log b$. Thus if we plot Y versus x on rectilinear scales, we will obtain a straight line whose slope is m and whose intercept is B. This is the same as plotting $\log y$ versus x on rectilinear scales, so we will obtain a straight line whose slope is m and whose intercept is $\log b$. This is equivalent to plotting y on a log axis and x on a rectilinear axis. Thus, if the data can be described by the exponential function, it will form a straight line when plotted on semilog axes (with the log axis used for the ordinate).

This property also holds for the other exponential form: $y = be^{mx}$. Taking the logarithm of both sides gives

$$\log y = \log(be^{mx}) = \log b + mx \log e$$

This has the form

$$Y = B + Mx$$

if we let $Y = \log y$, $B = \log b$, and $M = m \log e$. Thus if we plot Y versus x on rectilinear scales, we will obtain a straight line whose slope is M and whose intercept is B. This is the same as plotting $\log y$ versus x on rectilinear scales, so we will obtain a straight line whose slope is $m \log e$ and whose intercept is $\log b$. This is equivalent to plotting y on a log axis and x on a rectilinear axis. Thus, equivalent exponential form will also plot as a straight line on semilog axes.

STEPS FOR FUNCTION IDENTIFICATION

Here is a summary of the procedure to find a function that describes a given set of data. We assume that the data can be described by one of the three function types given above. Fortunately, many applications generate data that can be described by these functions. The procedure is

1. Examine the data near the origin. The exponential functions $y = b(10)^{mx}$ and $y = be^{mx}$ can never pass through the origin (unless, of course $b = 0$, which is a trivial case). See Figure 1.4.2 for examples with $b = 1$. The linear function $y = mx + b$ can pass through the origin only if $b = 0$. The power function $y = bx^m$ can pass through the origin but only if $m > 0$. See Figure 1.4.3 for examples.
2. Plot the data using rectilinear scales. If it forms a straight line, then it can be represented by the linear function, and you are finished. Otherwise, if you have data at $x = 0$, then
   a. If $y(0) = 0$, try the power function, or
   b. If $y(0) \neq 0$, try the exponential function.
   If data is not given for $x = 0$, proceed to step 3.

The Exponential Function $y = 10^{mx}$
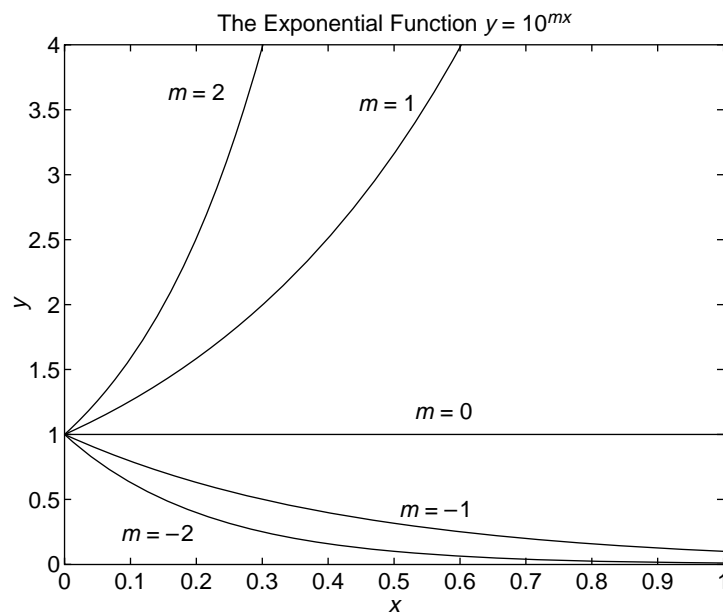
Figure 1.4.2 Examples of exponential functions.

Figure 1.4.3 Examples of
power functions.

The Power Function $y = x^m$
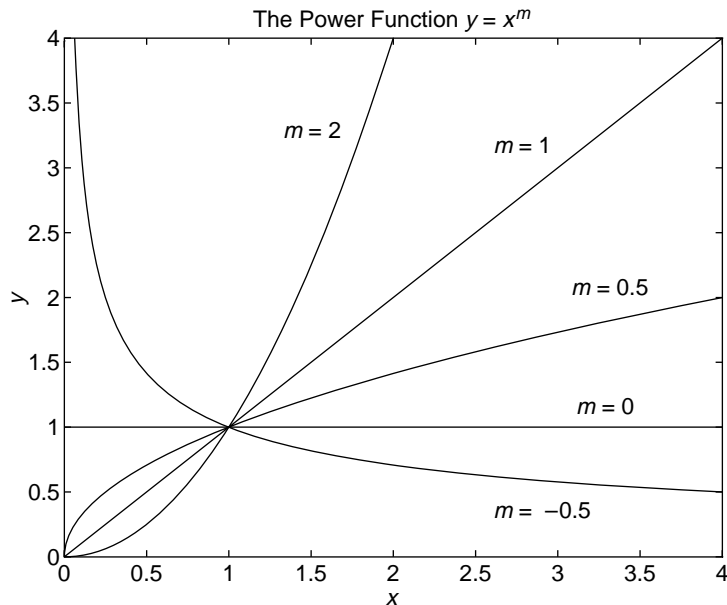
3. If you suspect a power function, plot the data using log-log scales. Only a power function will form a straight line. If you suspect an exponential function, plot it using semilog scales. Only an exponential function will form a straight line.

OBTAINING THE COEFFICIENTS

There are several ways to obtain the values of the coefficients b and m. If the data lie very close to a straight line, we can draw the line through the data using a straightedge and then read two points from the line. These points can be conveniently chosen to coincide with gridlines to eliminate interpolation error. Let these two points be denoted (x_1, y_1) and (x_2, y_2).

For the linear function $y = mx + b$, the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Once m is computed, b can be found by evaluating $y = mx + b$ at a given point, say the point (x_1, y_1). Thus $b = y_1 - mx_1$.

For the power function $y = bx^m$, we can write the following equations for the two chosen points.

$$y_1 = bx_1^m \quad y_2 = bx_2^m$$

These can be solved for m as follows.

$$m = \frac{\log(y_2/y_1)}{\log(x_2/x_1)}$$

Once m is computed, b can be found by evaluating $y = bx^m$ at a given point, say the point (x_1, y_1). Thus $b = y_1 x_1^{-m}$.

For the exponential function $y = b(10)^{mx}$ we can write the following equations for the two chosen points.

$$y_1 = b(10)^{mx_1} \quad y_2 = b(10)^{mx_2}$$

These can be solved for m as follows.

$$m = \frac{1}{x_2 - x_1} \log \frac{y_2}{y_1}$$

Once m is computed, b can be found by evaluating $y = b(10)^{mx}$ at a given point, say the point (x_1, y_1). Thus $b = y_1 10^{-mx_1}$.

A similar procedure can be used for the other exponential form, $y = be^{mx}$. The solutions are

$$m = \frac{1}{x_2 - x_1} \ln \frac{y_2}{y_1}$$

$$b = y_1 e^{-mx_1}$$

If the data are scattered about a straight line to the extent that it is difficult to identify a unique straight line that describes the data, we can use the *least-squares method* to obtain the function. This method finds the coefficients of a polynomial of specified degree n that best fits the data, in the so-called "least-squares sense." We discuss this method in Section 1.5. The MATLAB implementation of this method uses the `polyfit` function, which is discussed in Section 1.6.

Examples 1.4.1 and 1.4.2 feature experiments that you can easily perform on your own. Engineers are often required to make predictions about the temperatures that will occur in various industrial processes, for example. Example 1.4.1 illustrates how we can use function identification to predict the temperature dynamics of a cooling process.

Temperature Dynamics of Water | **EXAMPLE 1.4.1**

■ Problem
Water in a glass measuring cup was allowed to cool after being heated to 204°F. The ambient air temperature was 70°F. The measured water temperature at various times is given in the following table.

Time (sec)	0	120	240	360	480	600
Temperature (°F)	204	191	178	169	160	153

Time (sec)	720	840	960	1080	1200
Temperature (°F)	147	141	137	132	127

Obtain a functional description of the water temperature versus time.

■ Solution
Common sense tells us that the water temperature will eventually reach the air temperature of 70°. Thus we first subtract 70° from the temperature data T and seek to obtain a functional description of the relative temperature, $\Delta T = T - 70$. A plot of the relative temperature data is shown in Figure 1.4.4. We note that the plot has a distinct curvature and that it does not pass through the origin. Thus we can rule out the linear function and the power function as candidates.

To see if the data can be described by an exponential function, we plot the data on a semilog plot, which is shown in Figure 1.4.5. The straight line shown can be drawn by aligning a straight-edge so that it passes near most of the data points (note that this line is subjective; another person might draw a different line). The data lie close to a straight line, so we can use the exponential function to describe the relative temperature.

Figure 1.4.4 Plot of relative temperature versus time.
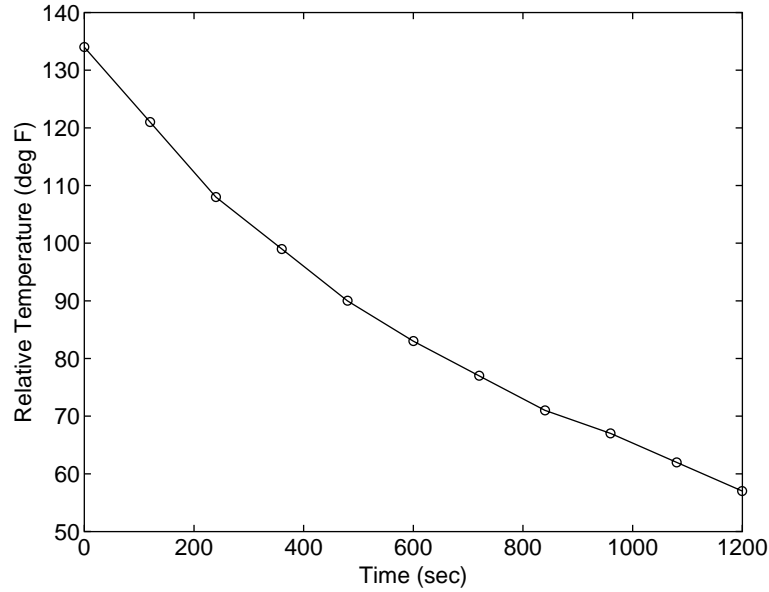
Figure 1.4.5 Semilog plot of relative temperature versus time.
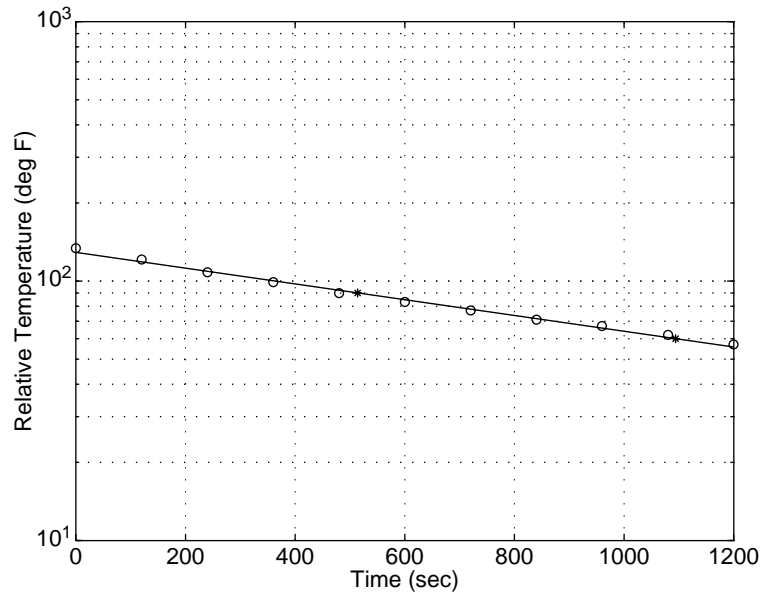
Using the second form of the exponential function, we can write $\Delta T = be^{mt}$. Next we select two points on the straight line to find the values of b and m. The two points indicated by an asterisk were selected to minimize interpolation error because they lie near grid lines. The accuracy of the values read from the plot obviously depends on the size of the plot. Two points read from the plot are $(1090, 60)$ and $(515, 90)$. Using the equations developed previously to compute b and m (with t replacing x and ΔT replacing y), we have

$$m = \frac{1}{1090 - 515} \ln \frac{60}{90} = -0.0007$$

$$b = 90e^{-0.0007(515)} = 129$$
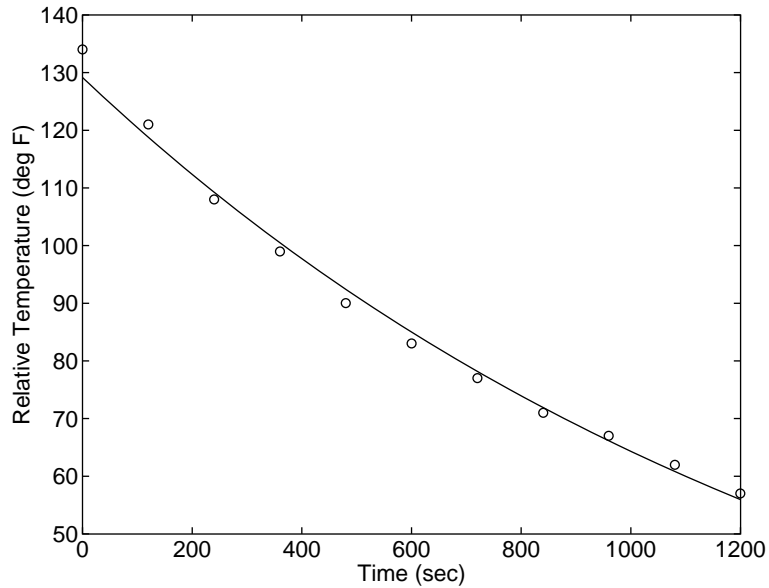
Figure 1.4.6 Comparison of the fitted function with the data.

Thus the estimated function is

$$\Delta T = 129e^{-0.0007t} \qquad \text{or} \qquad T = \Delta T + 70 = 129e^{-0.0007t} + 70$$

where ΔT and T are in °F and time t is in seconds. The plot of ΔT versus t is shown in Figure 1.4.6. From this we can see that the function provides a reasonably good description of the data. In Section 1.5 we will discuss how to quantify the quality of this description.

Engineers often need a model to calculate the flow rates of fluids under pressure. The coefficients of such models must often be determined from measurements.

| Orifice Flow | **EXAMPLE 1.4.2** |

■ **Problem**

A hole 6 mm in diameter was made in a translucent milk container (Figure 1.4.7). A series of marks 1 cm apart was made above the hole. While adjusting the tap flow to keep the water height constant, the time for the outflow to fill a 250-ml cup was measured ($1 \text{ ml} = 10^{-6} \text{ m}^3$). This was repeated for several heights. The data are given in the following table.

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

Obtain a functional description of the volume outflow rate f as a function of water height h above the hole.

■ **Solution**

First obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

$$f = \frac{250}{t}$$

Figure 1.4.7 An experiment to determine flow rate versus liquid height.
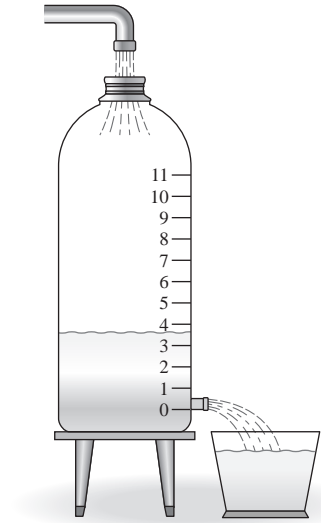
Figure 1.4.8 Plot of flow rate data.
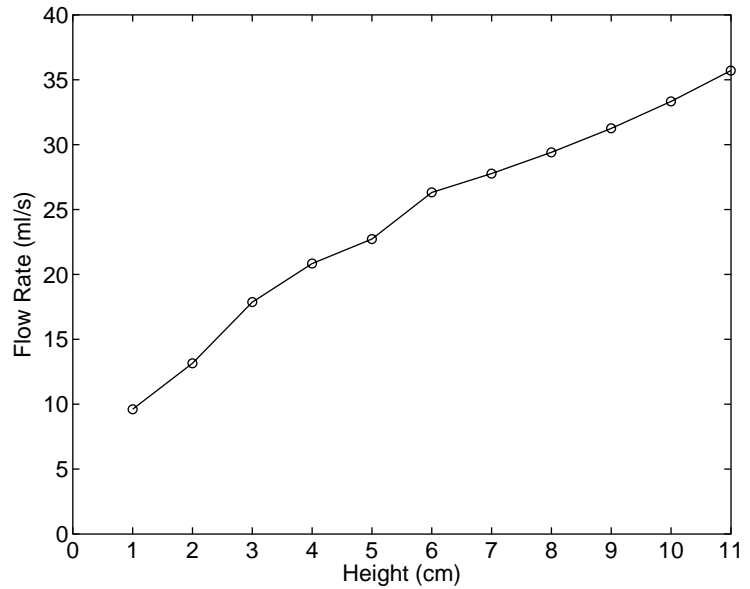
Figure 1.4.9 Log-log plot of flow rate data.
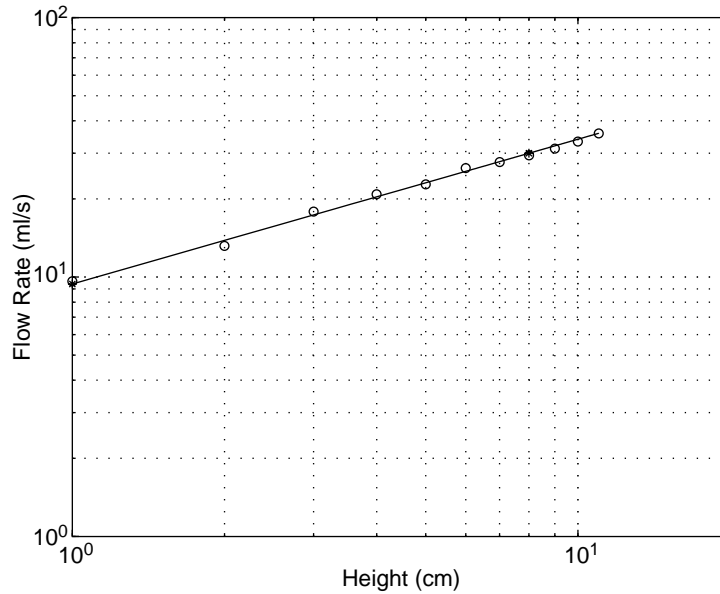
A plot of the resulting flow rate data is shown in Figure 1.4.8. There is some curvature in the plot, so we rule out the linear function. Common sense tells us that the outflow rate will be zero when the height is zero, so we can rule out the exponential function because it cannot pass through the origin.

The log-log plot shown in Figure 1.4.9 shows that the data lie close to a straight line, so we can use the power function to describe the flow rate as a function of height. Thus we can write

$$f = bh^m$$
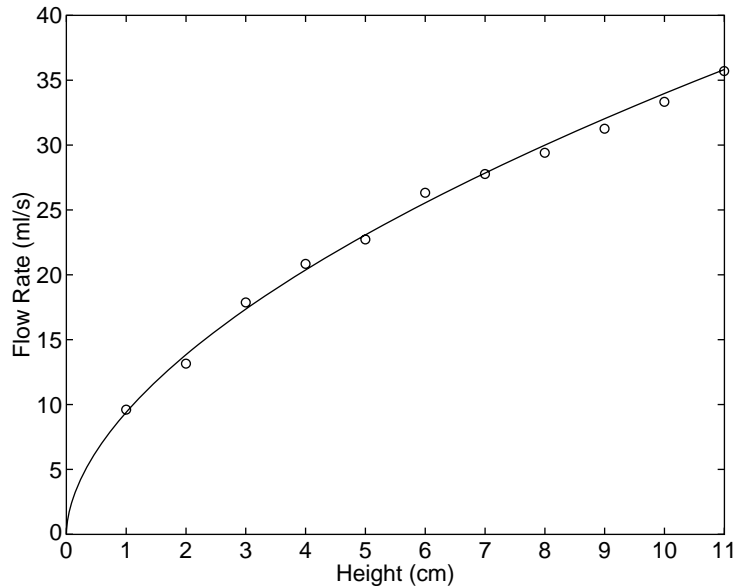
The straight line shown can be drawn by aligning a straightedge so that it passes near most of the data points (note that this line is subjective; another person might draw a different line). Next we select two points on the straight line to find the values of b and m. The two points indicated by an asterisk were selected to minimize interpolation error because they lie near grid lines. The accuracy of the values read from the plot obviously depends on the size of the plot. The values of the points as read from the plot are $(1, 9.4)$ and $(8, 30)$. Using the equations developed previously to compute b and m (with h replacing x and f replacing y), we have

$$m = \frac{\log(30/9.4)}{\log(8/1)} = 0.558$$

$$b = 9.4\,(1^{-0.558}) = 9.4$$

Thus the estimated function is $f = 9.4h^{0.558}$, where f is the outflow rate in ml/s and the water height h is in centimeters. The plot of f versus h is shown in Figure 1.4.10. From this we can see that the function provides a reasonably good description of the data. In Section 1.5 we will discuss how to quantify the quality of this description.

1.5 FITTING MODELS TO SCATTERED DATA

In practice the data often will not lie very close to a straight line, and if we ask two people to draw a straight line passing as close as possible to all the data points, we will probably receive two different answers. A systematic and objective way of obtaining a straight line describing the data is the *least-squares* method. Suppose we want to find the coefficients of the straight line $y = mx + b$ that best fits the following data.

x	0	5	10
y	2	6	11

According to the least-squares criterion, the line that gives the best fit is the one that minimizes J, the sum of the squares of the vertical differences between the line and the data points (see Figure 1.5.1). These differences are called the *residuals*. Here there are

Figure 1.5.1 Illustration of
the least-squares criterion.
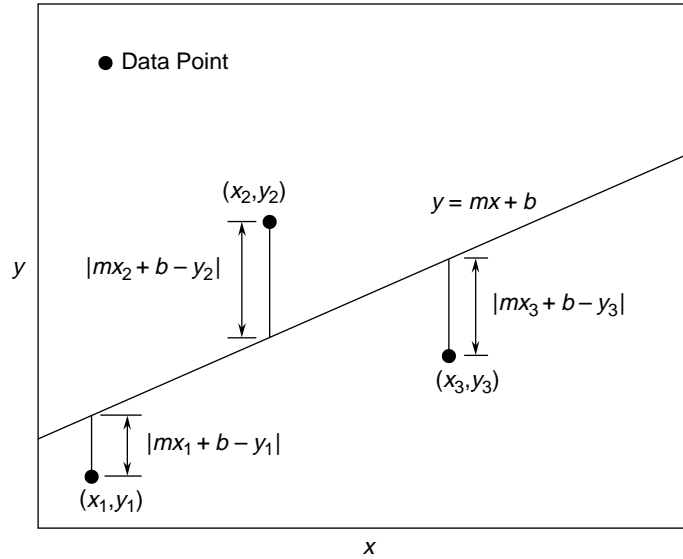
three data points and J is given by

$$J = \sum_{i=1}^{3} (mx_i + b - y_i)^2$$

Substituting the data values (x_i, y_i) given in the table, we obtain

$$J = (0m + b - 2)^2 + (5m + b - 6)^2 + (10m + b - 11)^2$$

The values of m and b that minimize J can be found from $\partial J / \partial m = 0$ and $\partial J / \partial b = 0$.

$$\frac{\partial J}{\partial m} = 2(5m + b - 6)(5) + 2(10m + b - 11)(10) = 250m + 30b - 280 = 0$$

$$\frac{\partial J}{\partial b} = 2(b - 2) + 2(5m + b - 6) + 2(10m + b - 11) = 30m + 6b - 38 = 0$$

These conditions give the following equations that must be solved for the two unknowns m and b.

$$250m + 30b = 280$$
$$30m + 6b = 38$$

The solution is $m = 9/10$ and $b = 11/6$. The best straight line in the least-squares sense is $y = (9/10)x + 11/6$. This is shown in Figure 1.5.2 along with the data.

If we evaluate this equation at the data values $x = 0$, 5, and 10, we obtain the values $y = 1.8333$, 6.3333, and 10.8333. These values are different than the given data values $y = 2$, 6, and 11 because the line is not a perfect fit to the data. The value of J is $J = (1.8333 - 2)^2 + (6.3333 - 6)^2 + (10.8333 - 11)^2 = 0.1666$. No other straight line will give a lower value of J for these data.
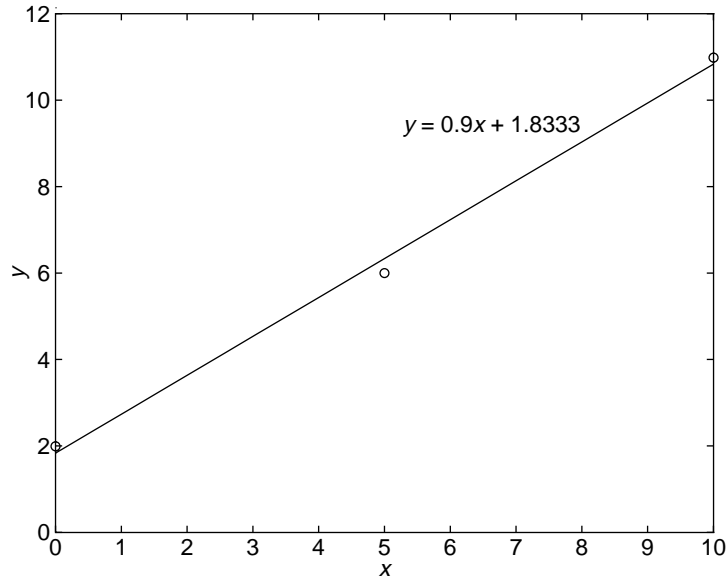
Figure 1.5.2 Example of a least-squares fit.

THE GENERAL LINEAR CASE

We can generalize the preceding results to obtain formulas for the coefficients m and b in the linear equation $y = mx + b$. Note that for n data points,

$$J = \sum_{i=1}^{n} (mx_i + b - y_i)^2$$

The values of m and b that minimize J are found from $\partial J/\partial m = 0$ and $\partial J/\partial b = 0$. These conditions give the following equations that must be solved for m and b:

$$\frac{\partial J}{\partial m} = 2\sum_{i=1}^{n} (mx_i + b - y_i)\, x_i = 2\sum_{i=1}^{n} mx_i^2 + 2\sum_{i=1}^{n} bx_i - 2\sum_{i=1}^{n} y_i x_i = 0$$

$$\frac{\partial J}{\partial b} = 2\sum_{i=1}^{n} (mx_i + b - y_i) = 2\sum_{i=1}^{n} mx_i + 2\sum_{i=1}^{n} b - 2\sum_{i=1}^{n} y_i = 0$$

These equations become

$$m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i x_i \tag{1.5.1}$$

$$m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i \tag{1.5.2}$$

These are two linear equations in terms of m and b.

Because the exponential and power functions form straight lines on semilog and log-log axes, we can use the previous results after computing the logarithms of the data.

| EXAMPLE 1.5.1 | Fitting Data with the Power Function |

■ Problem

Find a functional description of the following data:

x	1	2	3	4
y	5.1	19.5	46	78

■ Solution

These data do not lie close to a straight line when plotted on linear or semilog axes. However, they do when plotted on log-log axes. Thus a power function $y = bx^m$ can describe the data. Using the transformations $X = \log x$ and $Y = \log y$, we obtain the new data table:

$X = \log x$	0	0.3010	0.4771	0.6021
$Y = \log y$	0.7076	1.2900	1.6628	1.8921

From this table we obtain

$$\sum_{i=1}^{4} X_i = 1.3803 \quad \sum_{i=1}^{4} Y_i = 5.5525$$

$$\sum_{i=1}^{4} X_i Y_i = 2.3208 \quad \sum_{i=1}^{4} X_i^2 = 0.6807$$

Using X, Y, and $B = \log b$ instead of x, y, and b in (1.5.1) and (1.5.2) we obtain

$$0.6807m + 1.3803B = 2.3208$$

$$1.3803m + 4B = 5.5525$$

The solution is $m = 1.9802$ and $B = 0.7048$. This gives $b = 10^B = 5.068$. Thus, the desired function is $y = 5.068x^{1.9802}$.

CONSTRAINING MODELS TO PASS THROUGH A GIVEN POINT

Many applications require a model whose form is dictated by physical principles. For example, the force-extension model of a spring must pass through the origin $(0, 0)$ because the spring exerts no force when it is not stretched. Thus a linear model $y = mx + b$ sometimes must have a zero value for b. However, in general the least-squares method will give a nonzero value for b because of the scatter or measurement error that is usually present in the data.

To obtain a zero-intercept model of the form $y = mx$, we must derive the equation for m from basic principles. The sum of the squared residuals in this case is

$$J = \sum_{i=1}^{n} (mx_i - y_i)^2$$

Computing the derivative $\partial J / \partial m$ and setting it equal to zero gives the result

$$m \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \qquad (1.5.3)$$

which can be easily solved for m.

If the model is required to pass through a point not at the origin, say the point (x_0, y_0), subtract x_0 from all the x values, subtract y_0 from all the y values, and then use (1.5.3) to find the coefficient m. The resulting equation will be of the form

$$y = m(x - x_0) + y_0 \qquad (1.5.4)$$

Point Constraint | **EXAMPLE 1.5.2**

■ **Problem**

Consider the data given at the beginning of this section.

x	0	5	10
y	2	6	11

We found that the best-fit line is $y = (9/10)x + 11/6$. Find the best-fit line that passes through the point $x = 10$, $y = 11$.

■ **Solution**

Subtracting 10 from all the x values and 11 from all the y values, we obtain a new set of data in terms of the new variables $X = x - 10$ and $Y = y - 11$.

X	-10	-5	0
Y	-9	-5	0

Expressing (1.5.3) in terms of the new variables X and Y, we have

$$m \sum_{i=1}^{3} X_i^2 = \sum_{i=1}^{3} X_i Y_i$$

$$\sum_{i=1}^{3} X_i^2 = (-10)^2 + 5^2 + 0 = 125$$

$$\sum_{i=1}^{3} X_i Y_i = (-10)(-9) + (-5)(-5) + 0 = 115$$

Thus, $m = 115/125 = 23/25$ and the best-fit line is $Y = (23/25)X$. In terms of the original variables, this line is expressed as $y - 11 = (23/25)(x - 10)$ or $y = (23/25)x + 9/5$.

CONSTRAINING A COEFFICIENT

Sometimes we know from physical theory that the data can be described by a function with a specified form and specified values of one of more of its coefficients. For example, the fluid-drag relation states that $D = \rho C_D v^2/2$. In this case, we know that the relation is a power function with an exponent of 2, and we need to estimate the value of the drag coefficient C_D. In such cases, we can modify the least-squares method to find the best-fit function of a specified form.

Fitting a Power Function with a Known Exponent | **EXAMPLE 1.5.3**

■ **Problem**

Fit the power function $y = bx^m$ to the data y_i. The value of m is known.

■ Solution

The least-squares criterion is

$$J = \sum_{i=1}^{n}(bx^m - y_i)^2$$

To obtain the value of b that minimizes J, we must solve $\partial J/\partial b = 0$.

$$\frac{\partial J}{\partial b} = 2\sum_{i=1}^{n} x_i^m \left(bx_i^m - y_i\right) = 0$$

This gives

$$b = \frac{\sum_{i=1}^{n} x_i^m y_i}{\sum_{i=1}^{n} x_i^{2m}} \tag{1}$$

THE QUALITY OF A CURVE FIT

In general, if the arbitrary function $y = f(x)$ is used to represent the data, then the error in the representation is given by $e_i = f(x_i) - y_i$, for $i = 1, 2, 3, \ldots, n$. The error e_i is the difference between the data value y_i and the value of y obtained from the function; that is, $f(x_i)$. The least-squares criterion used to fit a function $f(x)$ is the sum of the squares of the residuals, J. It is defined as

$$J = \sum_{i=1}^{n} [f(x_i) - y_i]^2 \tag{1.5.5}$$

We can use this criterion to compare the quality of the curve fit for two or more functions used to describe the same data. The function that gives the smallest J value gives the best fit.

We denote the sum of the squares of the deviation of the y values from their mean $\bar{y}$ by S, which can be computed from

$$S = \sum_{i=1}^{n}(y_i - \bar{y})^2 \tag{1.5.6}$$

This formula can be used to compute another measure of the quality of the curve fit, the *coefficient of determination,* also known as the *r-squared value*. It is defined as

$$r^2 = 1 - \frac{J}{S} \tag{1.5.7}$$

For a perfect fit, $J = 0$ and thus $r^2 = 1$. Thus, the closer r^2 is to 1, the better the fit. The largest r^2 can be is 1. The value of S is an indication of how much the data is spread around the mean, and the value of J indicates how much of the data spread is left unaccounted for by the model. Thus, the ratio J/S indicates the fractional variation left unaccounted for by the model. It is possible for J to be larger than S, and thus it is possible for r^2 to be negative. Such cases, however, are indicative of a very poor model that should not be used. As a rule of thumb, a very good fit accounts for at least 99% of the data variation. This corresponds to $r^2 \geq 0.99$.

For example, the function $y = 9/10x + 11/6$ derived at the beginning of this section has the values $S = 40.6667$, $J = 0.1666$, and $r^2 = 0.9959$, which indicates a very good fit. The line $y = (23/25)x + 9/5$, which is constrained to pass through the point $x = 10$, $y = 11$ gives the values $S = 40.6667$, $J = 0.2$, and $r^2 = 0.9951$. So the constraint degraded the quality of the fit but very slightly.

The power function $y = 5.068x^{1.9802}$ derived in Example 1.5.1 has the values $S = 3085.8$, $J = 2.9192$, and $r^2 = 0.9991$. Thus its fit is very good.

When the least-squares method is applied to fit quadratic and higher-order polynomials, the resulting equations for the coefficents are linear algebraic equations, which are easily solved. Their solution forms the basis for MATLAB algorithm contained in the `polyfit` function, which is discussed in Section 1.6.

INTEGRAL FORM OF THE LEAST-SQUARES CRITERION

Sometimes we must obtain a linear description of a process over a range of the independent variable so large that linearization is impractical. In such cases we can apply the least-squares method to obtain the linear description. Because there are no data in such cases, we use the integral form of the least-squares criterion.

Fitting a Linear Function to a Power Function | **EXAMPLE 1.5.4**

■ **Problem**

a.   Fit the linear function $y = mx$ to the power function $y = ax^n$ over the range $0 \leq x \leq L$. The values of a and n are given.

b.   Apply the result to the Aerobee drag function $D = 0.00056v^2$ over the range $0 \leq v \leq 1000$, discussed in Example 1.3.4.

■ **Solution**

a.   The appropriate least-squares criterion is the integral of the square of the difference between the linear model and the power function over the stated range. Thus,

$$J = \int_0^L (mx - ax^n)^2 \, dx$$

To obtain the value of m that minimizes J, we must solve $\partial J/\partial m = 0$.

$$\frac{\partial J}{\partial m} = 2 \int_0^L x(mx - ax^n) \, dx = 0$$

This gives

$$m = \frac{3a}{n + 2} L^{n-1} \tag{1}$$

b.   For the Aerobee drag function $D = 0.00056v^2$, $a = 0.00056$, $n = 2$, and $L = 1000$. Thus,

$$m = \frac{3(0.00056)}{2 + 2} 1000^{2-1} = 0.42$$

and the linear description is $D = 0.42v$, where D is in pounds and v is in ft/sec. This is the linear model that minimizes the integral of the squared error over $0 \leq v \leq 1000$ ft/sec.

1.6 MATLAB AND THE LEAST-SQUARES METHOD

We now show how to use MATLAB's `polyfit` function to fit polynomials and functions that can be transformed into polynomials. The `polyfit` function is based on the least-squares method. Its syntax is `p = polyfit(x,y,n)`. The function fits a polynomial of degree n to data described by the vectors x and y, where x is the independent

variable. The result p is the row vector of length $n + 1$ that contains the polynomial coefficients in order of descending powers.

EXAMPLE 1.6.1

Fitting First and Second Degree Polynomials

■ Problem

Use the `polyfit` function to find the first and second degree polynomials that fit the following data in the least-squares sense. Evaluate the quality of fit for each polynomial.

x	0	1	2	3	4	5	6	7	8	9	10
y	48	49	52	63	76	98	136	150	195	236	260

■ Solution

The following MATLAB program computes the polynomial coefficients.

```
x = (0:10);
y = [48, 49, 52, 63, 76, 98, 136, 150, 195, 236, 260];
p_first = polyfit(x,y,1)
p_second = polyfit(x,y,2)
```

The polynomial coefficients of the first degree polynomial are contained in the vector p_first, and the coefficients of the second degree polynomial are contained in the vector p_second. The results are p_first = [22.1909, 12.0455], which corresponds to the polynomial $y = 22.1909x + 12.0455$, and p_second = [2.1993, 0.1979, 45.035], which corresponds to the polynomial $y = 2.1993x^2 + 0.1979x + 45.035$.

We can use MATLAB to plot the polynomials and to evaluate the "quality of fit" quantities J, S, and r^2. The following script file does this.

```
x = (0:10);
y = [48, 49, 52, 63, 76, 98, 136, 150, 195, 236, 260];
mu = mean(y);
xp = (0:0.01:10);
for k = 1:2
   yp(k,:) = polyval(polyfit(x,y,k),xp);
   J(k) = sum((polyval(polyfit(x,y,k),x)-y).^2);
   S(k) = sum((polyval(polyfit(x,y,k),x)- mu).^2);
   r2(k) = 1-J(k)/S(k);
end
subplot(2,1,1)
plot(xp,yp(1,:),x,y,'o'),axis([0 10 0 300]),xlabel('x'),...
   ylabel('y'),title('First-degree fit')
subplot(2,1,2)
plot(xp,yp(2,:),x,y,'o'),axis([0 10 0 300]),xlabel('x'),...
   ylabel('y'),title('Second-degree fit')
disp('The J values are:'),J
disp('The S values are:'),S
disp('The r^2 values are:'),r2
```
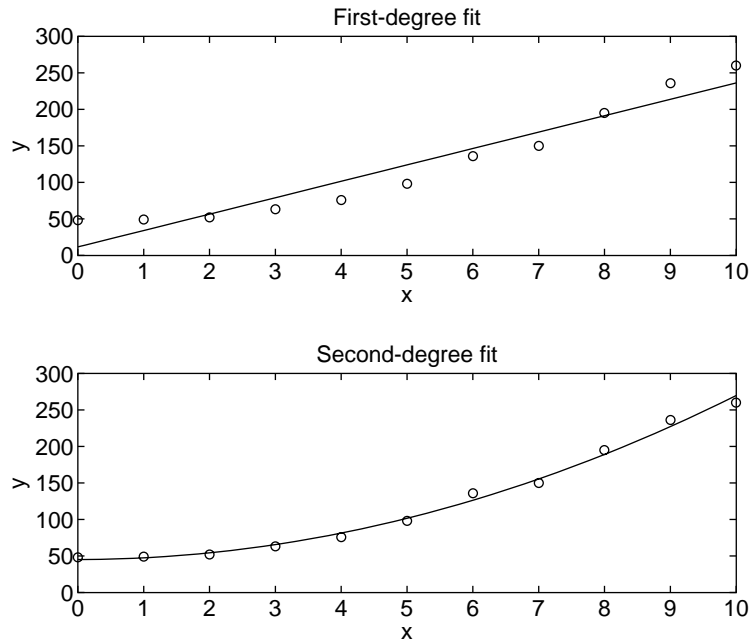
Figure 1.6.1 Plots of first-
and second-degree polynomial
curve fits.

The polynomial coefficients in the above script file are contained in the vector `polyfit` `(x,y,k)`. If you need the polynomial coefficients, say for the second-degree polynomial, type `polyfit(x,y,2)` after the program has been run.

The plots are shown in Figure 1.6.1. The following table gives the values of J, S, and r^2 for each polynomial.

Degree n	J	S	r^2
1	4348	54,168	0.9197
2	197.9	58,318	0.9997

Because the second-degree polynomial has the largest r^2 value, it represents the data better than the first-degree polynomial, according to the r^2 criterion. This is also obvious from the plots.

When we type `p = polyfit(z,w,1)`, MATLAB will fit a linear function $w = p_1 z + p_2$. The coefficients p_1 and p_2 are the first and second elements in the vector `p`; that is, `p` will be $[p_1, p_2]$. With a suitable transformation, the power and exponential functions can be transformed into a linear function, but the polynomial $w = p_1 z + p_2$ has a different interpretation in each of the three cases.

The linear function: $y = mx + b$. In this case the variables w and z in the polynomial $w = p_1 z + p_2$ are the original data variables, and we can find the linear function that fits the data by typing `p = polyfit(x,y,1)`. The first element p_1 of the vector `p` will be m, and the second element p_2 will be b.

The power function: $y = bx^m$. In this case $\log y = m \log x + \log b$, which has the form $w = p_1 z + p_2$, where the polynomial variables w and z are related to the original data variables x and y by $w = \log y$ and $z = \log x$. Thus, we can find the power function that fits the data by typing `p = polyfit(log10(x), log10(y),1)`. The first element p_1 of the vector `p` will be m, and the second element p_2 will be $\log b$. We can find b from $b = 10^{p_2}$.

The exponential function: $y = be^{mx}$. In this case, $\ln y = mx + \ln b$, which has the form $w = p_1 z + p_2$, where the polynomial variables w and z are related to the original data variables x and y by $w = \ln y$ and $z = x$. Thus, we can find the exponential function that fits the data by typing p = polyfit(x,log(y),1). The first element p_1 of the vector p will be m, and the second element p_2 will be $\ln b$. We can find b from $b = e^{p_2}$.

Note

The notation for logarithms used by MATLAB is different than that used in mathematical expressions. Do not make the common mistake of using the MATLAB function log to represent the base-ten logarithm. The *natural* logarithm $\ln x$ is expressed in MATLAB by log(x), whereas the *base-ten* logarithm $\log x$ is expressed as log10(x) in MATLAB.

Example 1.6.2 illustrates how to use MATLAB to estimate the force-deflection characteristics of the cantilever support beam treated in Example 1.3.1.

EXAMPLE 1.6.2

A Cantilever Beam Deflection Model

■ Problem
The force-deflection data from Example 1.3.1 for the cantilever beam shown in Figure 1.4.1 is given in the following table.

Force f (lb)	0	100	200	300	400	500	600	700	800
Deflection x (in.)	0	0.15	0.23	0.35	0.37	0.5	0.57	0.68	0.77

Use MATLAB to obtain a linear relation between x and f, estimate the stiffness k of the beam, and evaluate the quality of the fit.

■ Solution
In the following MATLAB script file the data are entered in the arrays x and f. The arrays xp and fp are created to plot the straight line at many points.

```
x = [0, 0.15, 0.23, 0.35, 0.37, 0.5, 0.57, 0.68, 0.77];
f = (0:100:800);
p = polyfit(f,x,1)
k = 1/p(1)
fp = (0:800);
xp = p(1)*fp+p(2);
plot(fp,xp,f,x,'o'), xlabel('Applied Force f (lb)'), ...
    ylabel('Deflection x (in.)'), ...
    axis([0 800 0 0.8])
J = sum((polyval(p,f)-x).^2)
S = sum((polyval(p,f)-mean(x)).^2)
r2 = 1 - J/S
```

The computed values in the array p are $p = [9.1667 \times 10^{-4}, 3.5556 \times 10^{-2}]$. Thus the fitted straight line is $x = 9.1667 \times 10^{-4} f + 3.5556 \times 10^{-2}$. Note that this line, which is shown in Figure 1.6.2, does not pass through the origin as required, but it is close (it predicts that
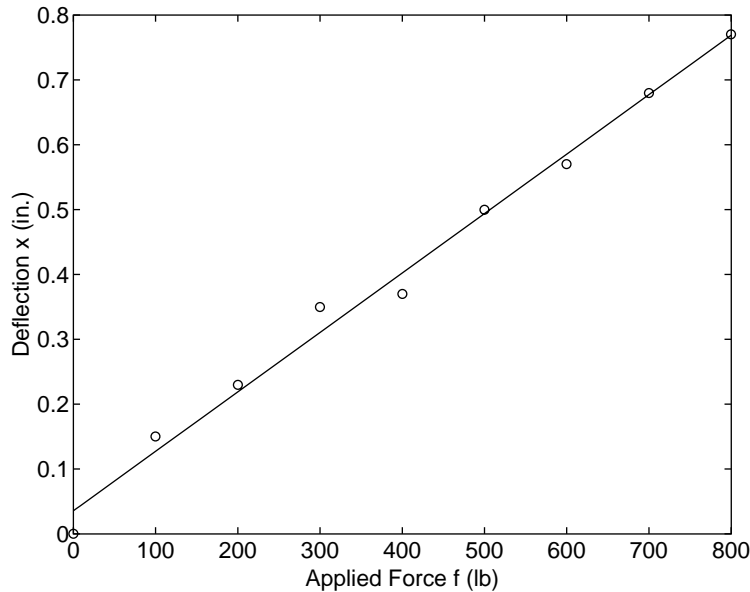
Figure 1.6.2 Models of beam deflection. Unconstrained linear function.

$x = 0.035556$ in. when $f = 0$). The quality-of-fit values are $J = 0.0048$, $S = 0.5042$, and $r^2 = 0.9905$, which indicates a very good fit.

Solving for f gives $f = (x - 3.5556 \times 10^{-2})/9.1667 \times 10^{-4} = 1091x - 38.7879$. The computed value of the stiffness k is the coefficient of x; thus $k = 1091$ lb/in.

Constraining the Curve Fit

EXAMPLE 1.6.3

■ **Problem**

Use MATLAB to fit a straight line to the beam force-deflection data given in Example 1.6.2, but constrain the line to pass through the origin.

■ **Solution**

We can apply (1.5.3), noting here that the measured variable is the deflection x and the independent variable is the force f. Thus (1.5.3) becomes

$$m \sum_{i=1}^{n} f_i^2 = \sum_{i=1}^{n} f_i x_i$$

The MATLAB program to solve this equation for m and k is

```
x = [0, 0.15, 0.23, 0.35, 0.37, 0.5, 0.57, 0.68, 0.77];
f = (0:100:800);
m = sum(f.*x)/sum(f.^2);
k = 1/m
J = sum((m*f-x).^2)
S = sum((m*f-mean(x)).^2)
r2 = 1 - J/S
```

The answer is $k = 1021$ lb/in. The corresponding line is shown in Figure 1.6.3. The quality-of-fit values are $J = 0.0081$, $S = 0.5765$, and $r^2 = 0.9859$, which indicates a good fit.

Figure 1.6.3 Linear function constrained to pass through the origin.
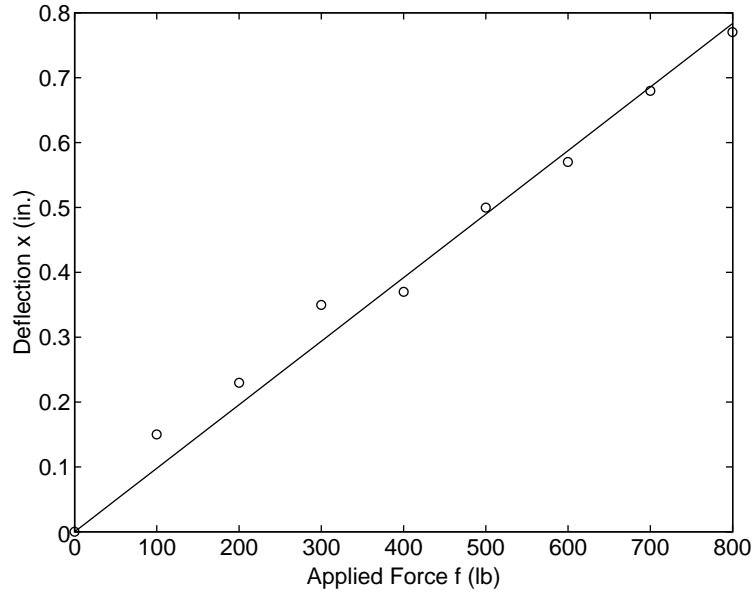
EXAMPLE 1.6.4

Temperature Dynamics of Water

■ Problem

Consider again Example 1.4.1. Water in a glass measuring cup was allowed to cool after being heated to 204°F. The ambient air temperature was 70°F. The measured water temperature at various times is given in the following table.

Time (sec)	0	120	240	360	480	600
Temperature (°F)	204	191	178	169	160	153

Time (sec)	720	840	960	1080	1200
Temperature (°F)	147	141	137	132	127

Obtain a functional description of the water temperature versus time.

■ Solution

From Example 1.4.1, we learned that the relative temperature, $\Delta T = T - 70$ has the exponential form

$$\Delta T = be^{mt}$$

We can find values of m and b by using $p = \text{polyfit}(x,\log(y),1)$. The first element p_1 of the vector p will be m, and the second element p_2 will be $\ln b$. We can find b from $b = e^{p_2}$. The following MATLAB program performs the calculations.

```
time = (0:120:1200);
temp = [204,191,178,169,160,153,147,141,137,132,127];
rel_temp = temp - 70;
log_rel_temp = log(rel_temp);
p = polyfit(time,log_rel_temp,1);
m = p(1),b = exp(p(2))
DT = b*exp(m*time);
```

```
J = sum((DT-rel_temp).^2)
S = sum((rel_temp - mean(rel_temp)).^2
r2 = 1 - J/S
```

The results are $m = -6.9710 \times 10^{-4}$ and $b = 1.2916 \times 10^2$, and the corresponding function is

$$\Delta T = be^{mt} \quad \text{or} \quad T = \Delta T + 70 = be^{mt} + 70$$

The quality-of-fit values are $J = 47.4850$, $S = 6.2429 \times 10^3$, and $r^2 = 0.9924$, which indicates a very good fit.

Orifice Flow | **EXAMPLE 1.6.5**

■ Problem

Consider again Example 1.4.2. A hole 6 mm in diameter was made in a translucent milk container (Figure 1.4.7). A series of marks 1 cm apart was made above the hole. While adjusting the tap flow to keep the water height constant, the time for the outflow to fill a 250 ml cup was measured ($1 \text{ ml} = 10^{-6} \text{ m}^3$). This was repeated for several heights. The data are given in the following table.

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

Obtain a functional description of the volume outflow rate f as a function of water height h above the hole.

■ Solution

First obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

$$f = \frac{250}{t}$$

In Example 1.4.2, we learned that the following power function can describe the data:

$$f = bh^m$$

We can find the values of m and b by using $p = polyfit(log10(x),log10(y),1)$. The first element p_1 of the vector p will be m, and the second element p_2 will be $\log b$. We can find b from $b = 10^{p_2}$. The following MATLAB program performs the calculations.

```
h = (1:11);
time = [26, 19, 14, 12, 11, 9.5, 9, 8.5, 8, 7.5, 7];
flow = 250./time;
logflow = log10(flow);logheight = log10(h);
p = polyfit(logheight,logflow,1);
m = p(1),b = 10^p(2)
F = b*h.^m;
J = sum((F - flow).^2)
S = sum((flow - mean(flow)).^2)
r2 = 1 - J/S
```

The results are $m = 0.5499$ and $b = 9.4956$, and the corresponding function is

$$f = bh^m = 9.4956h^{0.5499}$$

The quality-of-fit values are $J = 2.5011$, $S = 698.2203$, and $r^2 = 0.9964$, which indicates a very good fit.

Sometimes we know from physical theory that the data can be described by a power function with a specified exponent. For example, Torricelli's principle of hydraulic resistance states that the volume flow rate f of a liquid through a restriction is proportional to the square root of the pressure drop p across the restriction; that is, $q = c\sqrt{p} = cp^{1/2}$. In many applications, the pressure drop is due to the weight of liquid in a container. This is the case for water in the milk container of Example 1.6.5. In such situations, Torricelli's principle states that the flow rate is proportional to the square root of the height h of the liquid above the orifice. Thus,

$$f = b\sqrt{h} = bh^{1/2}$$

where b is a constant that must be determined from data.

EXAMPLE 1.6.6

Orifice Flow with Constrained Exponent

■ Problem
Consider the data of Example 1.6.5. Determine the best-fit value of the coefficient b in the square-root function

$$f = bh^{1/2}$$

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

■ Solution
First obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

$$f = \frac{250}{t}$$

Referring to Example 1.5.3, whose model is $y = bx^m$, we see here that $y = f, h = x, m = 0.5$, and $a = b$. From Equation (1) of that example,

$$a = \frac{\sum_{i=1}^{n} h_i^{0.5} y_i}{\sum_{i=1}^{n} h_i} \tag{1}$$

The MATLAB program to carry out these calculations is shown next.

```
h = (1:11);
time = [26, 19, 14, 12, 11, 9.5, 9, 8.5, 8, 7.5, 7];
flow = 250./time;
a = sum(sqrt(h).*flow)/sum(h)
f = a*sqrt(h);
J = sum((f - flow).^2)
S = sum((flow - mean(flow)).^2)
r2 = 1 - J/S
```

The result is $a = 10.4604$ and the flow model is $f = 10.4604\sqrt{h}$. The quality-of-fit values are $J = 5.5495$, $S = 698.2203$, and $r^2 = 0.9921$, which indicates a very good fit.

1.7 CHAPTER REVIEW

This chapter introduced the basic terminology of system dynamics, which includes the notions of system, static and dynamic elements, input, and output. The chapter also introduced the foot-pound-second (FPS) and the metric (SI) systems of units, which will be used throughout this text. We developed methods for obtaining algebraic models of input-output relations. We saw how to obtain algebraic models of specified form, how to use the methods of function identification and parameter estimation to develop models from data, and how to apply the least-squares method. We then showed how to apply MATLAB for this purpose.

Review of Objectives

Now that you have finished this chapter, you should be able to

1. Define the terms: static and dynamic elements, static and dynamic systems, input, and output.
2. Discuss the principle of integral causality.
3. Apply the basic steps used for engineering problem solving.
4. Understand the steps in developing a computer solution.
5. Use both FPS and SI units.
6. Develop linearized models from given algebraic expressions.
7. Identify the algebraic form and obtain the coefficient values of a model, given a set of data.
8. Apply the least-squares method to obtain an algebraic model of a specified form, given a set of data.
9. Use MATLAB to implement the least-squares method.

PROBLEMS

Section 1.2 Units

1.1   Calculate the weight in pounds of an object whose mass is 3 slugs. Then convert the weight to newtons and the mass to kilograms.

1.2   Folklore has it that Sir Isaac Newton formulated the law of gravitation supposedly after being hit on the head by a falling apple. The weight of an apple depends strongly on its variety, but a typical weight is 1 newton! Calculate the total mass of 100 apples in kilograms. Then convert the total weight to pounds and the total mass to slugs.

1.3   A ball is thrown a distance of 50 feet, 5 inches. Calculate the distance in meters.

1.4   How many 60 watt lightbulbs are equivalent to one horsepower?

1.5   Convert the temperature of 70°F to °C.

1.6   A particular motor rotates at 3000 revolutions per minute (rpm). What is its speed in rad/sec, and how many seconds does it take to make one revolution?

1.7   The displacement of a certain object is described by $y(t) = 23 \sin 5t$, where t is measured in seconds. Compute its period and its oscillation frequency in rad/sec and in hertz.

Section 1.3 Developing Linear Models

1.8   The distance a spring stretches from its "free length" is a function of how much tension is applied to it. The following table gives the spring length y that was

produced in a particular spring by the given applied force f. The spring's free length is 4.7 in. Find a functional relation between f and x, the extension from the free length ($x = y - 4.7$).

Force f (pounds)	Spring length y (inches)
0	4.7
0.47	7.2
1.15	10.6
1.64	12.9

1.9   The following "small angle" approximation for the sine is used in many engineering applications to obtain a simpler model that is easier to understand and analyze. This approximation states that $\sin x \approx x$, where x must be in radians. Investigate the accuracy of this approximation by creating three plots. For the first plot, plot $\sin x$ and x versus x for $0 \leq x \leq 1$. For the second plot, plot the approximation error $\sin(x) - x$ versus x for $0 \leq x \leq 1$. For the third plot, plot the percent error $[\sin(x) - x]/\sin(x)$ versus x for $0 \leq x \leq 1$. How small must x be for the approximation to be accurate within 5%?

1.10 Obtain two linear approximations of the function $f(\theta) = \sin\theta$, one valid near $\theta = \pi/4$ rad and the other valid near $\theta = 3\pi/4$ rad.

1.11 Obtain two linear approximations of the function $f(\theta) = \cos\theta$, one valid near $\theta = \pi/3$ rad and the other valid near $\theta = 2\pi/3$ rad.

1.12 Obtain a linear approximation of the function $f(h) = \sqrt{h}$, valid near $h = 25$.

1.13 Obtain two linear approximations of the function $f(r) = r^2$, one valid near $r = 5$ and the other valid near $r = 10$.

1.14 Obtain a linear approximation of the function $f(h) = \sqrt{h}$, valid near $h = 16$. Noting that $f(h) \geq 0$, what is the value of h below which the linearized model loses its meaning?

1.15 The flow rate f in m³/s of water through a particular pipe, as a function of the pressure drop p across the ends of the pipe (in N/m²) is given by $f = 0.002\sqrt{p}$. Obtain a linear model of f as a function of p that always *underestimates* the flow rate over the range $0 \leq p \leq 900$.

Section 1.4 Function Identification and Parameter Estimation

In the following problems for Section 1.4, solve the problem by drawing a straight line by eye using a straightedge.

1.16 In each of these problems, plot the data and determine the best function $y(x)$ (linear, exponential, or power function) to describe the data.

a.

x	25	30	35	40	45
y	0	250	500	750	1000

b.

x	2.5	3	3.5	4	4.5	5	5.5	6	7	8	9	10
y	1500	1220	1050	915	810	745	690	620	520	480	410	390

c.

x	550	600	650	700	750
y	41.2	18.62	8.62	3.92	1.86

1.17 The population data for a certain country are given here.

Year	1990	1991	1992	1993	1994	1995
Population (millions)	10	10.5	11.1	11.6	12.2	12.8

Plot the data and obtain a function that describes the data. Estimate when the population will be double its 1990 size.

1.18 The *half-life* of a radioactive substance is the time it takes to decay by half. The half-life of carbon-14, which is used for dating previously living things, is 5500 years. When an organism dies, it stops accumulating carbon-14. The carbon-14 present at the time of death decays with time. Let $C(t)/C(0)$ be the fraction of carbon-14 remaining at time t. In radioactive carbon dating, it is usually assumed that the remaining fraction decays exponentially according to the formula

$$\frac{C(t)}{C(0)} = e^{-bt}$$

   a. Use the half-life of carbon-14 to find the value of the parameter b and plot the function.
   b. Suppose we estimate that 90% of the original carbon-14 remains. Estimate how long ago the organism died.
   c. Suppose our estimate of b is off by $\pm 1\%$. How does this affect the age estimate in part (b)?

1.19 *Quenching* is the process of immersing a hot metal object in a bath for a specified time to improve properties such as hardness. A copper sphere 25 mm in diameter, initially at 300°C, is immersed in a bath at 0°C. Measurements of the sphere's temperature versus time are shown here. Plot the data and find a functional description of the data.

Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6
Temperature (°C)	300	150	75	35	12	5	0

1.20 The useful life of a machine bearing depends on its operating temperature, as shown by the following data. Plot the data and obtain a functional description of the data. Estimate a bearing's life if it operates at 150°F.

Temperature (°F)	100	120	140	160	180	200	220
Bearing life (hours $\times 10^3$)	28	21	15	11	8	6	4

1.21 A certain electric circuit has a resistor and a capacitor. The capacitor is initially charged to 100 V. When the power supply is detached, the capacitor voltage decays with time as shown in the following data table. Find a functional description of the capacitor voltage v as a function of time t.

Time (s)	0	0.5	1	1.5	2	2.5	3	3.5	4
Voltage (V)	100	62	38	21	13	7	4	2	3

1.22   Water (of volume 425 ml) in a glass measuring cup was allowed to cool after
being heated to 207°F. The ambient air temperature was 70°F. The measured
water temperature at various times is given in the following table.

Time (sec)	0	300	600	900	1200	1500
Temperature (°F)	207	182	167	155	143	135

Time (sec)	1800	2100	2400	2700	3000
Temperature (°F)	128	123	118	114	109

Obtain a functional description of the relative water temperature ($\Delta T =
T - 70$) versus time.

1.23   Consider the milk container of Example 1.4.2 (Figure 1.4.7). A straw 19 cm
long was inserted in the side of the container. While adjusting the tap flow to
keep the water height constant, the time for the outflow to fill a 250-ml cup was
measured. This was repeated for several heights. The data are given in the
following table.

Height (cm)	11	10	9	8	7	6	5	4	3	2	1
Time (s)	7	7	7	8	9	10	11	13	15	17	23

Obtain a functional description of the volume outflow rate f through the
straw as a function of water height h above the hole.

1.24   Consider the milk container of Example 1.4.2 (Figure 1.4.7). A straw 9.5 cm
long was inserted in the side of the container. While adjusting the tap flow to
keep the water height constant, the time for the outflow to fill a 250-ml cup was
measured. This was repeated for several heights. The data are given in the
following table.

Height (cm)	11	10	9	8	7	6	5	4	3	2	1
Time (s)	6	6	6	7	8	9	9	11	13	17	21

Obtain a functional description of the volume outflow rate f through the straw
as a function of water height h above the hole.

Section 1.5 Fitting Models to Scattered Data

1.25   Use the least-squares method to fit the linear function $y = mx + b$ to the data
given in the following table. Evaluate the quality of the fit by computing J, S,
and r^2.

x	0	2	4	6
y	4.5	39	72	94

1.26   Use the least-squares method to fit the power function $y = bx^m$ to the data
given in the following table. Evaluate the quality of the fit by computing J, S,
and r^2.

x	0	1	2	3	4
y	1	8	50	178	490

1.27   Use the least-squares method to fit the exponential function $y = be^{mx}$ to the
data given in the following table. Evaluate the quality of the fit by computing
J, S, and r^2.

x	0	0.4	0.8	1.2
y	6.3	22	60	215

1.28 Use the least-squares method to fit the linear function $y = mx + b$ to the data given in the following table. Constrain the function to pass through the point $(0, 0)$. Evaluate the quality of the fit by computing J, S, and r^2.

x	0	2	4	6
y	4.5	39	72	94

1.29 Use the least-squares method to fit the power function $y = bx^m$ to the data given in the following table. Constrain the exponent of the function to be $m = 3$. Evaluate the quality of the fit by computing J, S, and r^2.

x	0	1	2	3	4
y	1	8	50	178	490

1.30 (a) Use the least-squares method to derive the equation for b to fit the exponential function $y = be^{mx}$ to a given set of data y_i, where the exponent m is constrained to a specified value. (b) Fit the function $y = be^{-3x}$ to the data given in the following table. Evaluate the quality of the fit by computing J, S, and r^2.

x	0	0.4	0.8	1.2
y	6.3	22	60	215

1.31 Use the least-squares method to fit the linear function $y = mx + b$ to the function $y = 5x^2$ over the range $0 \leq x \leq 4$.

1.32 (a) Use the least-squares method to fit the linear function $y = mx + b$ to the function $y = ax^2 + bx$ over the range $0 \leq x \leq L$. (b) Apply the results to the case where $a = 3$, $b = 5$, and $L = 2$.

1.33 (a) Use the least-squares method to fit the linear function $y = mx + b$ to the exponential function $y = be^{mx}$ over the range $0 \leq x \leq L$. (b) Apply the results to the case where $m = -5$, $b = 15$, and $L = 1$.

Section 1.6 MATLAB and the Least-Squares Method

In Problems 1.34 through 1.48, work the problem using MATLAB.

1.34 Do Problem 1.16.
1.35 Do Problem 1.17.
1.36 Do Problem 1.18.
1.37 Do Problem 1.19.
1.38 Do Problem 1.20.
1.39 Do Problem 1.21.
1.40 Do Problem 1.22.
1.41 Do Problem 1.23.
1.42 Do Problem 1.24.
1.43 Do Problem 1.25.
1.44 Do Problem 1.26.
1.45 Do Problem 1.27.
1.46 Do Problem 1.28.
1.47 Do Problem 1.29.
1.48 Do Problem 1.30.

Modeling of Rigid-Body Mechanical Systems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Obtain the equations of motion for an object consisting of a single mass undergoing simple translation or simple rotation.

2. Solve the equations of motion when the applied forces or moments are either constants or simple functions of time.

3. Apply the principle of conservation of mechanical energy to analyze systems acted on by conservative forces.

4. Apply the concepts of equivalent mass and equivalent inertia to obtain a simpler model of a multimass system whose motions are directly coupled.

5. Obtain the equation of motion of a body in planar motion involving simultaneous translation and rotation.

W hen modeling the motion of objects, the bending or twisting of the object is often negligible, and we can model the object as a rigid body. We begin this chapter by reviewing Newton's laws of motion and applying them to rigid bodies where the object's motion is relatively uncomplicated, namely, simple translations and simple rotation about a fixed axis. We then introduce energy-based methods and the concepts of equivalent mass and equivalent inertia, which simplify the modeling of systems having both translating and rotating components. Following

that we treat the case of general motion in a plane, involving simultaneous translation and rotation.

In Chapter 4 we will consider systems that have nonrigid, or *elastic,* behavior. ■

2.1 TRANSLATIONAL MOTION

A *particle* is a mass of negligible dimensions. We can consider a body to be a particle if its dimensions are irrelevant for specifying its position and the forces acting on it. For example, we normally need not know the size of an earth satellite to describe its orbital path. *Newton's first law* states that a particle originally at rest, or moving in a straight line with a constant speed, will remain that way as long as it is not acted upon by an unbalanced external force. *Newton's second law* states that the acceleration of a mass particle is proportional to the vector resultant force acting on it and is in the direction of this force. *Newton's third law* states that the forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear. The third law is summarized by the commonly used statement that every action is opposed by an equal reaction.

For an object treated as a particle of mass m, the second law can be expressed as

$$m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \mathbf{f} \tag{2.1.1}$$

where $\mathbf{a}$ and $\mathbf{v}$ are the acceleration and velocity vectors of the mass and $\mathbf{f}$ is the force vector acting on the mass (Figure 2.1.1). Note that the acceleration vector and the force vector lie on the same line.

If the mass is constrained to move in only one direction, say along the direction of the coordinate x, then the equation of motion is the scalar equation
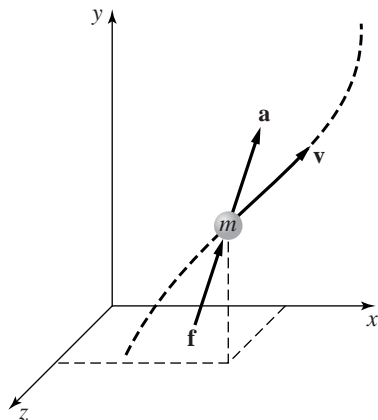
$$ma = m\frac{dv}{dt} = f \tag{2.1.2}$$

or

$$\frac{dv}{dt} = \frac{f}{m} = a \tag{2.1.3}$$

It will be convenient to use the following abbreviated "dot" notation for time derivatives:

$$\dot{x}(t) = \frac{dx}{dt} \qquad \ddot{x}(t) = \frac{d^2x}{dt^2}$$

Thus, we can express the scalar form of Newton's law as $m\dot{v} = f$.

Figure 2.1.1 Particle motion showing the coordinate system, the applied force **f**, and the resulting acceleration **a**, velocity **v**, and path.

If we assume that the object is a rigid body and we neglect the force distribution within the object, we can treat the object as if its mass were concentrated at its mass center. This is the *point mass* assumption, which makes it easier to obtain the translational equations of motion, because the object's dimensions can be ignored and all external forces can be treated as if they acted through the mass center. If the object can rotate, then the translational equations must be supplemented with the rotational equations of motion, which are treated in Sections 2.2 and 2.4.

MECHANICAL ENERGY

Conservation of mechanical energy is a direct consequence of Newton's second law. Consider the scalar case (2.1.2), where the force f can be a function of displacement x.

$$m\dot{v} = f(x)$$

Multiply both sides by $v\,dt$ and use the fact that $v = dx/dt$.

$$mv\,dv = vf(x)\,dt = \frac{dx}{dt} f(x)\,dt = f(x)\,dx$$

Integrate both sides.

$$\int mv\,dv = \frac{mv^2}{2} = \int f(x)\,dx + C \tag{2.1.4}$$

where C is a constant of integration.

Work is force times displacement, so the integral on the right represents the total work done on the mass by the force $f(x)$. The term on the left-hand side of the equal sign is called the *kinetic energy* (KE).

If the work done by $f(x)$ is independent of the path and depends only on the end points, then the force $f(x)$ is derivable from a function $V(x)$ as follows:

$$f(x) = -\frac{dV}{dx} \tag{2.1.5}$$

Then, in this case, $f(x)$ is called a *conservative force*. If we integrate both sides of the last equation, we obtain

$$V(x) = \int dV = -\int f(x)\,dx$$

or from (2.1.4),

$$\frac{mv^2}{2} + V(x) = C \tag{2.1.6}$$

This equation shows that $V(x)$ has the same units as kinetic energy. $V(x)$ is called the *potential energy* (PE) *function*.

Equation (2.1.6) states that the sum of the kinetic and potential energies must be constant, if no force other than the conservative force is applied. If v and x have the values v_0 and x_0 at the time t_0, then

$$\frac{mv_0^2}{2} + V(x_0) = C$$

Comparing this with (2.1.6) gives

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} + V(x) - V(x_0) = 0 \tag{2.1.7}$$

which can be expressed as

$$\Delta \text{KE} + \Delta \text{PE} = 0 \tag{2.1.8}$$

where the change in kinetic energy is $\Delta KE = m(v^2 - v_0^2)/2$ and the change in potential energy is $\Delta PE = V(x) - V(x_0)$. For some problems, the following form of the principle is more convenient to use:

$$\frac{mv_0^2}{2} + V(x_0) = \frac{mv^2}{2} + V(x) \tag{2.1.9}$$

In the form (2.1.8), we see that conservation of mechanical energy states that the change in kinetic energy plus the change in potential energy is zero. Note that the potential energy has a relative value only. The choice of reference point for measuring x determines only the value of C, which (2.1.7) shows to be irrelevant.

Gravity is an example of a conservative force, for which $f = -mg$. The gravity force is conservative because the work done lifting an object depends only on the change in height and not on the path taken. Thus, if x represents vertical displacement,

$$V(x) = mgx$$

and

$$\frac{mv^2}{2} + mgx = C \tag{2.1.10}$$

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} + mg(x - x_0) = 0 \tag{2.1.11}$$

Speed of a Falling Object $\qquad$ **EXAMPLE 2.1.1**

■ **Problem**

An object with a mass of $m = 2$ slugs drops from a height of 30 ft above the ground (see Figure 2.1.2). Determine its speed after it drops 20 ft to a platform that is 10 ft above the ground.

■ **Solution**

Measuring x from the ground gives $x_0 = 30$ ft and $x = 10$ ft at the platform. If the object is dropped from rest, then $v_0 = 0$. From (2.1.11)

$$\frac{m}{2}(v^2 - 0) + mg(10 - 30) = 0$$

or $v^2 = 40g$. Using $g = 32.2$ ft/sec^2, we obtain $v = \sqrt{644} = 25.4$ ft/sec. This is the speed of the object when it reaches the platform.

Note that if we had chosen to measure x from the platform instead of the ground, then $v_0 = 0$, $x_0 = 20$, and $x = 0$ at the platform. Equation (2.1.11) gives

$$\frac{m}{2}(v^2 - 0) + mg(0 - 20) = 0$$

or $v^2 = 40g$, which gives the same answer as before. When x is measured from the platform, (2.1.10) gives $C = 20mg$. When x is measured from the ground, $C = 30mg$, but the two values of C are irrelevant for solving the problem because it is the *change* in kinetic and potential energies that governs the object's dynamics.

Figure 2.1.2 A falling object.
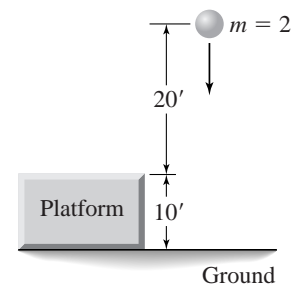
CONSTANT FORCE CASE

For the point mass model, $ma = f$, (2.1.9) can be used to find the speed v as a function of displacement x. If f is a constant, then

$$\frac{mv^2}{2} = f(x - x_0) + \frac{mv_0^2}{2} \tag{2.1.12}$$

Noting that work equals force times displacement, the work done on the mass by the force f is $f(x - x_0)$. Thus (2.1.12) says that the final energy of the mass, $mv^2/2$, equals the initial energy, $mv_0^2/2$, plus the work done by the force f. This is a statement of conservation of energy.

DRY FRICTION FORCE

Not every constant force is conservative. A common example of a *non-conservative force* is the *dry* friction force. This force is non-conservative because the work done by the force depends on the path taken. The dry friction force F is directly proportional to the force N normal to the frictional surface. Thus $F = \mu N$. The proportionality constant is μ, the *coefficient of friction*.

The dry friction force that exists before motion begins is called *static friction* (sometimes shortened to *stiction*). The static friction coefficient has the value μ_s to distinguish it from the *dynamic* friction coefficient μ_d, which describes the friction after motion begins. Dynamic friction is also called *sliding* friction, *kinetic* friction, or *Coulomb* friction. In general, $\mu_s > \mu_d$, which explains why it is more difficult to start an object sliding than to keep it moving. We will use the symbol μ rather than μ_d because most of our applications involve motion.

Because Coulomb friction cannot be derived from a potential energy function, the conservation of mechanical energy principle does not apply. This makes sense physically because the friction force dissipates the energy as heat, and thus mechanical energy, which consists of kinetic plus potential energy, is not conserved. Total energy, of course, is conserved.

EXAMPLE 2.1.2

Equation of Motion with Friction

■ Problem
Derive the equation of motion (a) for the block of mass m shown in Figure 2.1.3a and (b) for the mass m on an incline, shown in Figure 2.1.3b. In both cases, a force f_1, which is not the friction force, is applied to move the mass.
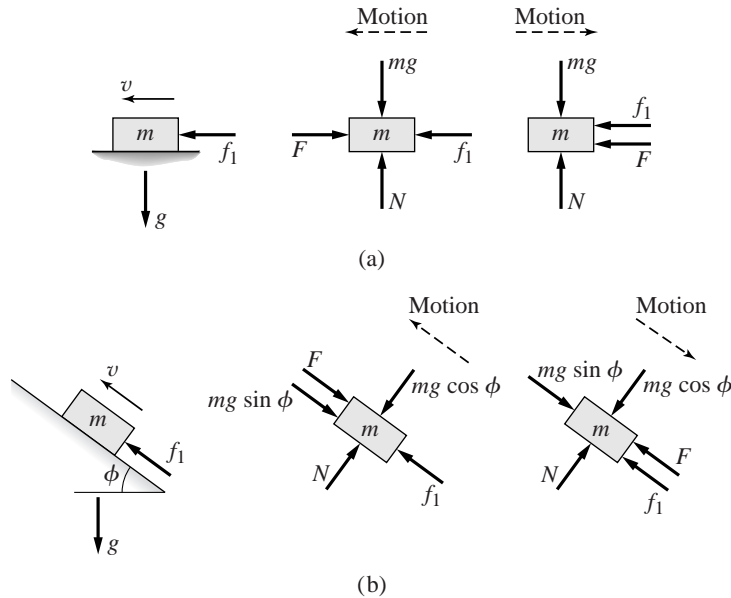
■ Solution
a. The free body diagrams are shown in Figure 2.1.3a for the two cases: $v > 0$ and $v < 0$. The normal force N is the weight mg. Thus the friction force F is μN, or $F = \mu mg$. If $v > 0$, the equation of motion is

$$m\dot{v} = f_1 - \mu mg \qquad v > 0 \tag{1}$$

Dry friction always opposes the motion. So, for $v < 0$,

$$m\dot{v} = f_1 + \mu mg \qquad v < 0 \tag{2}$$

Equations (1) and (2) are the desired equations of motion.

b. The friction force depends on the force normal to the surface. For the mass m shown in Figure 2.1.3b, the normal force N must equal $mg \cos \phi$ as long as the mass is in contact with the surface. The free body diagrams are shown in the figure for the two cases: $v > 0$ and $v < 0$. Newton's second law applied in the direction parallel to the surface gives, for $v > 0$,

$$m\dot{v} = f_1 - mg \sin \phi - \mu mg \cos \phi \qquad v > 0 \qquad (3)$$

For $v < 0$,

$$m\dot{v} = f_1 - mg \sin \phi + \mu mg \cos \phi \qquad v < 0 \qquad (4)$$

Equations (3) and (4) are the desired equations of motion.

Motion with Friction on an Inclined Plane | **EXAMPLE 2.1.3**

■ **Problem**

For the mass shown in Figure 2.1.3b, $m = 2$ kg, $\phi = 30°$, $v(0) = 3$ m/s, and $\mu = 0.5$. Determine whether the mass comes to rest if (a) $f_1 = 50$ N and (b) $f_1 = 5$ N.

■ **Solution**

Because the velocity is initially positive $[v(0) = 3]$, we use equation (3) of Example 2.1.2:

$$2\dot{v} = f_1 - (\sin 30° + 0.5 \cos 30°)(2)(9.81) = f_1 - 18.3$$

For part (a), $f_1 = 50$ and thus $\dot{v} = (50 - 18.3)/2 = 15.85$ and the acceleration is positive. Thus, because $v(0) > 0$, the speed is always positive for $t \geq 0$ and the mass never comes to rest.

For part (b), $f_1 = 5$, $\dot{v} = (5 - 18.3)/2 = -6.65$, and thus the mass is decelerating. Because $v(t) = -6.65t + 3$, the speed becomes zero at $t = 3/6.65 = 0.45$ s.

2.2 ROTATION ABOUT A FIXED AXIS

In this section, we consider the dynamics of rigid bodies whose motion is constrained to allow only rotation about an axis through a nonaccelerating point. In Section 2.4 we will treat the case of rotation about an axis through an accelerating point.

For planar motion, which means that the object can translate in two dimensions and can rotate only about an axis that is perpendicular to the plane, Newton's second law can be used to show that

$$I_O \dot{\omega} = M_O \tag{2.2.1}$$

where ω is the *angular velocity* of the mass about an axis through a point O fixed in an inertial reference frame and attached to the body (or the body extended), I_O is the *mass moment of inertia* of the body about the point O, and M_O is the sum of the moments applied to the body about the point O.

This situation is illustrated in the Figure 2.2.1. The angular displacement is θ, and $\dot{\theta} = \omega$. The term *torque* and the symbol T are often used instead of *moment* and M. Also, when the context is unambiguous, we use the term "inertia" as an abbreviation for "mass moment of inertia."

CALCULATING INERTIA

The mass moment of inertia I about a specified reference axis is defined as

$$I = \int r^2 \, dm \tag{2.2.2}$$

where r is the distance from the reference axis to the mass element dm. The expressions for I for some common shapes are given in Table 2.2.1.

If the rotation axis of a homogeneous rigid body does not coincide with the body's axis of symmetry, but is parallel to it at a distance d, then the mass moment of inertia about the rotation axis is given by the *parallel-axis theorem,*
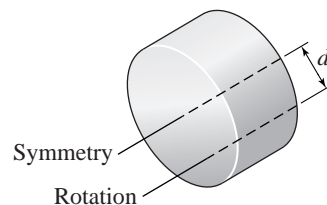
$$I = I_s + md^2 \tag{2.2.3}$$

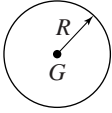where I_s is the inertia about the symmetry axis (see Figure 2.2.2).
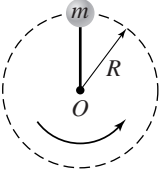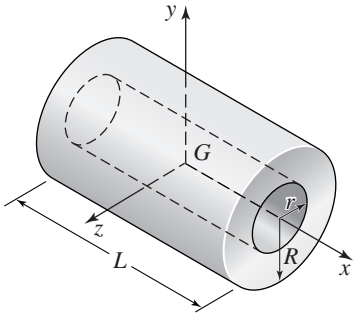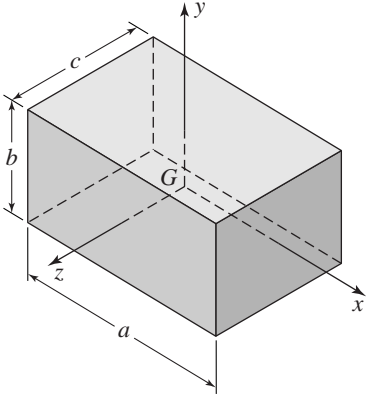
Figure 2.2.1 An object rotating about a fixed axis.
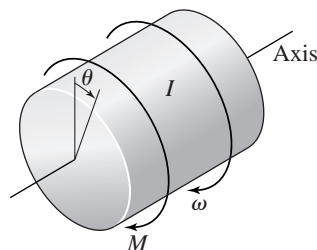
Figure 2.2.2 Illustration of the parallel-axis theorem.

Table 2.2.1 Mass moments of inertia of common elements

Sphere

$$I_G = \frac{2}{5}mR^2$$

Mass rotating about point O

$$I_O = mR^2$$

Hollow cylinder

$$I_x = \frac{1}{2}m(R^2 + r^2)$$

$$I_y = I_z = \frac{1}{12}m(3R^2 + 3r^2 + L^2)$$

Rectangular prism


$$I_x = \frac{1}{12}m(b^2 + c^2)$$

A Rod-and-Bob Pendulum | **EXAMPLE 2.2.1**

■ **Problem**

The pendulum shown in Figure 2.2.3a consists of a concentrated mass m_C (the bob) a distance L_C from point O, attached to a rod of length L_R and inertia I_{RG} about its mass center. (a) Obtain its equation of motion. (b) Discuss the case where the rod's mass m_R is small compared to the concentrated mass. (c) Determine the equation of motion for small angles θ.

Figure 2.2.3 A rod-and-bob pendulum.

(a)   (b)   (c)   (d)
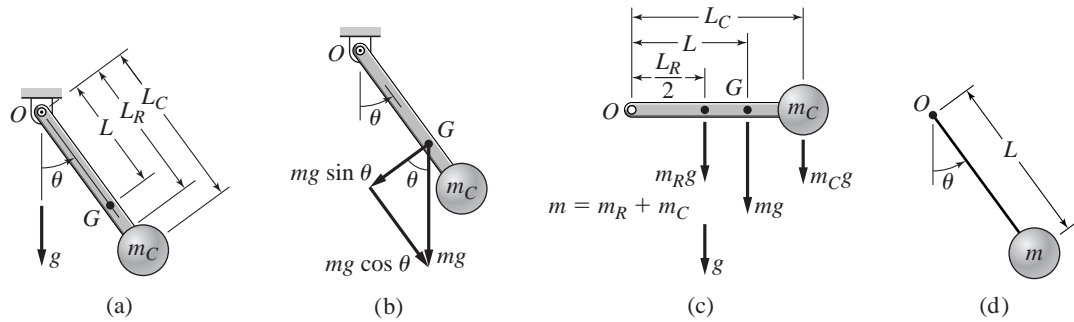
■ Solution

a. For the pendulum shown in Figure 2.2.3a, the inertia of the concentrated mass m_C about point O is $m_C L_C^2$ (see Table 2.2.1). From the parallel-axis theorem, the rod's inertia about point O is

$$I_{RO} = I_{RG} + m_R \left(\frac{L_R}{2} \right)^2$$

and thus the entire pendulum's inertia about point O is

$$I_O = I_{RO} + m_C L_C^2 = I_{RG} + m_R \left(\frac{L_R}{2} \right)^2 + m_C L_C^2$$

With the moment equation (2.2.1) about point O, the moment M_O is caused by the perpendicular component of the weight mg acting through the mass center at G (see Figure 2.2.3b). Thus the desired equation of motion is

$$I_O \ddot{\theta} = -mgL \sin \theta \tag{1}$$

The distance L between point O and the mass center G of the entire pendulum is not given, but can be calculated as follows (Figure 2.2.3c). If the entire pendulum mass were concentrated at G, the weight force would produce the same moment about point O as the real pendulum. Thus, taking moments about point O, we have

$$mgL = m_C g L_C + m_R g \frac{L_R}{2}$$

where $m = m_C + m_R$. Solve for L to obtain

$$L = \frac{m_C L_C + m_R (L_R/2)}{m_C + m_R} \tag{2}$$

b. If we neglect the rod's mass m_R compared to the concentrated mass m_C, then we can take $m_R = I_{RG} = 0$, $m = m_C$, $L = L_C$, and $I_O = mL^2$. In this case, the equation of motion reduces to

$$L\ddot{\theta} + g \sin \theta = 0 \tag{3}$$

This is a model for a pendulum whose mass is concentrated at a distance L from the pivot point, like that shown in Figure 2.2.3d. Note that this equation of motion is independent of the value of m.

c. For small angles, $\sin \theta \approx \theta$ if θ is in radians. Substituting this approximation into equation (3) gives

$$L\ddot{\theta} + g\theta = 0 \tag{4}$$

We will learn how to solve this equation for $\theta(t)$ in Chapter 3, and we will see that it is easier to solve than equation (3).

ENERGY AND ROTATIONAL MOTION

The work done by a moment M causing a rotation through an angle θ is

$$W = \int_0^\theta M \, d\theta \qquad (2.2.4)$$

Multiply both sides of (2.2.1) by $\omega \, dt$, and note that $\omega = d\theta/dt$.

$$I\omega \, d\omega = M \, d\theta$$

Integrating both sides gives

$$\int_0^\omega I\omega \, d\omega = \frac{1}{2} I\omega^2 = \int_0^\theta M \, d\theta \qquad (2.2.5)$$

We thus see that the work done by the moment M produces the kinetic energy of rotation: $KE = I\omega^2/2.$

PULLEY DYNAMICS

Figure 2.2.4 Pulley forces.

Pulleys can be used to change the direction of an applied force or to amplify forces. In our examples, we will assume that the cords, ropes, chains, and cables drive the pulleys without slipping and are inextensible; if not, then they must be modeled as springs. Figure 2.2.4 shows a pulley of inertia I whose center is fixed to a support. Assume that the tension forces in the cable are different on each side of the pulley. Then application of (2.2.1) gives

$$I\ddot{\theta} = RT_1 - RT_2 = R(T_1 - T_2)$$

An immediate result of practical significance is that the tension forces are approximately equal if $I\ddot{\theta}$ is negligible. This condition is satisfied if either the pulley rotates at a constant speed or if the pulley inertia is negligible compared to the other inertias in the system. The pulley inertia will be negligible if either its mass or its radius is small. Thus, when we neglect the mass, radius, or inertia of a pulley, the tension forces in the cable may be taken to be the same on both sides of the pulley.

   The force on the support at the pulley center is $T_1 + T_2$. If the mass, radius, or inertia of the pulley are negligible, then the support force is $2T_1$.

Energy Analysis of a Pulley System   **EXAMPLE 2.2.2**

■ Problem

Figure 2.2.5a shows a pulley used to raise the mass m_2 by hanging a mass m_1 on the other side of the pulley. If pulley inertia is negligible then it is obvious that m_1 will lift m_2 if $m_1 > m_2$. How does a nonnegligible pulley inertia I change this result? Also, investigate the effect of the pulley inertia on the speed of the masses.

■ Solution

Define the coordinates x and y such that $x = y = 0$ at the start of the motion. If the pulley cable is inextensible, then $x = y$ and thus $\dot{x} = \dot{y}$. If the cable does not slip, then $\dot{\theta} = \dot{x}/R$. Because we were asked about the speed and because the only applied force is a conservative force (gravity),

Figure 2.2.5 A pulley system for lifting a mass.

(a)   (b)
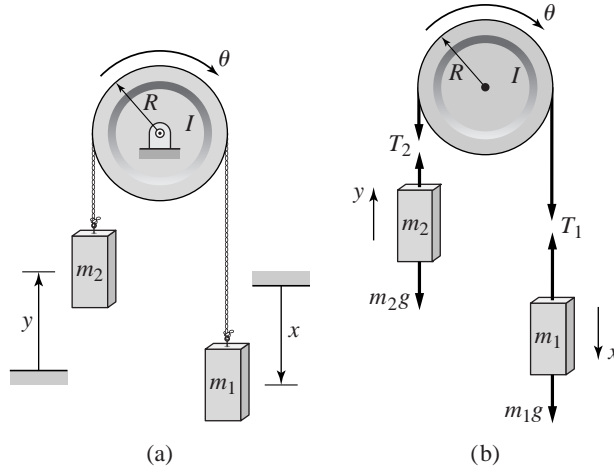
this suggests that we use an energy-based analysis. If the system starts at rest at $x = y = 0$, then the kinetic energy is initially zero. We take the potential energy to be zero at $x = y = 0$. Thus, the total mechanical energy is initially zero, and from conservation of energy we obtain

$$\text{KE} + \text{PE} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + m_2gy - m_1gx = 0$$

Note that the potential energy of m_1 has a negative sign because m_1 loses potential energy when x is positive.

Substituting $y = x$, $\dot{y} = \dot{x}$, and $\dot{\theta} = \dot{x}/R$ into this equation and collecting like terms gives

$$\frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_2 - m_1)gx = 0$$

and thus

$$\dot{x} = \sqrt{\frac{2(m_1 - m_2)gx}{m_1 + m_2 + I/R^2}} \tag{1}$$

The mass m_2 will be lifted if $\dot{x} > 0$; that is, if $m_2 < m_1$. So the pulley inertia does not affect this result. However, because I appears in the denominator of the expression for $\dot{x}$, the pulley inertia does decrease the speed with which m_1 lifts m_2.

In Example 2.2.2, it is inconvenient to use an energy-based analysis to compute $x(t)$ or the tensions in the cable. To do this it is easier to use Newton's law directly.

EXAMPLE 2.2.3

Equation of Motion of a Pulley System

■ **Problem**

Consider the pulley system shown in Figure 2.2.5a. Obtain the equation of motion in terms of x and obtain an expression for the tension forces in the cable.

■ **Solution**

The free body diagrams of the three bodies are shown in part (b) of the figure. Newton's law for mass m_1 gives

$$m_1\ddot{x} = m_1g - T_1 \tag{1}$$

Newton's law for mass m_2 gives

$$m_2\ddot{y} = T_2 - m_2 g \tag{2}$$

Equation (2.2.1) applied to the inertia I gives

$$I\ddot{\theta} = RT_1 - RT_2 = R(T_1 - T_2) \tag{3}$$

Because the cable is assumed inextensible, $x = y$ and thus $\ddot{x} = \ddot{y}$. We can then solve (1) and (2) for the tension forces.

$$T_1 = m_1 g - m_1\ddot{x} = m_1(g - \ddot{x}) \tag{4}$$

$$T_2 = m_2\ddot{y} + m_2 g = m_2(\ddot{y} + g) = m_2(\ddot{x} + g) \tag{5}$$

Substitute these expressions into (3).

$$I\ddot{\theta} = (m_1 - m_2)gR - (m_1 + m_2)R\ddot{x} \tag{6}$$

Because $x = R\theta$, $\ddot{x} = R\ddot{\theta}$, and (6) becomes

$$I\frac{\ddot{x}}{R} = (m_1 - m_2)gR - (m_1 + m_2)R\ddot{x}$$

which can be rearranged as

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g \tag{7}$$

This is the desired equation of motion. We can solve it for $\ddot{x}$ and substitute the result into equations (4) and (5) to obtain T_1 and T_2 as functions of the parameters m_1, m_2, I, R, and g.

Equation (7) can be solved for $\dot{x}(t)$ and $x(t)$ by direct integration.
Let

$$A = \frac{(m_1 - m_2)gR^2}{(m_1 + m_2)R^2 + I}$$

Then (7) becomes $\ddot{x} = A$, whose solutions are $\dot{x} = At + \dot{x}(0)$ and $x = At^2/2 + \dot{x}(0)t + x(0)$. Note that if we use the solutions to express $\dot{x}$ as a function of x, we will obtain the same expression as equation (1) in Example 2.2.2.

PULLEY-CABLE KINEMATICS

Consider Figure 2.2.6. Suppose we need to determine the relation between the velocities of mass m_A and mass m_B. Define x and y as shown from a common reference line attached to a fixed part of the system. Noting that the cable lengths wrapped around the pulleys are constant, we can write $x + 3y = $ constant. Thus $\dot{x} + 3\dot{y} = 0$. So the speed of point A is three times the speed of point B, and in the opposite direction.

In many problems, we neglect the inertia of the pulleys so as to keep the resulting model as simple as possible. As we will see, the models of many practical engineering applications are challenging enough without introducing pulley dynamics. Example 2.2.4 illustrates such an application.

Figure 2.2.6 A multiple-pulley system.
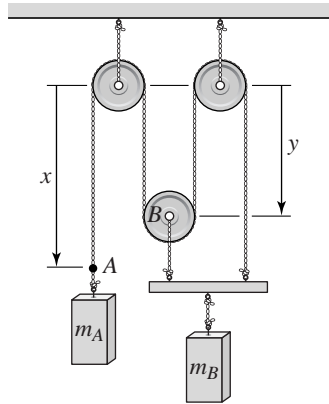
EXAMPLE 2.2.4

Lifting a Mast

■ Problem

A mast weighing 500 lb is hinged at its bottom to a fixed support at point O (see Figure 2.2.7). The mast is 70 ft long and has a uniform mass distribution, so its center of mass is 35 ft from O. The winch applies a force $f = 380$ lb to the cable. The mast is supported initially at the $30°$ angle, and the cable at A is initially horizontal. Derive the equation of motion of the mast. You may assume that the pulley inertias are negligible and that the pulley diameter d is very small compared to the other dimensions.

■ Solution

Part (b) of the figure shows the geometry of the mast at some angle $\theta > 30°$, with the diameter d neglected. From the law of sines,

$$\sin\phi = P\frac{\sin(180° - \mu - \theta)}{Q} = P\frac{\sin(\mu + \theta)}{Q}$$

From the law of cosines,

$$Q = \sqrt{P^2 + L^2 - 2PL\cos(180° - \mu - \theta)} = \sqrt{P^2 + L^2 + 2PL\cos(\mu + \theta)} \tag{1}$$

Figure 2.2.7 A system for lifting a mast.

(a)   (b)
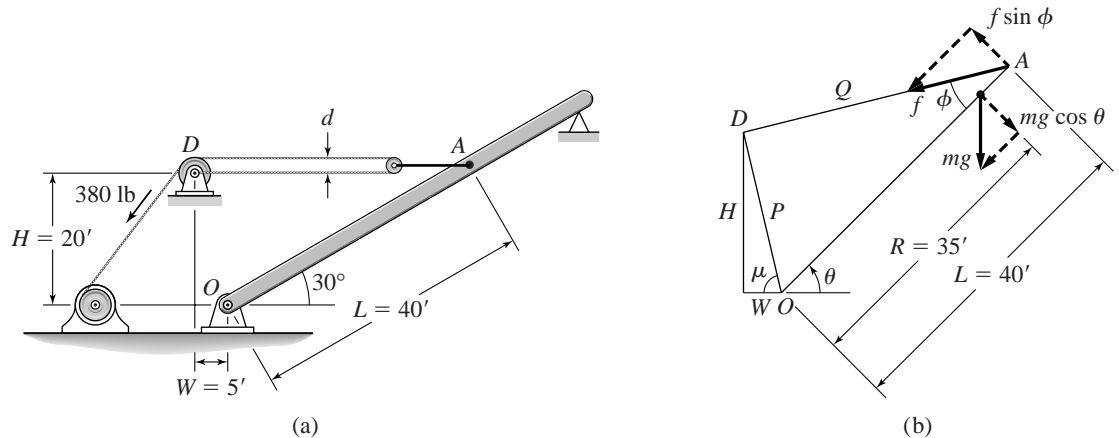
where $H = 20$ ft, $W = 5$ ft, and

$$\mu = \tan^{-1}\left(\frac{H}{W}\right) = \tan^{-1}\left(\frac{20}{5}\right) = 76° = 1.33 \text{ rad}$$

$$P = \sqrt{H^2 + W^2} = 20.6 \text{ m}$$

The moment equation about the fixed point O is

$$I_O\ddot{\theta} = -mgR\cos\theta + \frac{fLP}{Q}\sin(\mu + \theta) \tag{2}$$

The moment of inertia is

$$I_O = \frac{1}{3}m(70)^2 = \frac{1}{3}\frac{500}{32.2}70^2 = 25{,}400 \text{ slug-ft}^2$$

The force f at point A is twice the applied force of 380 lb, because of the action of the small pulley. Thus $f = 760$ lb. With the given values, the equation of motion becomes

$$25{,}400\,\ddot{\theta} = -17{,}500\cos\theta + \frac{626{,}000}{Q}\sin(1.33 + \theta) \tag{3}$$

where

$$Q = \sqrt{2020 + 1650\cos(1.33 + \theta)} \tag{4}$$

Equation (3) cannot be solved in closed-form to find $\theta(t)$, so it must be solved numerically using the methods to be introduced in Chapter 5.

2.3 EQUIVALENT MASS AND INERTIA

Some systems composed of translating and rotating parts whose motions are directly coupled can be modeled as a purely translational system or as a purely rotational system, by using the concepts of equivalent mass and equivalent inertia. These models can be derived using kinetic energy equivalence.

Equivalent mass and equivalent inertia are complementary concepts. A system should be viewed as an equivalent mass if an external force is applied, and as an equivalent inertia if an external torque is applied. Examples 2.3.1–2.3.5 will illustrate this approach.

A Vehicle on an Incline: Energy Analysis | **EXAMPLE 2.3.1**

■ Problem
A tractor pulls a cart up a slope, starting from rest and accelerating to 20 m/s in 15 s (Figure 2.3.1a). The force in the cable is f, and the body of the cart has a mass m. The cart has two identical wheels, each with radius R, mass m_w, and inertia I_w about the wheel center. The two wheels are coupled with an axle whose mass is negligible. Assume that the wheels do not slip or bounce. Derive an expression for the force f using kinetic energy equivalence.

■ Solution
The assumption of no slip and no bounce means that the wheel rotation is directly coupled to the cart translation. This means that if we know the cart translation x, we also know the wheel rotation θ, because $x = R\theta$ if the wheels do not slip or bounce. Because the input is the force f, we will derive an equivalent mass, and thus we will visualize the system as a block of mass m_e being pulled up the incline by the force f, as shown in Figure 2.3.1b. The wheels will roll without

Figure 2.3.1 A vehicle on an incline.

(a)

(b)

(c)
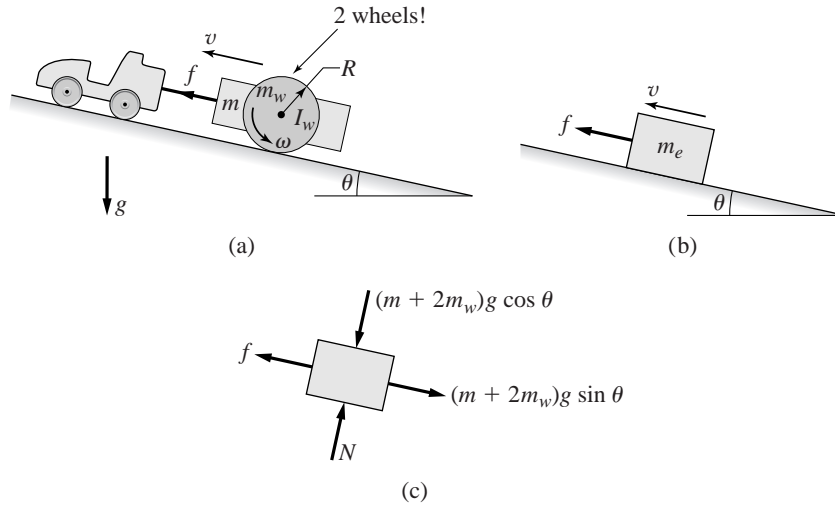
slipping (pure rolling) if the tangential force between the wheel and the surface is smaller than the static friction force. In this case the tangential force does no work because it does not act through a distance (see Section 2.4), and thus there is no energy loss due to friction. Therefore, in our equivalent model, Figure 2.3.1b, there is no friction between the block and the surface.

The kinetic energy of the system is

$$\text{KE} = \frac{1}{2}mv^2 + \frac{1}{2}\left(2m_w v^2\right) + \frac{1}{2}\left(2I_w \omega^2\right)$$

Because $v = R\omega$, we obtain

$$\text{KE} = \frac{1}{2}\left(m + 2m_w + 2\frac{I_w}{R^2}\right)v^2 \tag{1}$$

For the block in Figure 2.3.1b, $\text{KE} = 0.5m_e v^2$. Comparing this with equation (1) we see that the equivalent mass is given by

$$m_e = m + 2m_w + 2\frac{I_w}{R^2} \tag{2}$$

The free body diagram is shown in part (c) of the figure. Note that the gravity force is determined, not from the equivalent mass m_e, but from the actual mass $m + 2m_w$. The reason for this can be seen from the potential energy expression. Assuming that the center of mass of the cart coincides with the axle location, we can express the potential energy of the system as $(m + 2m_w)gx \sin\theta$, where $\dot{x} = v$. Therefore, the actual mass $m + 2m_w$ must be used to compute the gravity force.

From the free body diagram we obtain the following equation of motion.

$$m_e \dot{v} = f - (m + 2m_w)g \sin\theta \tag{3}$$

where m_e is given by equation (2).

The acceleration is $\dot{v} = 20/15 = 4/3 \text{ m/s}^2$. Substitute this into equation (3) and solve for f:

$$f = \frac{4}{3}m_e + (m + 2m_w)g \sin\theta \tag{4}$$

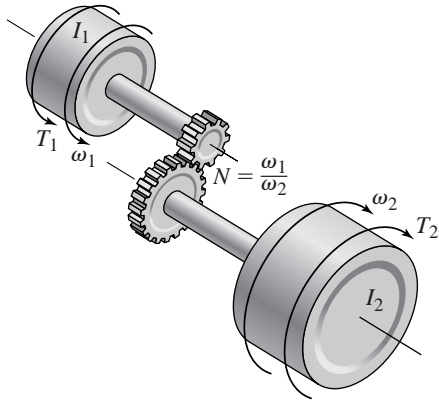This is the force required to provide the specified acceleration.

Figure 2.3.2 A spur-gear pair.

MECHANICAL DRIVES

Gears, belts, levers, and pulleys transform an input motion, force, or torque into another motion, force, or torque at the output. For example, a gear pair can be used to reduce speed and increase torque, and a lever can increase force.

Several types of gears are used in mechanical drives. These include helical, spur, rack-and-pinion, worm, bevel, and planetary gears. Other mechanical drives use belts or chains. We now use a spur gear pair, a rack-and-pinion gear pair, and a belt drive to demonstrate the use of kinetic energy equivalence to obtain a model. This approach can be used to analyze other gear and drive types.

A pair of spur gears is shown in Figure 2.3.2. The input shaft (shaft 1) is connected to a motor that produces a torque T_1 at a speed ω_1, and drives the output shaft (shaft 2). One use of such a system is to increase the effective motor torque. The gear ratio N is defined as the ratio of the input rotation θ_1 to the output rotation θ_2. Thus, $N = \theta_1/\theta_2$. From geometry we can see that N is also the speed ratio $N = \omega_1/\omega_2$. Thus, the pair is a speed *reducer* if $N > 1$. The gear ratio is also the diameter ratio $N = D_2/D_1$, and the gear tooth ratio $N = n_2/n_1$, where n is the number of gear teeth.

If the gear inertias are negligible or if there is zero acceleration, and if we neglect energy loss due to friction, such as that between the gear teeth, then the input work $T_1\theta_1$ must equal the output work $T_2\theta_2$. Thus, under these conditions, $T_2 = T_1(\theta_1/\theta_2) = NT_1$, and the output torque is greater than the input torque for a speed reducer. For cases that involve acceleration and appreciable gear inertia, the output torque is less than NT_1.

Equivalent Inertia of Spur Gears | **EXAMPLE 2.3.2**

■ Problem
Consider the spur gears shown in Figure 2.3.2. Derive the expression for the equivalent inertia I_e felt on the input shaft.

■ Solution
Let I_1 and I_2 be the total moments of inertia on the shafts. The kinetic energy of the system is then

$$\text{KE} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\left(\frac{\omega_1}{N}\right)^2$$

or

$$KE = \frac{1}{2}\left(I_1 + \frac{1}{N^2}I_2\right)\omega_1^2$$

Therefore the equivalent inertia felt on the input shaft is

$$I_e = I_1 + \frac{I_2}{N^2} \tag{1}$$

This means that the dynamics of the system can be described by the model $I_e\dot{\omega}_1 = T_1$.

A spur gear pair consists of only rotating elements. However, a rack-and-pinion consists of a rotating component (the pinion gear) and a translating component (the rack). The input to such a device is usually the torque applied to the shaft of the pinion. If so, then we should model the device as an equivalent inertia. The following example shows how to do this.

EXAMPLE 2.3.3

Equivalent Inertia of a Rack-and-Pinion

■ Problem
A rack-and-pinion, shown in Figure 2.3.3, is used to convert rotation into translation. The input shaft rotates through the angle θ as a result of the torque T produced by a motor. The pinion rotates and causes the rack to translate. Derive the expression for the equivalent inertia I_e felt on the input shaft. The mass of the rack is m, the inertia of the pinion is I, and its mean radius is R.

■ Solution
The kinetic energy of the system is (neglecting the inertia of the shaft)

$$KE = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

where $\dot{x}$ is the velocity of the rack and $\dot{\theta}$ is the angular velocity of the pinion and shaft. From geometry, $x = R\theta$, and thus $\dot{x} = R\dot{\theta}$. Substituting for $\dot{x}$ in the expression for KE, we obtain
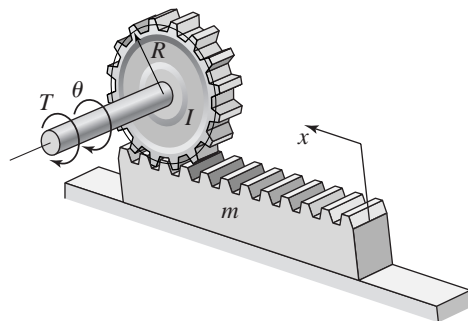
$$KE = \frac{1}{2}m\left(R\dot{\theta}\right)^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(mR^2 + I\right)\dot{\theta}^2$$

Thus the equivalent inertia felt on the shaft is

$$I_e = mR^2 + I \tag{1}$$

and the model of the system's dynamics is $I_e\ddot{\theta} = T$, which can be expressed in terms of x as $I_e\ddot{x} = RT$.

Figure 2.3.3 A rack-and-pinion gear.

Belt and chain drives are good examples of devices that can be difficult to analyze by direct application of Newton's laws but can be easily modeled using kinetic energy equivalence.

Equivalent Inertia of a Belt Drive

EXAMPLE 2.3.4

■ **Problem**

Belt drives and chain drives, like those used on bicycles, have similar characteristics and can be analyzed in a similar way. A belt drive is shown in Figure 2.3.4. The input shaft (shaft 1) is connected to a device (such as a bicycle crank) that produces a torque T_1 at a speed ω_1, and drives the output shaft (shaft 2). The mean sprocket radii are r_1 and r_2, and their inertias are I_1 and I_2. The belt mass is m.

Derive the expression for the equivalent inertia I_e felt on the input shaft.

■ **Solution**

The kinetic energy of the system is

$$KE = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}mv^2$$

If the belt does not stretch, the translational speed of the belt is $v = r_1\omega_1 = r_2\omega_2$. Thus we can express KE as

$$KE = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\left(\frac{r_1\omega_1}{r_2}\right)^2 + \frac{1}{2}m(r_1\omega_1)^2 = \frac{1}{2}\left[I_1 + I_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2\right]\omega_1^2$$

Therefore, the equivalent inertia felt on the input shaft is

$$I_e = I_1 + I_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2 \tag{1}$$

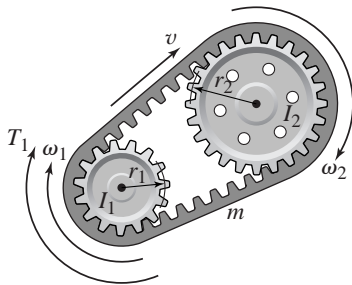This means that the dynamics of the system can be described by the model $I_e\dot{\omega}_1 = T_1$.


Figure 2.3.4 A belt drive.

A Robot-Arm Link

EXAMPLE 2.3.5

■ **Problem**

A single link of a robot arm is shown in Figure 2.3.5. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through a pair of spur gears. The values of m and L depend on the payload being carried in the hand and thus can be different for each application. The gear ratio is $N = 2$ (the motor shaft has the greater speed). The motor and gear rotation axes are fixed by bearings.

Figure 2.3.5 A robot-arm link.
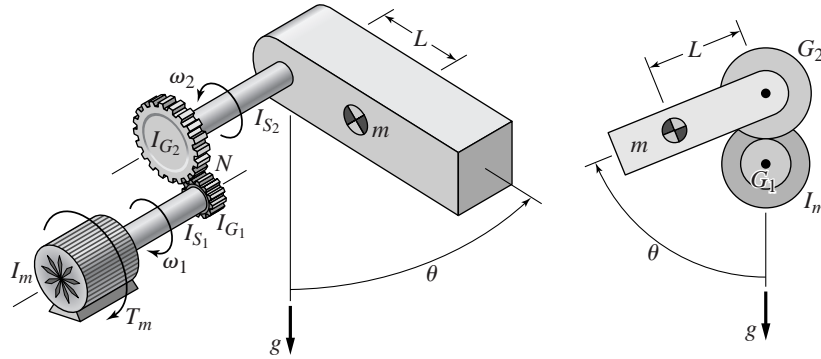
To control the motion of the arm we need to have its equation of motion. Obtain this equation in terms of the angle θ. The given values for the motor, shaft, and gear inertias are

$$I_m = 0.05 \text{ kg} \cdot \text{m}^2 \qquad I_{G_1} = 0.025 \text{ kg} \cdot \text{m}^2 \qquad I_{S_1} = 0.01 \text{ kg} \cdot \text{m}^2$$

$$I_{G_2} = 0.1 \text{ kg} \cdot \text{m}^2 \qquad I_{S_2} = 0.02 \text{ kg} \cdot \text{m}^2$$

■ Solution

Our approach is to model the system as a single inertia rotating about the motor shaft with a speed ω_1. To find the equivalent inertia about this shaft we first obtain the expression for the kinetic energy of the total system and express it in terms of the shaft speed ω_1. Note that the mass m is translating with a speed $L\omega_2$.

$$\text{KE} = \frac{1}{2}\left(I_m + I_{S_1} + I_{G_1}\right)\omega_1^2 + \frac{1}{2}\left(I_{S_2} + I_{G_2}\right)\omega_2^2 + \frac{1}{2}m\left(L\omega_2\right)^2$$

But $\omega_2 = \omega_1/N = \omega_1/2$. Thus,

$$\text{KE} = \frac{1}{2}\left[I_m + I_{S_1} + I_{G_1} + \frac{1}{2^2}\left(I_{S_2} + I_{G_2} + mL^2\right)\right]\omega_1^2$$

Therefore, the equivalent inertia referenced to the motor shaft is

$$I_e = I_m + I_{S_1} + I_{G_1} + \frac{1}{2^2}\left(I_{S_2} + I_{G_2} + mL^2\right) = 0.115 + 0.25mL^2$$

The equation of motion for this equivalent inertia can be obtained in the same way as that of a pendulum, by noting that the gravity moment $mgL\sin\theta$, which acts on shaft 2, is also felt on the motor shaft, but *reduced* by a factor of N due to the gear pair. Thus,

$$I_e\dot{\omega}_1 = T_m - \frac{1}{N}mgL\sin\theta$$

But $\omega_1 = N\omega_2 = N\dot{\theta}$. Thus

$$I_e N\ddot{\theta} = T_m - \frac{1}{N}mgL\sin\theta$$

Substituting the given values, we have

$$2(0.115 + 0.25mL^2)\ddot{\theta} = T_m - \frac{9.8}{2}mL\sin\theta$$

or

$$(0.23 + 0.5mL^2)\ddot{\theta} = T_m - 4.9mL\sin\theta \tag{1}$$

We will discuss techniques for solving this equation in Chapter 3.

2.4 GENERAL PLANAR MOTION

In Section 2.1 we limited our attention to systems undergoing pure translation, and in Section 2.2 we analyzed systems rotating about a single nonaccelerating axis. As demonstrated by the examples in Section 2.3, we can use energy equivalence to model a system as if it were in pure translation or pure rotation, but only if the motions of the rotating and translating components are directly coupled. We now consider the general case of an object undergoing translation and rotation about an accelerating axis.

We will restrict our attention to motion in a plane. This means that the object can translate in two dimensions and can rotate only about an axis that is perpendicular to the plane. Many practical engineering problems can be handled with the plane motion methods covered here. The completely general motion case involves translation in three dimensions, and rotation about three axes. This type of motion is considerably more complex to analyze, and is beyond the scope of our coverage.

FORCE EQUATIONS

Newton's laws for plane motion are derived in basic dynamics references, and we will review them here. We assume that the object in question is a rigid body that moves in a plane passing through its mass center, and that it is symmetrical with respect to that plane. Thus it can be thought of as a slab with its motion confined to the plane of the slab. We assume that the mass center and all forces acting on the mass are in the plane of the slab.

We can describe the motion of such an object by its translational motion in the plane and by its rotational motion about an axis perpendicular to the plane. Two force equations describe the translational motion, and a moment equation is needed to describe the rotational motion. Consider the slab shown in Figure 2.4.1, where we arbitrarily assume that three external forces f_1, f_2, and f_3 are acting on the slab. Define an x-y coordinate system as shown with its origin located at a nonaccelerating point. Then the two force equations can be written as

$$f_x = ma_{G_x} \qquad\qquad (2.4.1)$$

$$f_y = ma_{G_y} \qquad\qquad (2.4.2)$$
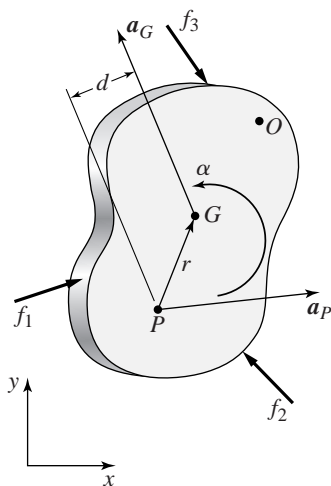
Figure 2.4.1 Planar motion of a slab.

where f_x and f_y are the net forces acting on the mass m in the x and y directions, respectively. The mass center is located at point G. The quantities a_{G_x} and a_{G_y} are the accelerations of the mass center in the x and y directions relative to the fixed x-y coordinate system.

MOMENT EQUATIONS

Recall that in Section 2.2 we treated the case where an object is constrained to rotate about an axis that passes through a fixed point O. For this case, we can apply the following moment equation:

$$I_O \alpha = M_O \tag{2.4.3}$$

where α is the angular acceleration of the mass about the axis through point O, I_O is the mass moment of inertia of the body about the point O, and M_O is the sum of the moments applied to the body about the point O.

The following moment equation applies regardless of whether or not the axis of rotation is constrained:

$$M_G = I_G \alpha \tag{2.4.4}$$

where M_G is the net moment acting on the body about an axis that passes through the mass center G and is perpendicular to the plane of the slab. I_G and α are the mass moment of inertia and angular acceleration of the body about this axis. The net moment M_G is caused by the action of the external forces $f_1, f_2, f_3, \ldots$ and any couples applied to the body. The positive direction of M_G is determined by the right-hand rule (counterclockwise if the x-y axes are chosen as shown).

Note that point G in the preceding equation must be the mass center of the object; no other point may be used. However, in many problems the acceleration of some point P is known, and sometimes it is more convenient to use this point rather than the mass center or a fixed point. The following moment equation applies for an accelerating point P, which need not be fixed to the body:

$$M_P = I_G \alpha + m a_G d \tag{2.4.5}$$

where the moment M_P is the net moment acting on the body about an axis that passes through P and is perpendicular to the plane of the slab, a_G is the magnitude of the acceleration vector $\mathbf{a}_G$, and d is the distance between $\mathbf{a}_G$ and a parallel line through point P (see Figure 2.4.1).

An alternative form of this equation is

$$M_P = I_P \alpha + m r_x a_{P_y} - m r_y a_{P_x} \tag{2.4.6}$$

where a_{P_x} and a_{P_y} are the x and y components of the acceleration of point P relative to the x-y coordinate system. The terms r_x and r_y are the x and y components of the location of G relative to P, and I_P is the mass moment of inertia of the body about an axis through P. Note that in general, M_P does not equal M_G, and I_P does not equal I_G. If point P is fixed at some point O, then $a_{P_x} = a_{P_y} = 0$, and the moment equation (2.4.6) simplifies to (2.4.3), because $M_O = M_P$ and $I_O = I_P$. Note that the angular acceleration α is the same regardless of whether point O, G, or P is used.

SLIDING VERSUS ROLLING MOTION

Wheels are common examples of systems undergoing general plane motion with both translation and rotation. The wheel shown in Figure 2.4.2 can have one of three possible

Figure 2.4.2 Motion of a wheel.
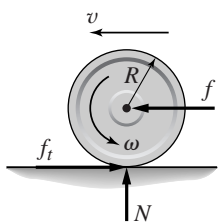
motion types:

1. *Pure rolling motion.* This occurs when there is no slipping between the wheel and the surface. In this case, $v = R\omega$.
2. *Pure sliding motion.* This occurs when the wheel is prevented from rotating (such as when a brake is applied). In this case, $\omega = 0$ and $v \neq R\omega$.
3. *Sliding and rolling motion.* In this case $\omega \neq 0$. Because slipping occurs in this case, $v \neq R\omega$.

The wheel will roll without slipping (pure rolling) if the tangential force f_t is smaller than the static friction force $\mu_s N$, where N is the force of the wheel normal to the surface. In this case, the tangential force does no work because it does not act through a distance. If the static friction force is smaller than f_t, the wheel will slip.

In Example 2.3.1, we modeled the wheeled vehicle as an equivalent translating mass. We can "lump" the rotating elements with the translating elements only when the translational and rotational motions are directly related to one another. If the vehicle's wheels slip, we cannot express the vehicle's translational speed in terms of the wheels' rotational speed. We must then treat the vehicle body and the wheels as separate masses, and apply Newton's laws to each separately. We must also take this approach if we need to compute any forces internal to the system.

A Vehicle on an Incline: Force Analysis | **EXAMPLE 2.4.1**

■ Problem

Consider again the system of Example 2.3.1, in which a tractor pulls a cart up a slope (Figure 2.3.1a). The force in the cable is f, and the body of the cart has a mass m. The cart has two identical wheels, each with radius R, mass m_w, and inertia I_w about the wheel center. The two wheels are coupled with an axle whose mass is negligible. Assume that the center of mass of the cart coincides with the axis of the axle. In Example 2.3.1, we assumed that the wheels do not slip. Now we want to develop a model that can be used to examine this assumption and also to compute the forces on the axle. Apply Newton's laws to develop a model of the system. (a) Show that the model gives the same result as equation (3) of Example 2.3.1, assuming no slip and no bounce. (b) Use the model to discuss the no-slip assumption.

■ Solution

a. The free body diagrams of the cart and the wheel-axle subsystem are shown in Figure 2.4.3. The forces R_x and R_y are the reaction forces between the cart and the axle. The forces f_t and N are the tangential and normal forces acting on both wheels as a result of their contact with the road's surface. Newton's second law applied to the cart gives

$$m\ddot{x} = R_x - mg \sin\theta + f \tag{1}$$

$$m\ddot{y} = R_y - mg \cos\theta \tag{2}$$
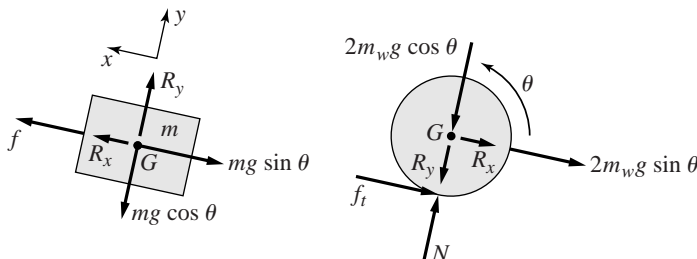
Figure 2.4.3 Free-body diagrams of the cart body and the wheel-axle subsystem.

Newton's second law applied to the wheel-axle subsystem gives

$$2m_w \ddot{x} = -R_x - 2m_w g \sin\theta - f_t \tag{3}$$

$$2m_w \ddot{y} = N - R_y - 2m_w g \cos\theta \tag{4}$$

Application of the moment equation about the center of mass of the wheel-axle subsystem gives

$$2I_w \dot{\omega} = R f_t \tag{5}$$

The assumption of no slip means that

$$\ddot{x} = \dot{v} = R\dot{\omega} \tag{6}$$

Substitute this into equation (5) and solve for f_t:

$$f_t = \frac{2I_w}{R^2} \ddot{x} \tag{7}$$

Substitute this into equation (3) and solve for R_x:

$$R_x = -\left(2m_w + \frac{2I_w}{R^2}\right)\ddot{x} - 2m_w g \sin\theta \tag{8}$$

Substitute this into equation (1) and collect terms to obtain

$$\left(m + 2m_w + 2\frac{I_w}{R^2}\right)\ddot{x} = f - (m + 2m_w)g \sin\theta \tag{9}$$

which is the same as equation (3) of Example 2.3.1 since $\ddot{x} = \dot{v}$.

b.   The assumption of no bounce means that $\ddot{y} = 0$. For this condition, equations (2) and (4) can be solved for N to obtain

$$N = (m + 2m_w)g \cos\theta \tag{10}$$

Slip occurs if the tangential force f_t is greater than the maximum static friction force, which is $\mu_s N$. Thus, if we solve equation (9) for $\ddot{x}$ and substitute it into equation (7), the resulting value of f_t must be no greater than $\mu_s N$ or else slip will occur. Thus, slip will not occur if

$$\mu_s(m + 2m_w)g \cos\theta \geq 2I_w \frac{f - (m + 2m_w)g \sin\theta}{(m + 2m_w)R^2 + 2I_w} \tag{11}$$

EXAMPLE 2.4.2   ## A Rolling Cylinder

■ Problem
A solid cylinder of mass m and radius r starts from rest and rolls down the incline at an angle θ (Figure 2.4.4). The static friction coefficient is μ_s. Determine the acceleration of the center of mass a_{G_x} and the angular acceleration α. Assume that the cylinder rolls without bouncing, so that $a_{G_y} = 0$. Assume also that the cylinder rolls without slipping. Use two approaches to solve the problem: (a) Use the moment equation about G, and (b) use the moment equation about P. (c) Obtain the frictional condition required for the cylinder to roll without slipping.

■ Solution
The free body diagram is shown in the figure. The friction force is F and the normal force is N. The force equation in the x direction is

$$ma_{G_x} = f_x = mg \sin\theta - F \tag{1}$$
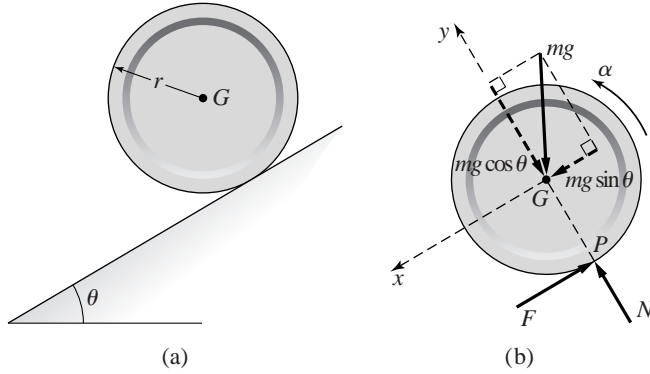
Figure 2.4.4 A cylinder rolling down an inclined plane.

In the y direction,

$$ma_{G_y} = f_y = N - mg \cos\theta \tag{2}$$

If the cylinder does not bounce, then $a_{G_y} = 0$ and thus from (2),

$$N = mg \cos\theta \tag{3}$$

a. The moment equation about the center of mass gives

$$I_G \alpha = M_G = Fr \tag{4}$$

Solve for F and substitute it into (1):

$$ma_{G_x} = mg \sin\theta - \frac{I_G \alpha}{r} \tag{5}$$

If the cylinder does not slip, then

$$a_{G_x} = r\alpha \tag{6}$$

Solve this for α and substitute into (5):

$$ma_{G_x} = mg \sin\theta - \frac{I_G a_{G_x}}{r^2}$$

Solve this for a_{G_x}:

$$a_{G_x} = \frac{mgr^2 \sin\theta}{mr^2 + I_G} \tag{7}$$

For a solid cylinder, $I_G = mr^2/2$, and the last expression reduces to

$$a_{G_x} = \frac{2}{3} g \sin\theta \tag{8}$$

b. We could have used instead the moment equation (2.4.5) about the point P, which is accelerating. This equation is

$$M_P = I_G \alpha + ma_{G_x} d$$

where $d = r$ and $M_P = (mg \sin\theta)r$. Thus

$$mgr \sin\theta = I_G \frac{a_{G_x}}{r} + ma_{G_x} r$$

Solving this for a_{G_x} we obtain the same expression as (7). The angular acceleration is found from (6).

c.   The maximum possible friction force is $F_{\max} = \mu_s N = \mu_s mg \cos\theta$. From (4), (6), and (7),

$$F = \frac{I_G \alpha}{r} = \frac{I_G a_{G_x}}{r^2} = \frac{I_G mg \sin\theta}{I_G + mr^2}$$

If $F_{\max} > F$, the cylinder will not slip. The condition of no slip is therefore given by

$$\mu_s \cos\theta > \frac{I_G \sin\theta}{I_G + mr^2} \tag{9}$$

For a solid cylinder, this reduces to

$$\mu_s \cos\theta > \frac{1}{3} \sin\theta \tag{10}$$

EXAMPLE 2.4.3

Maximum Vehicle Acceleration

■ Problem

It is required to determine, as a function of the friction coefficient μ_s, the maximum acceleration of the rear-wheel drive vehicle shown in Figure 2.4.5. The vehicle mass is 1800 kg, and its dimensions are $L_A = 1.3$ m, $L_B = 1$ m, and $H = 0.5$ m.

Assume that the mass of the wheels is small compared with the total vehicle mass and neglect the effects of the vehicle suspension. These assumptions enable us to ignore the rolling and vertical motions of the wheels and thus to treat the vehicle as a single rigid body translating in only one direction, with no rotating parts. The assumption of negligible wheel mass implies that there is no tangential force on the front wheels.

Assume that each front wheel experiences the same reaction force $N_A/2$. Similarly, each rear wheel experiences the same reaction force $N_B/2$. Thus the traction force $\mu_s N_B$ is the total traction force due to both driving wheels. The traction force is due to the torque T applied from the engine through the axles to the rear wheels. Thus, when the rear wheels are on the verge of slipping, $\mu_s N_B = T/R$, where R is the radius of the rear wheels.

Figure 2.4.5 Vehicle acceleration.

(a)

(b)
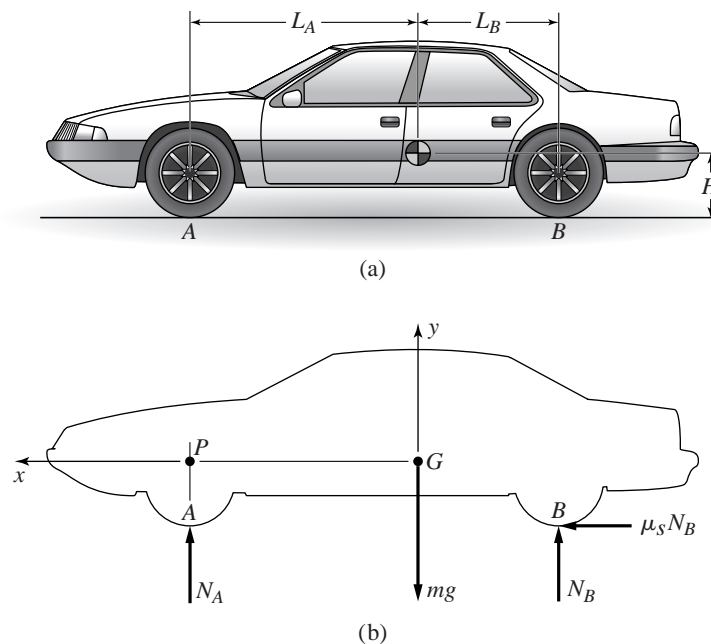
■ Solution

The key to solving this problem is to recognize that the maximum traction force, and therefore the maximum acceleration, is obtained when the driving tires are just on the verge of slipping relative to the road surface. In this condition, the friction force, which is the traction force, is given by $\mu_s N_B$. From the free body diagram shown in Figure 2.4.5b, Newton's law applied in the x direction gives

$$f_x = ma_{G_x} \qquad \text{or} \qquad \mu_s N_B = ma_{G_x} \tag{1}$$

In the y direction, if the vehicle does not leave the road, Newton's law gives

$$f_y = ma_{G_y} \qquad \text{or} \qquad N_A + N_B - mg = 0 \tag{2}$$

The moment equation about the mass center G gives

$$M_G = I_G \alpha \qquad \text{or} \qquad N_B L_B - \mu_s N_B H - N_A L_A = 0 \tag{3}$$

because $\alpha = 0$ if the vehicle body does not rotate.

Equation (1) shows that we need to find N_B to determine the acceleration a_{G_x}. Equations (2) and (3) can be solved for N_B. The solution is

$$N_B = \frac{mg L_A}{L_A + L_B - \mu_s H} = \frac{9.8(1.3)m}{1.3 + 1 - 0.5\mu_s} = \frac{25.5}{4.6 - \mu_s}m \tag{4}$$

and thus the maximum acceleration is

$$a_{G_x} = \frac{\mu_s N_B}{m} = \frac{25.5\mu_s}{4.6 - \mu_s}$$

An alternative approach to the problem is to use the moment equation (2.4.5) about the accelerating point P shown in the figure. This approach avoids the need to solve two equations to obtain N_B. This equation gives

$$M_P = I_G \alpha + ma_G d \qquad \text{or} \qquad \mu_s N_B H - N_B(L_A + L_B) + mg L_A = 0 \tag{5}$$

because $\alpha = 0$ and $d = 0$. This gives the same solution as equation (4).

The analysis in Example 2.4.3 ignored the effects of the vehicle suspension. This simplification results in the assumption that the vehicle body does not rotate. You may have noticed, however, that a vehicle undergoing acceleration will have a pitching motion that is made possible because of the suspension springs. So a complete analysis of this problem would include this effect. However, it is always advisable to begin with a simplified version of a problem, to make sure you understand the problem's basic features. If you cannot solve the problem with the suspension effects ignored, then you certainly will not be able to solve the more complex problem that includes the suspension dynamics!

DYNAMICS OF A PERSONAL TRANSPORTER

Personal transporters are small vehicles designed to carry usually only one person. They have become more available because of the advent of less expensive sensors and more powerful microprocessor control systems to handle the complex calculations required to balance the vehicle. There are a variety of designs, including unicycles, but Figure 2.4.6 illustrates a two-wheel version. The transporter's motors drive the wheels to balance the vehicle with the help of a computer-control system using tilt sensors and gyroscopes.

One type of gyroscope used is a *vibrating structure gyroscope* that functions much like the *halteres* of some winged insects. These are small knobbed cantilever–beam-like

Figure 2.4.6 A personal transporter.
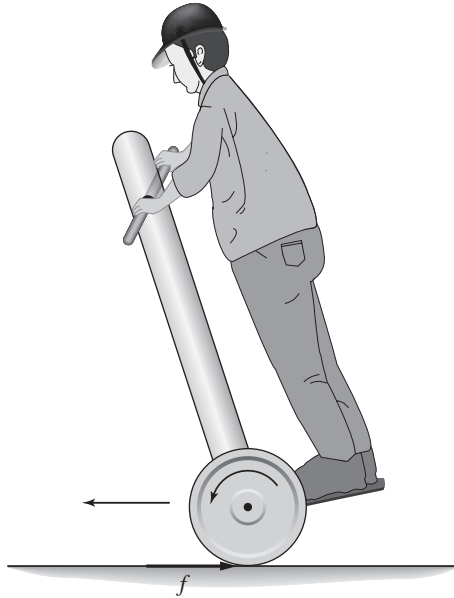
structures that are flapped rapidly to maintain stability when flying. The vibrating beam tends to keep vibrating in the same plane even though its support rotates. When used as a sensor, the transducer attached to the beam detects the bending strain that results from the *Coriolis acceleration*. These sensors are simpler and less expensive than a rotating gyroscope.

We have restricted our coverage of dynamics to inertial (for example, non-rotating) coordinate systems. The Coriolis acceleration is a term that appears in the equations of motion of an object when expressed in a rotating frame of reference (see [Meriam, 2002]). The Coriolis acceleration depends on the velocity of the object. A reference frame fixed to the Earth is actually rotating and therefore non-inertial. Thus a projectile, due to the Coriolis acceleration, appears to curve to the right in the northern hemisphere and to the left in the southern hemisphere. For many applications, the effect of the Coriolis term is negligible because of the relatively small velocities. However, in the vibrating gyroscope, the oscillation, and thus the velocity, of the beam are large, and thus the Coriolis effect is detectable.

The transporter will be balanced (kept nearly upright) as long as the wheels stay under the center of gravity. Thus the transporter can be balanced by driving the wheels in the direction of leaning. This means that to accelerate forward, the person should lean forward. A representation of the transporter is shown in Figure 2.4.6. The drive motor applies a torque to the wheel-axle subsystem. The tangential force between the wheel and the road is f. This force acts in the opposite direction on the vehicle body and propels the transporter forward (to the left in the figure).

The dynamics of a personal transporter are similar to a classic control problem called the *inverted pendulum*[1].

---

[1]This problem is a good example of why you should be careful to check any technical information, especially any obtained from the Internet. The author found several websites, including an entry in a well-known online encyclopedia, that contained erroneous equations of motion for the inverted pendulum. The equations derived here are equivalent to those derived in [Cannon, 1967].

| Transporter Equations of Motion | **EXAMPLE 2.4.4** |

■ Problem

We may model the transporter as a cart of mass M (which includes the equivalent mass of the wheel-axle subsystem) and an inverted pendulum attached to the cart by a pivot at point P (see Figure 2.4.7a). The pendulum mass is m and its center of mass G is a distance L from P. The inertia of the pendulum about G is I_G. For generality we include an applied torque T about the pivot, which is due to a motor at the pivot in some applications. Derive the equations of motion with f and T as the inputs, and x and ϕ as the outputs.

■ Solution

The free body diagrams are shown in Figure 2.4.7b. First consider the pendulum. The vertical and horizontal components of the pendulum's mass center are $L \cos \phi$ and $x - L \sin \phi$, respectively. For the horizontal direction Newton's law gives:

$$m \frac{d^2}{dt^2}(x - L \sin \phi) = H \tag{2.4.7}$$

where H is the horizontal component of the reaction force at the pivot. The moment equation (2.4.6) about the pendulum's pivot point P gives:

$$(I_G + mL^2)\ddot{\phi} - mL\ddot{x} \cos \phi = T + mgL \sin \phi \tag{2.4.8}$$

where $I_G + mL^2$ is the pendulum's moment of inertia about the pivot point.

Now consider the base. Newton's law in the horizontal direction gives:

$$M\ddot{x} = -f - H \tag{2.4.9}$$

Because we are assuming that the base does not rotate or move vertically, we need not consider the moments and vertical forces on the base, unless we need to compute the reaction forces V, R_1, and R_2.

Evaluate the derivative in (2.4.7) to obtain:

$$m\ddot{x} - mL \frac{d}{dt}(\cos \phi \, \dot{\phi}) = m\ddot{x} - mL(-\sin \phi \, \dot{\phi}^2 + \cos \phi \, \ddot{\phi}) = H \tag{2.4.10}$$

Figure 2.4.7 (a) model of a personal transporter. (b) free-body diagram of the base and the inverted pendulum.
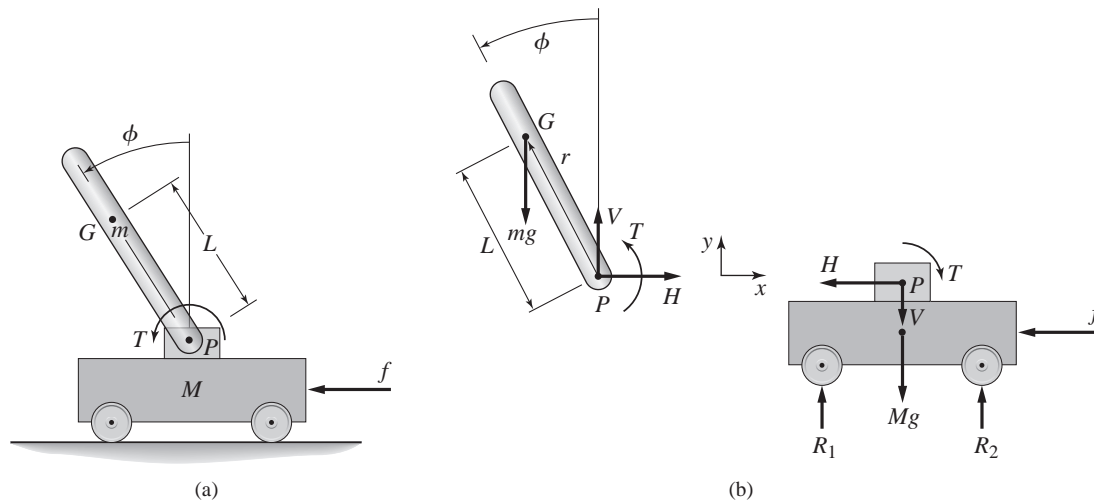
Solve (2.4.9) for the reaction force H: $H = -f - M\ddot{x}$. Substitute this into (2.4.10) and collect terms to obtain:

$$(m + M)\ddot{x} - mL(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2) = -f \qquad (2.4.11)$$

The equations of motion are (2.4.8) and (2.4.11).

2.5 CHAPTER REVIEW

In this chapter we reviewed Newton's laws of motion and applied them first to situations where the object's motion is relatively simple—simple translations or simple rotations—and then to the case of translation and rotation in a plane. We assumed that the masses are rigid and we restricted our analysis to forces that are constant or functions of time. We introduced the concepts of equivalent mass and equivalent inertia. These concepts simplify the modeling of systems having both translating and rotating components whose motions are directly coupled.

Now that you have finished this chapter, you should be able to

1. Obtain the equations of motion for an object consisting of a single mass undergoing simple translation or rotation, and solve them when the applied forces or moments are either constants or simple functions of time.
2. Apply the principle of conservation of mechanical energy to analyze systems acted on by conservative forces.
3. Apply the concepts of equivalent mass and equivalent inertia to obtain a simpler model of a multimass system whose motions are directly coupled.
4. Obtain the equation of motion of a body in planar motion, involving simultaneous translation and rotation.

REFERENCES

[Cannon, 1967] R. H. Cannon, Jr., *Dynamics of Physical Systems*, McGraw-Hill, New York, 1967.

[Meriam, 2002] J. L. Meriam and L. G. Kraige, *Engineering Mechanics: Dynamics*, Fifth edition, John Wiley & Sons, New York, 2002.

PROBLEMS

Section 2.1 Translational Motion

2.1   Consider the falling mass in Example 2.1.1 and Figure 2.1.2. Find its speed and height as functions of time. How long will it take to reach (a) the platform and (b) the ground?

2.2   A baseball is thrown horizontally from the pitcher's mound with a speed of 90 mph. Neglect air resistance, and determine how far the ball will drop by the time it crosses home plate 60 ft away.

2.3   For the mass shown in Figure 2.1.3b, $m = 10$ kg, $\phi = 25°$, $v(0) = 2$ m/s, and $\mu = 0.3$. Determine whether the mass comes to rest if (a) $f_1 = 100$ N and (b) $f_1 = 50$ N. If the mass comes to rest, compute the time at which it stops.

2.4   A particle of mass m slides down a frictionless ramp starting from rest (see Figure P2.4). The lengths L and H and the angle θ are given. Derive an expression for the distance D of the impact point.
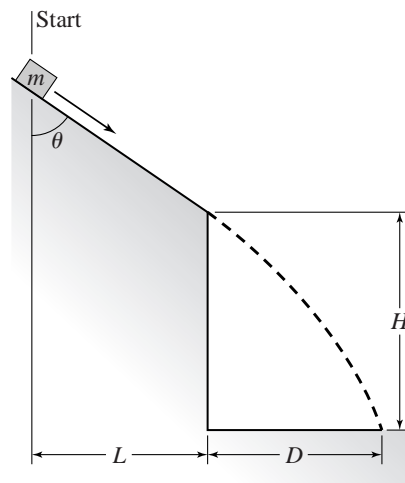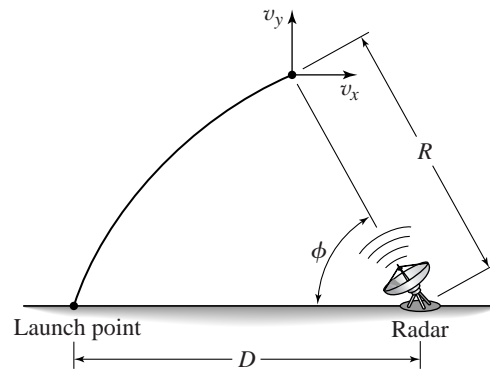
Figure P2.4

Figure P2.5

2.5 A radar tracks the flight of a projectile (see Figure P2.5). At time t, the radar measures the horizontal component $v_x(t)$ and the vertical component $v_y(t)$ of the projectile's velocity and its range $R(t)$ and elevation $\phi(t)$. Are these measurements sufficient to compute the horizontal distance D from the radar to the launch point of the projectile? If so, derive the expression for D as a function of g and the measured values $v_x(t)$, $v_y(t)$, $R(t)$, and $\phi(t)$.

Section 2.2 Rotation About a Fixed Axis

2.6 Table 2.2.1 gives the inertia I_O for a point mass in rotation. Suppose instead that a sphere of radius r is in rotation about a fixed point O a distance R from its center of mass. Find the inertia I_O of the sphere about the point O.

2.7 The pendulum shown in Figure P2.7 consists of a slender rod weighing 3 lb and a block weighing 10 lb.
   a. Determine the location of the center of mass.
   b. Derive the equation of motion in terms of θ.

2.8 The scale shown in Figure P2.8 measures the weight mg of an object placed on the scale, by using a counterweight of mass m_c. Friction in the pivot point at A

Figure P2.7
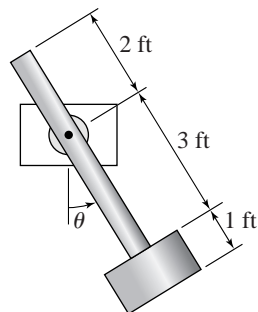
Figure P2.8
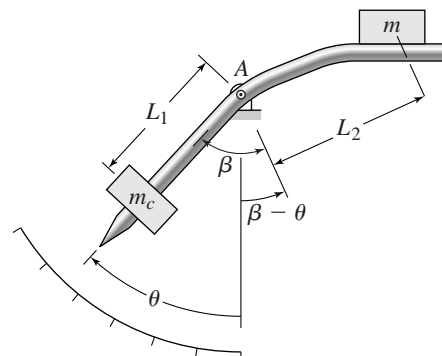
causes the pointer to eventually come to rest at an angle θ, which indicates the measured value of the weight mg. The angle β has a fixed value that depends on the shape of the scale arm. (a) Neglect the friction in the pivot and neglect the mass of the scale arm, and obtain the scale's equation of motion. (b) Find the equilibrium relation between the weight mg and the angle θ. (c) Find the weight mg if $m_c = 5$ kg, $L_1 = 0.2$ m, $L_2 = 0.15$ m, $\beta = 30°$, and $\theta = 20°$.

2.9   The motor in Figure P2.9 lifts the mass m_L by winding up the cable with a force F_A. The center of pulley B is fixed. The pulley inertias are I_B and I_C. Pulley C has a mass m_C. Derive the equation of motion in terms of the speed v_A of point A on the cable with the force F_A as the input.
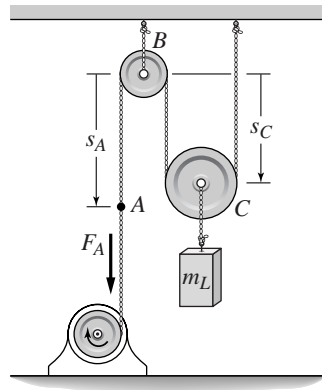
Figure P2.9   **Figure P2.10**
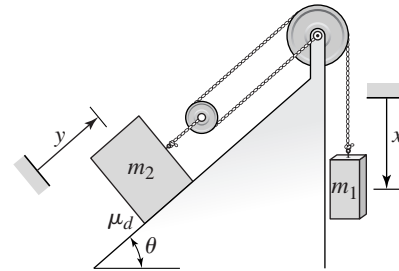
2.10   Instead of using the system shown in Figure 2.2.5a to raise the mass m_2, an engineer proposes to use two simple machines, the pulley and the inclined plane, to reduce the weight required to lift m_2. The proposed design is shown in Figure P2.10. The pulley inertias are negligible. The available horizontal space limits the angle of the inclined plane to no less than $30°$.

   a.   Suppose that the friction between the plane and the mass m_2 is negligible. Determine the smallest value m_1 can have to lift m_2. Your answer should be a function of m_2 and θ.

   b.   In practice, the coefficient of dynamic friction μ_d between the plane and the mass m_2 is not known precisely. Assume that the system can be started to overcome static friction. For the value of $m_1 = m_2/2$, how large can μ_d be before m_1 cannot lift m_2?

Section 2.3 Equivalent Mass and Inertia

2.11   Derive the expression for the equivalent inertia I_e felt on the input shaft, for the spur gears treated in Example 2.3.2, where the shaft inertias are I_{s_1} and I_{s_2}.

2.12   Draw the free body diagrams of the two spur gears shown in Figure 2.3.2. Use the resulting equations of motion to show that $T_2 = NT_1$ if the gear inertias are negligible or if there is zero acceleration.

2.13   The geared system shown in Figure P2.13 represents an elevator system. The motor has an inertia I_1 and supplies a torque T_1. Neglect the inertias of the gears, and assume that the cable does not slip on the pulley. Derive an
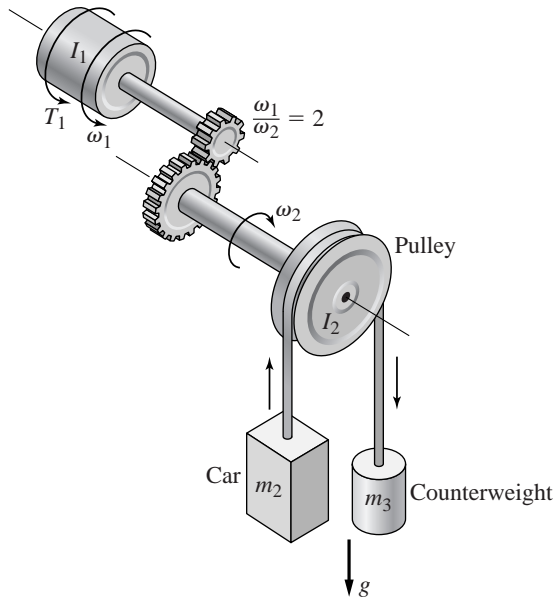
expression for the equivalent inertia I_e felt on the input shaft (shaft 1). Then derive the dynamic model of the system in terms of the speed ω_1 and the applied torque T_1. The pulley radius is R.

2.14 Derive the expression for the equivalent inertia I_e felt on the input shaft, for the rack and pinion treated in Example 2.3.3, where the shaft inertia is I_s.

2.15 Derive the expression for the equivalent inertia I_e felt on the input shaft, for the belt drive treated in Example 2.3.4, where the shaft inertias connected to the sprockets are I_{s_1} and I_{s_2}.

2.16 For the geared system shown in Figure P2.16, proper selection of the gear ratio N can maximize the load acceleration $\dot{\omega}_2$ for a given motor and load. Note that the gear ratio is defined such that $\omega_1 = N\omega_2$. Assuming that the inertias I_1 and I_2 and the torques T_1 and T_2 are given,

    a.   Derive the expression for the load acceleration $\dot{\omega}_2$.

    b.   Use calculus to determine the value of N that maximizes $\dot{\omega}_2$.

2.17 For the geared system shown in Figure P2.17, assume that shaft inertias and the gear inertias I_1, I_2, and I_3 are negligible. The motor and load inertias in $kg \cdot m^2$ are

$$I_4 = 0.03 \qquad I_5 = 0.15$$

The speed ratios are

$$\frac{\omega_1}{\omega_2} = \frac{\omega_2}{\omega_3} = 1.6$$

Derive the system model in terms of the speed ω_3, with the applied torque T as the input.

Figure P2.16
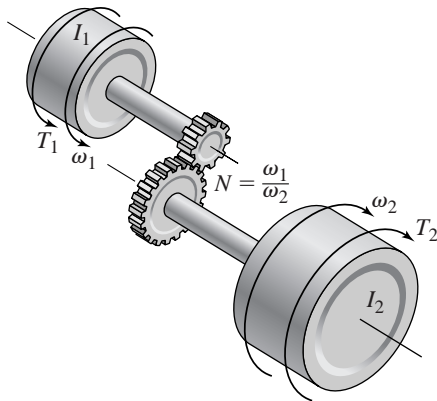
Figure P2.17
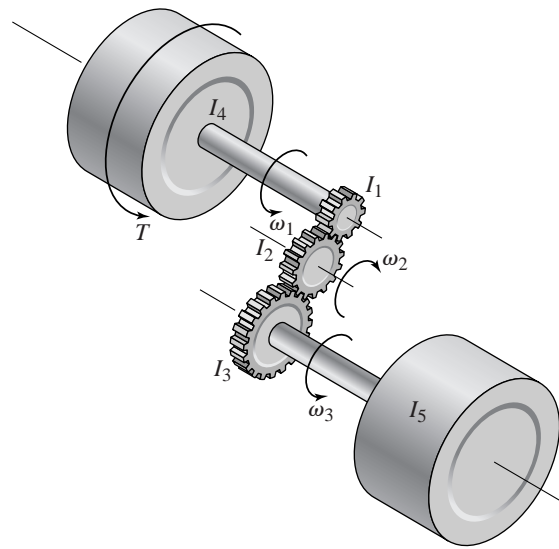
2.18 For the geared system discussed in Problem 2.17, shown in Figure P2.17, the inertias are given in $kg \cdot m^2$ as

$$I_1 = 10^{-3} \qquad I_2 = 3.84 \times 10^{-3} \qquad I_3 = 0.0148$$
$$I_4 = 0.03 \qquad I_5 = 0.15$$

The speed ratios are

$$\frac{\omega_1}{\omega_2} = \frac{\omega_2}{\omega_3} = 1.6$$

Derive the system model in terms of the speed ω_3, with the applied torque T as the input. The shaft inertias are negligible.

2.19 The geared system shown in Figure P2.19 is similar to that used in some vehicle transmissions. The speed ratios (which are the ratios of the gear radii) are

$$\frac{\omega_2}{\omega_1} = 3 \qquad \frac{\omega_3}{\omega_2} = \frac{3}{5} \qquad \frac{\omega_4}{\omega_3} = \frac{13}{11}$$

a. Determine the overall speed ratio ω_4/ω_1.
b. Derive the equation of motion in terms of the velocity ω_4, with the torque T_1 as the input. Neglect the inertias of the gears and the shafts.

2.20 The *lead screw* (also called a *power screw* or a *jack screw*) is used to convert the rotation of a motor shaft into a translational motion of the mass m (see Figure P2.20). For one revolution of the screw, the mass translates a distance L (called the *screw lead*). As felt on the motor shaft, the translating mass appears as an equivalent inertia. Use kinetic energy equivalence to derive an expression for the equivalent inertia. Let I_s be the inertia of the screw.

Figure P2.19
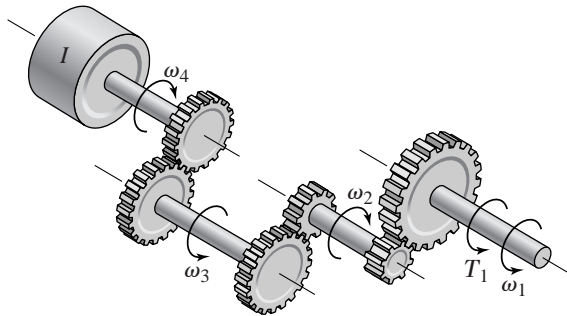
Figure P2.20
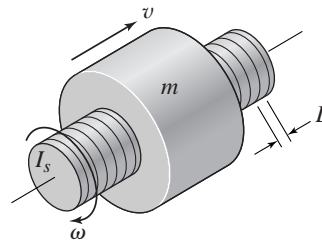
2.21 At time $t = 0$, the operator of the road roller disengages the transmission so that the vehicle rolls down the incline (see Figure P2.21). Determine an expression for the vehicle's speed as a function of time. The two rear wheels weigh 500 lb each and have a radius of 4 ft. The front wheel weighs 800 lb and has a radius of 2 ft. The vehicle body weighs 9000 lb. Assume the wheels roll without slipping.
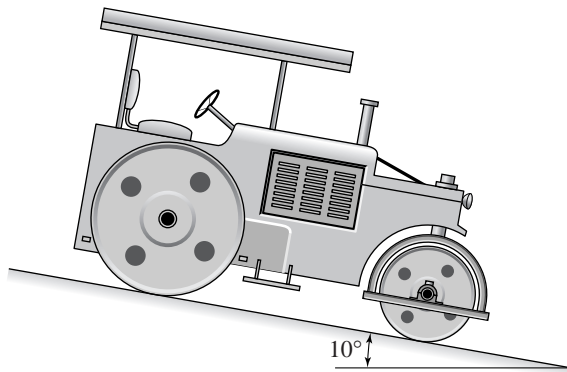
Figure P2.21

2.22 Derive the equation of motion of the block of mass m_1 in terms of its displacement x (see Figure P2.22). The friction between the block and the surface is negligible. The pulley has negligible inertia and negligible friction. The cylinder has a mass m_2 and rolls without slipping.
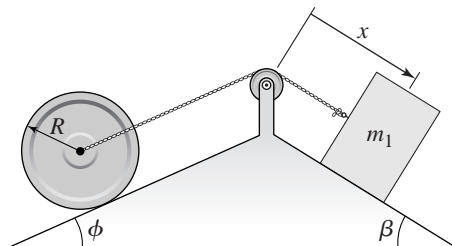
Figure P2.22

2.23 Assume the cylinder in Figure P2.23 rolls without slipping. Neglect the mass of the pulleys and derive the equation of motion of the system in terms of the displacement x.

Figure P2.23
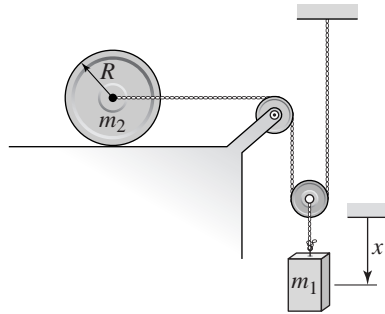
Figure P2.24
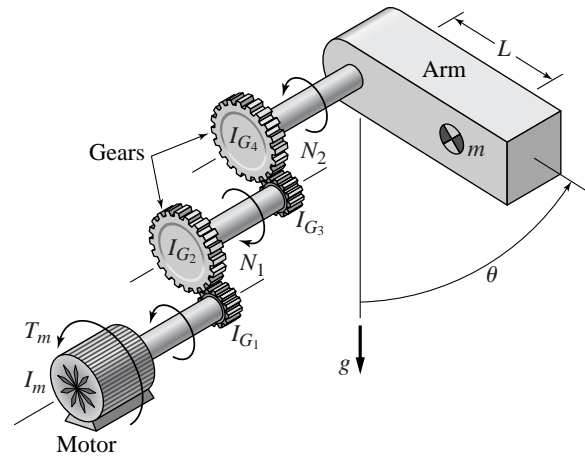
2.24 A single link of a robot arm is shown in Figure P2.24. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through two pairs of spur gears. The gear ratios are $N_1 = 2$ (the motor shaft has the greater speed) and $N_2 = 1.5$ (the shaft connected to the link has the slower speed). Obtain the equation of motion in terms of the angle θ, with T_m as the input. Neglect the shaft inertias relative to the other inertias. The given values for the motor and gear inertias are

$$I_m = 0.05 \text{ kg} \cdot \text{m}^2 \qquad I_{G_1} = 0.025 \text{ kg} \cdot \text{m}^2 \qquad I_{G_2} = 0.1 \text{ kg} \cdot \text{m}^2$$

$$I_{G_3} = 0.025 \text{ kg} \cdot \text{m}^2 \qquad I_{G_4} = 0.08 \text{ kg} \cdot \text{m}^2$$

The values for the link are

$$m = 10 \text{ kg} \qquad L = 0.3 \text{ m}$$

2.25 A conveyor drive system to produce translation of the load is shown in Figure P2.25a. The reducer is a gear pair that reduces the motor speed by a factor of 10:1. The motor inertia is $I_1 = 0.002 \text{ kg} \cdot \text{m}^2$. The reducer inertia as felt on the motor shaft is $I_2 = 0.003 \text{ kg} \cdot \text{m}^2$. Neglect the inertia of the tachometer, which is used to measure the speed for control purposes. The properties of the remaining elements are given here.
Sprockets:
  Sprocket 1: radius $= 0.05$ m   weight $= 9$ N
  Sprocket 2: radius $= 0.15$ m   weight $= 89$ N
Chain weight: 107 N
Drive shaft: radius $= 0.04$ m   weight $= 22$ N
Drive wheels (four of them): radius $= 0.2$ m   weight $= 89$ N each
Drive chains (two of them): weight $= 670$ N each
Load friction torque measured at the drive shaft: 54 N $\cdot$ m
Load weight: 450 N

a.   Derive the equation of motion of the conveyor in terms of the motor velocity ω_1, with the motor torque T_1 as the input.
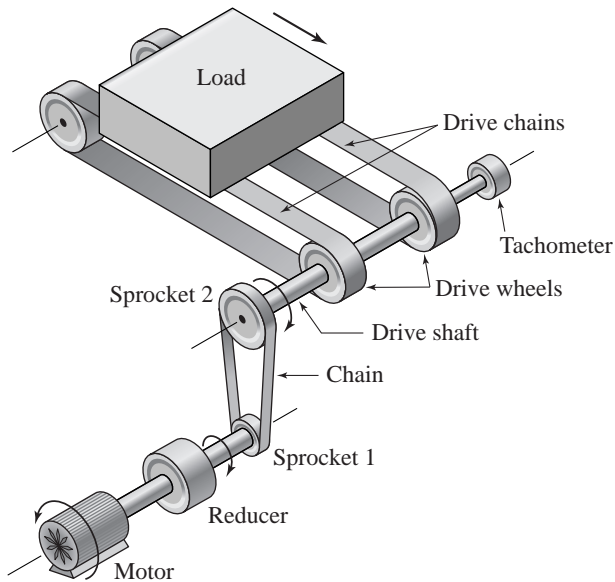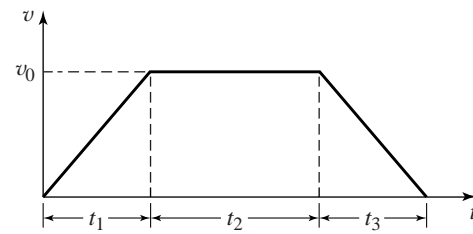
Figure P2.25a

Figure P2.25b

b. Suppose the motor torque is constant at $10\ \text{N}\cdot\text{m}$. Determine the resulting motor velocity ω_1 and load velocity v as functions of time, assuming the system starts from rest.

c. The profile of a desired velocity for the load is shown in Figure P2.25b, where $v_0 = 1$ m/s, $t_1 = t_3 = 0.5$ s, and $t_2 = 2$ s. Use the equation of motion found in part (a) to compute the required motor torque for each part of the profile.

Section 2.4 General Planar Motion

2.26 A person pushes a roller of radius R and inertia $mR^2/2$, with a force f applied at an angle of ϕ to the horizontal (see Figure P2.26). The roller weighs 800 N and has a diameter of 0.4 m. Assume the roller does not slip. Derive the equation of motion in terms of (a) the rotational velocity ω of the roller and (b) the displacement x.
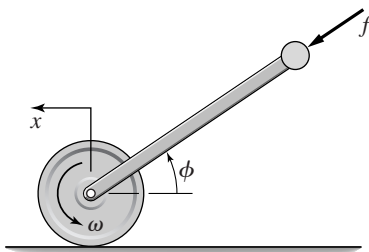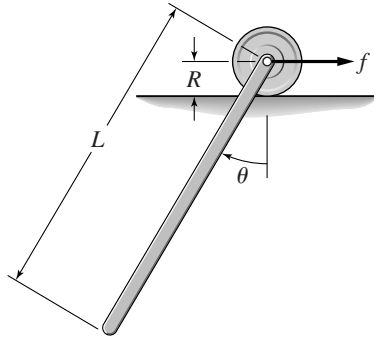
Figure P2.26

2.27   A slender rod 1.4 m long and of mass 20 kg is attached to a wheel of radius 0.05 m and negligible mass, as shown in Figure P2.27. A horizontal force f is applied to the wheel axle. Derive the equation of motion in terms of θ. Assume the wheel does not slip.

2.28   A slender rod 1.4 m long and of mass 20 kg is attached to a wheel of mass 3 kg and radius 0.05 m, as shown in Figure P2.27. A horizontal force f is applied to the wheel axle. Derive the equation of motion in terms of θ. Assume the wheel does not slip.

2.29   Consider the rolling cylinder treated in Example 2.4.2. Assume that the no-slip condition is not satisfied, so that the cylinder slips while it rolls. Derive expression for the translational acceleration a_{G_x} and the angular acceleration α. The coefficient of dynamic friction is μ_d.

2.30   A hoop of mass m and radius r starts from rest and rolls down an incline at an angle θ. The hoop's inertia is given by $I_G = mr^2$. The static friction coefficient is μ_s. Determine the acceleration of the center of mass a_{G_x} and the angular acceleration α. Assume that the hoop rolls without bouncing or slipping. Use two approaches to solve the problem: (a) Use the moment equation about the mass center G and (b) use the moment equation about the contact point P. (c) Obtain the frictional condition required for the hoop to roll without slipping.

2.31   It is required to determine the maximum acceleration of the rear-wheel-drive vehicle shown in Figure P2.31. The vehicle mass is 1700 kg, and its dimensions are $L_A = 1.2$ m, $L_B = 1.1$ m, and $H = 0.5$ m. Assume that each front wheel experiences the same reaction force $N_A/2$. Similarly, each rear
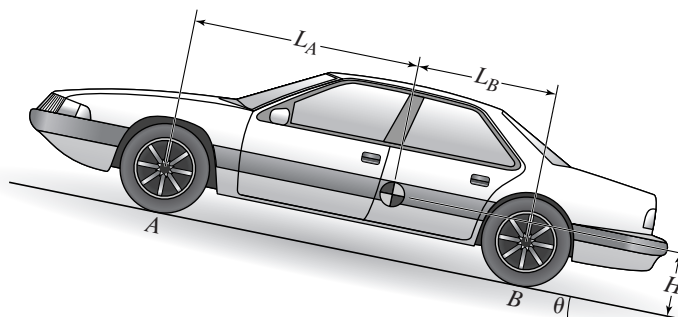
wheel experiences the same reaction force $N_B/2$. Thus the traction force $\mu_s N_B$ is the total traction force due to both driving wheels. Assume that the mass of the wheels is small compared with the total vehicle mass; this assumption enables us to treat the vehicle as a translating rigid body with no rotating parts.

Figure P2.32

2.32 Figure P2.32 illustrates a pendulum with a base that moves horizontally. This is a simple model of an overhead crane carrying a suspended load with cables. The load mass is m, the cable length is L, and the base acceleration is $a(t)$. Assuming that the cable acts like a rigid rod, derive the equation of motion in terms of θ with $a(t)$ as the input.

2.33 Figure P2.33 illustrates a pendulum with a base that moves vertically. The base acceleration is $a(t)$. Derive the equation of motion in terms of θ with $a(t)$ as the input. Neglect the mass of the rod.
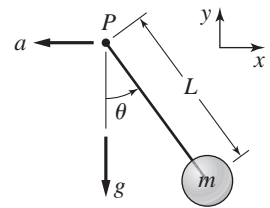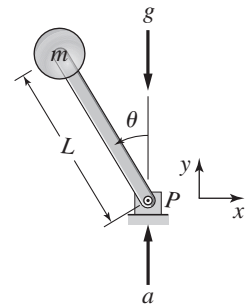
Figure P2.33

CHAPTER 3

Solution Methods for Dynamic Models

CHAPTER OBJECTIVES

When you have finished this chapter you should be able to do the following:

1. Apply the trial-solution method or the Laplace transform method, whichever is appropriate, to obtain the solution of a linear differential equation model.

2. When applying the Laplace transform method, be able to perform the appropriate expansion and apply the appropriate transform properties to obtain the inverse transform.

3. Identify the free, forced, transient, and steady-state components of the complete response.

4. Obtain transfer functions from models expressed as single equations or as sets of equations.

5. Evaluate the effects of impulse inputs and input derivatives on the response.

6. Use MATLAB to apply the chapter's methods.

D ynamic models are differential equations that describe a dynamic system. In Chapter 2, we encountered differential equations of the form $\dot{x} = f(t)$ and $\ddot{x} = c$, which were easily solved by integrating the right-hand side with respect to time. In this chapter, we develop methods for obtaining analytical solutions to other differential equations commonly found in engineering applications. In Section 3.1, we introduce some important concepts and terminology associated with differential equations, and present methods for quickly solving simple differential equations. Section 3.2 introduces the concepts of free, forced, transient, and steady-state response, and explains the important concept of stability.

An important tool for obtaining solutions in general is the Laplace transform, introduced in Section 3.3. It also forms the basis for the important concept of the transfer function, covered in Section 3.4. The partial-fraction expansion, used with the Laplace transform, is covered in Section 3.5. The effects of impulse inputs and input derivatives on a system's dynamic response are treated in Section 3.6. Section 3.7 presents additional, in-depth examples.

Sections 3.8 and 3.9 show how to use MATLAB to obtain partial-fraction expansions and to obtain responses to step, impulse, and other input function types. A review of the chapter's main concepts is given in Section 3.10. ■

3.1 DIFFERENTIAL EQUATIONS

An *ordinary differential equation (ODE)* is an equation containing ordinary, but not partial, derivatives of the dependent variable. Because the subject of system dynamics is time-dependent behavior, the independent variable in our ODEs will be time t.

In a standard form for expressing an ODE, all functions of the dependent variable are placed on the left-hand side of the equal ($=$) sign, and all isolated constants and isolated functions of time are placed on the right-hand side. The quantities in the right-hand side are called the *input*, or *forcing function*. The dependent variable is called the solution or the *response*. For example, consider the equation $3\ddot{x} + 7\dot{x} + 2t^2 x = 5 + \sin t$, where x is the dependent variable. The input is $5 + \sin t$ and the response is $x(t)$. If the right-hand side is zero, the equation is said to be *homogeneous;* otherwise, it is *nonhomogeneous*.

INITIAL CONDITIONS

An example of an ODE is $2\dot{x} + 6x = 3$, where x is the dependent variable. "Solving the equation" means to obtain the function $x(t)$ that satisfies the equation. For this example, the function is $x(t) = Ce^{-3t} + 0.5$, where C is a constant. We cannot determine a numerical value for C unless we are given a specified value for x at some time t. Most commonly x is specified at some starting time, usually denoted t_0. The specified value of x at t_0 is denoted x_0 and is called the *initial condition*. Often the starting time t_0 is taken to be at $t = 0$.

When solving differential equations, you need never wonder if your answer is correct, because you can always check your answer by substituting it into the differential equation and by evaluating the solution at $t = t_0$.

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

We can categorize differential equations as *linear* or *nonlinear*. Linear differential equations are recognized by the fact that they contain only linear functions of the dependent variable and its derivatives. Nonlinear functions of the *independent* variable do *not* make a differential equation nonlinear. For example, the following equations are linear.

$$\dot{x} + 3x = 5 + t^2 \qquad \dot{x} + 3t^2 x = 5 \qquad 3\ddot{x} + 7\dot{x} + 2t^2 x = \sin t$$

whereas the following equations are nonlinear:

$$2\ddot{x} + 7\dot{x} + 6x^2 = 5 + t^2, \text{ because of } x^2$$

$$3\ddot{x} + 5\dot{x}^2 + 8x = 4, \text{ because of } \dot{x}^2$$

$$\ddot{x} + 4x\dot{x} + 3x = 10, \text{ because of } x\dot{x}$$

The equation $\dot{x} + 3t^2x = 5$ is a *variable-coefficient* differential equation, so named because one of its coefficients is a function of the independent variable t. By contrast, the equation $\dot{x} + 2x = 5$ is a *constant-coefficient* differential equation. When solving constant-coefficient equations, the initial time t_0 can always be chosen to be 0. This simplifies the solution form.

The *order* of the equation is the order of the highest derivative of the dependent variable in the equation. The equation $3\ddot{x} + 7\dot{x} + 2x = 5$ is thus called a *second-order* differential equation.

A model can consist of more than one equation. For example, the model

$$3\dot{x}_1 + 5x_1 - 7x_2 = 5$$

$$\dot{x}_2 + 4x_1 + 6x_2 = 0$$

consists of two equations that must be solved *simultaneously* to obtain the solution for the two dependent variables $x_1(t)$ and $x_2(t)$. The equations are said to be *coupled*. Although each equation is first order, the set can be converted into a single differential equation of second order. Thus, solving a set of two coupled first-order equations is equivalent to solving a single second-order equation. In general, a coupled set of differential equations can be reduced to a single differential equation whose order is the sum of the orders of the individual equations in the set.

SEPARATION OF VARIABLES

You can solve the equation

$$\dot{x} = g(t)f(x) \tag{3.1.1}$$

by separating the variables x and t as follows. First write the equation as

$$\frac{dx}{f(x)} = g(t)\,dt$$

Then integrate both sides to obtain

$$\int_{x(0)}^{x(t)} \frac{dx}{f(x)}\,dx = \int_0^t g(t)\,dt$$

The solution $x(t)$ can be found if the integrals on the left and on the right can be evaluated and if the resulting expression can be solved for x as a function of t.
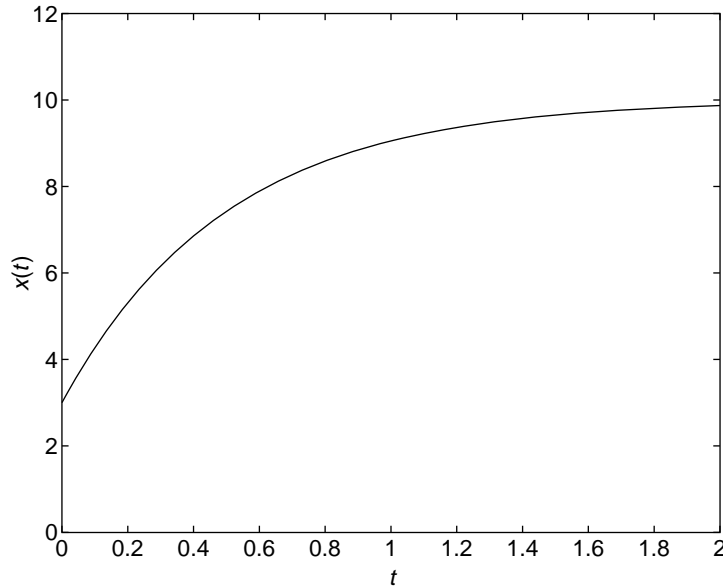

EXAMPLE 3.1.1

Separation of Variables for a Linear Equation

■ **Problem**

Use separation of variables to solve the following problem for $t \geq 0$.

$$\dot{x} + 2x = 20 \qquad x(0) = 3 \tag{1}$$

Figure 3.1.1 Response for Example 3.1.1.

■ **Solution**

Comparing this with (3.1.1) we see that $f(x) = 20 - 2x$ and $g(t) = 1$. First write the equation as

$$\frac{dx}{dt} = 20 - 2x \qquad \text{or} \qquad \frac{dx}{20 - 2x} = dt$$

Integrate both sides to obtain

$$\int_{3}^{x(t)} \frac{dx}{20 - 2x} \, dx = \int_{0}^{t} dt = t$$

The integral on the left can be evaluated as follows:

$$\ln[20 - 2x(t)] - \ln[20 - 2(3)] = -2t$$

Solve for $x(t)$ to obtain

$$x(t) = 10 - 7e^{-2t} \tag{2}$$

The plot of this function is shown in Figure 3.1.1.

TRIAL-SOLUTION METHOD

We can use the results of Example 3.1.1 to gain insight into the solution of the equation $\dot{x} + ax = b$, where $a \neq 0$. Its solution has the form

$$x(t) = C + De^{st} \tag{3.1.2}$$

where C, D, and s are constants to be determined. We can verify that this form is the solution by substituting $x(t)$ into the differential equation, as follows:

$$\dot{x} + ax = sDe^{st} + a\left(C + De^{st}\right) = (s + a)De^{st} + aC = b$$

The only way this equation can be true is if $s + a = 0$ and $aC = b$. Thus $s = -a$ and $C = b/a$. The remaining constant, D, can determined from the initial value $x(0)$ as follows. Substituting $t = 0$ into the solution form gives $x(0) = C + De^0 = C + D$. Thus $D = x(0) - C = x(0) - b/a$, and the solution can be written as

$$x(t) = \frac{b}{a} + \left[x(0) - \frac{b}{a}\right]e^{-at} \tag{3.1.3}$$

The exponential coefficient s is called the *characteristic* root, and its equation $s + a = 0$ is called the *characteristic equation*. We will soon see that characteristic roots are of great use in determining the form of the trial solution.

This insight leads to the *trial-solution* method for solving equations. It can be used to solve higher-order equations as well. The method is useful for quickly obtaining solutions of common ODEs whose solution forms are already known from experience.

EXAMPLE 3.1.2

Two Distinct, Real Roots

■ **Problem**

Use the trial-solution method to solve the following problem.

$$\ddot{x} + 7\dot{x} + 10x = 20 \qquad x(0) = 5 \qquad \dot{x}(0) = 3$$

■ **Solution**

Substitute the trial-solution form

$$x(t) = C + De^{st} \tag{1}$$

into the ODE and collect terms to obtain

$$\left(s^2 + 7s + 10\right)De^{st} + 10C = 20$$

The only way this equation can be true is if $C = 20/10 = 2$ and $s^2 + 7s + 10 = 0$, which is the characteristic equation. The roots of this equation can be found with the quadratic formula given in Table 3.1.1. So we now have two characteristic roots, $s = -2$ and $s = -5$. The form given by equation (1) with $C = 2$ will solve the ODE with either value of s, but it cannot satisfy *both* initial conditions. Therefore, we need an additional term with an arbitrary constant. It has been found that the appropriate trial-solution form is

$$x(t) = C + D_1 e^{s_1 t} + D_2 e^{s_2 t} \tag{2}$$

where s_1 and s_2 are the two characteristic roots. Substituting this form into the ODE and collecting terms gives

$$\left(s_1^2 + 7s_1 + 10\right)D_1 e^{s_1 t} + \left(s_2^2 + 7s_2 + 10\right)D_2 e^{s_2 t} + 10C = 20 \tag{3}$$

Because both s_1 and s_2 satisfy the characteristic equation $s^2 + 7s + 10 = 0$, equation (3) is satisfied if $C = 2$. Thus the solution is

$$x(t) = 2 + D_1 e^{-2t} + D_2 e^{-5t} \tag{4}$$

Table 3.1.1 Roots and complex numbers.

The quadratic formula

The roots of $as^2 + bs + c = 0$ are given by

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For complex roots, $s = \sigma \pm j\omega$, the quadratic can be expressed as

$$as^2 + bs + c = a\left[(s + \sigma)^2 + \omega^2\right] = 0$$

Complex numbers

Rectangular representation:

$$z = x + jy, \; j = \sqrt{-1}$$

Magnitude and angle:

$$|z| = \sqrt{x^2 + y^2} \qquad \theta = \angle z = \tan^{-1}\frac{y}{x}$$

Polar and exponential representation:

$$z = |z|\angle\theta = |z|e^{j\theta}$$

Equality: If $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$z_1 = z_2 \text{ if } x_1 = x_2 \text{ and } y_1 = y_2$$

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication:

$$z_1 z_2 = |z_1||z_2|\angle(\theta_1 + \theta_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

Complex-Conjugate Multiplication:

$$(x + jy)(x - jy) = x^2 + y^2$$

Division:

$$\frac{1}{z} = \frac{1}{x + yj} = \frac{x - jy}{x^2 + y^2}$$
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$
$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 + jy_1}{x_2 + jy_2}\frac{x_2 - jy_2}{x_2 - jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$

The two constants, D_1 and D_2, are determined from the initial conditions as follows:

$$x(0) = 2 + D_1 + D_2 = 5$$
$$\dot{x}(0) = \left(-2D_1 e^{-2t} - 5D_2 e^{-5t}\right)\big|_{t=0} = -2D_1 - 5D_2 = 3$$

The solution of these two equations is $D_1 = 6$ and $D_2 = -3$. Thus, the solution is

$$x(t) = 2 + 6e^{-2t} - 3e^{-5t}$$

The plot of this function is shown in Figure 3.1.2. The "hump" is caused by the positive value of $\dot{x}(0)$. Note that $e^{-5t} < 0.02$ for $t > 4/5$. Thus the response is approximately given by $x(t) = 2 + 6e^{-2t}$ for $t > 4/5$. Because $e^{-2t} < 0.02$ for $t > 2$, the response is essentially constant for $t > 2$.

Figure 3.1.2 Response for
Example 3.1.2.
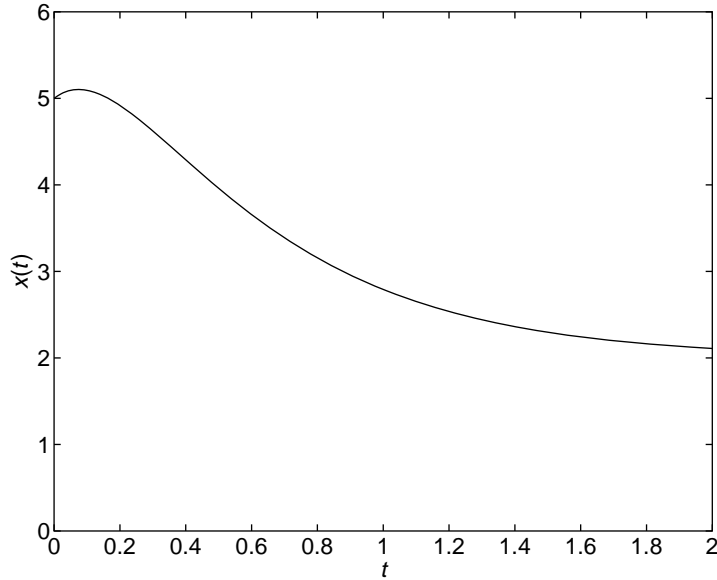
Calculator Tip

Now is a good time to learn how to use your calculator to solve for polynomial roots. Determine
whether and how your calculator can find complex roots and determine how it expresses com-
plex numbers; for example, on some calculators the number $3 + 7j$ is expressed as (3, 7). Try
calculating the roots of the following equations: $s^2 + 6s + 13 = 0$ (Answer: $s = -3 \pm 2j$) and
$s^3 + 12s^2 + 45s + 50 = 0$ (Answer: $s = -2$, $s = -5$, and $s = -5$).

The trial solution $x(t) = C + D_1 e^{s_1 t} + D_2 e^{s_2 t}$ always gives the correct solution of
the equation $\ddot{x} + a\dot{x} + bx = c$ if $b \neq 0$ and if the two characteristic roots are distinct.
Example 3.1.3 shows how repeated, real roots are handled.

EXAMPLE 3.1.3   ## Two Repeated, Real Roots

■ Problem
Use the trial-solution method to solve the following problem.

$$5\ddot{x} + 20\dot{x} + 20x = 28 \qquad x(0) = 5 \qquad \dot{x}(0) = 8$$

■ Solution
If we substitute the trial-solution form

$$x(t) = C + De^{st} \tag{1}$$

into the ODE and collect terms, we obtain

$$\left(5s^2 + 20s + 20\right)De^{st} + 20C = 28$$

The only way this equation can be true is if $C = 28/20 = 1.4$ and $5s^2 + 20s + 20 = 0$. This
gives two identical roots, $s = -2$ and $s = -2$. The form given by equation (1) with $C = 1.4$ will

solve the ODE with $s = -2$, but it cannot satisfy *both* initial conditions. Therefore, we need an additional term with an arbitrary constant. From experience it has been found that when the roots are identical (or *repeated*), the appropriate trial-solution form is

$$x(t) = C + (D_1 + D_2 t)e^{s_1 t} \tag{2}$$

Substituting this form into the ODE and collecting terms gives

$$\left(5s_1^2 + 20s_1 + 20\right) D_1 e^{s_1 t} + \left(5s_1^2 + 20s_1 + 20\right) D_2 t e^{s_1 t}$$
$$+ \ (10s_1 + 20)D_2 e^{s_1 t} + 20C = 28 \tag{3}$$

Because $s_1 = -2$, $5s_1^2 + 20s_1 + 20 = 0$ and $10s_1 + 20 = 0$. Thus equation (3) is satisfied if $C = 28/20 = 1.4$. The solution is, therefore,

$$x(t) = 1.4 + (D_1 + D_2 t)e^{-2t} \tag{4}$$

The two constants, D_1 and D_2, are determined from the initial conditions as follows:

$$x(0) = 1.4 + D_1 = 5$$
$$\dot{x}(0) = \left[D_2 e^{-2t} - 2(D_1 + D_2 t)e^{-2t} \right]\Big|_{t=0} = D_2 - 2D_1 = 8$$

The solution of these two equations is $D_1 = 3.6$ and $D_2 = 15.2$. Thus the ODE solution is

$$x(t) = 1.4 + (3.6 + 15.2t)e^{-2t}$$

The plot of this function is shown in Figure 3.1.3. The "hump" is caused by the positive value of D_2 along with the factor t that multiplies e^{-2t}. For this model, a hump in the response will occur whenever $D_2 > 0$, even if $\dot{x}(0)$ is zero or negative.
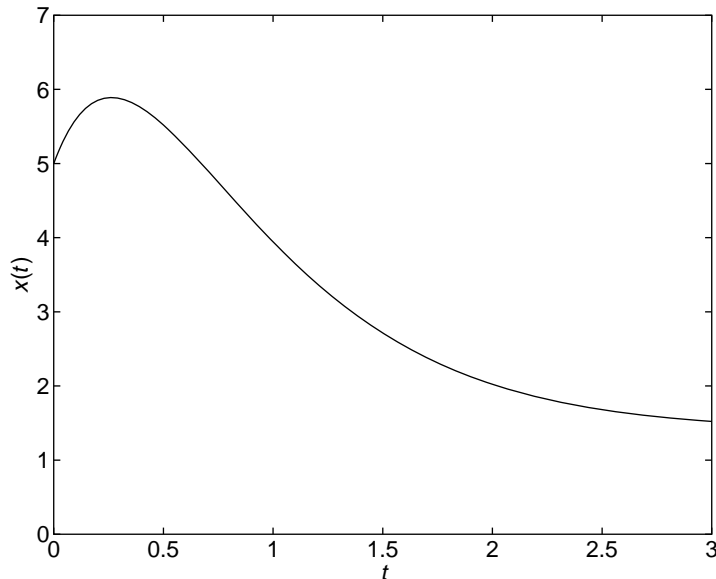
Figure 3.1.3 Response for Example 3.1.3.

An exponential trial-solution form often leads to the correct answer, even though we may want to express the final answer in a more convenient form. This situation occurs when the roots are imaginary or complex, as illustrated by Examples 3.1.4 and 3.1.5.

EXAMPLE 3.1.4

Two Imaginary Roots

■ Problem

Use the trial-solution method to solve the following problem:

$$\ddot{x} + 16x = 144 \qquad x(0) = 5 \qquad \dot{x}(0) = 12$$

■ Solution

Substituting $x(t) = C + De^{st}$ into the ODE gives

$$\left(s^2 + 16\right)De^{st} + 16C = 144$$

Thus $C = 144/16 = 9$ and $s^2 + 16 = 0$, so the roots are $s = \pm 4j$, where $j = \sqrt{-1}$. Because two initial conditions must be satisfied, from Example 3.1.2 we know that the general solution form is

$$x(t) = C + D_1 e^{4jt} + D_2 e^{-4jt} \tag{1}$$

This solution is difficult to interpret until we use *Euler's identities* (Table 3.1.2):

$$e^{4jt} = \cos 4t + j \sin 4t \qquad e^{-4jt} = \cos 4t - j \sin 4t$$

If we substitute these two identities into equation (1) and collect terms, we find that the solution has the form

$$x(t) = C + (D_1 + D_2)\cos 4t + j(D_1 - D_2)\sin 4t \tag{2}$$

Because $x(t)$ must be real, this equation can be used to show that D_1 and D_2 must be complex conjugates. Thus equation (2) reduces to

$$x(t) = C + B_1 \cos 4t + B_2 \sin 4t$$

Table 3.1.2 The exponential function.

Taylor series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

Euler's identities

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Limits

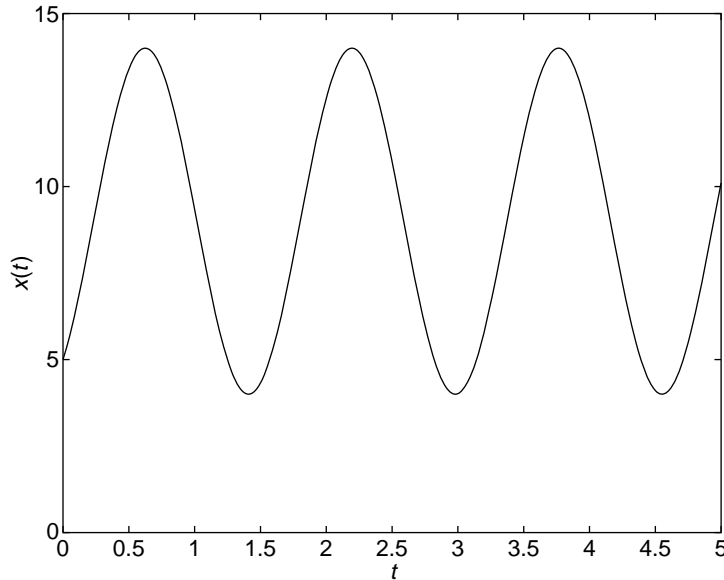$$\lim_{x\to\infty} xe^{-x} = 0 \quad \text{if } x \text{ is real.}$$
$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if the real part of } s \text{ is positive.}$$

If a is real and positive,

$$e^{-at} < 0.02 \text{ if } t > 4/a.$$
$$e^{-at} < 0.01 \text{ if } t > 5/a.$$

The *time constant* is $\tau = 1/a$.

Figure 3.1.4 Response for Example 3.1.4.

where $B_1 = D_1 + D_2$ and $B_2 = j(D_1 - D_2)$ are real constants that depend on the initial conditions. Evaluating $x(t)$ and $\dot{x}(t)$ at $t = 0$, we obtain

$$x(0) = C + B_1 = 5 \qquad \dot{x}(0) = 4B_2 = 12$$

or $B_1 = 5 - C = 5 - 9 = -4$ and $B_2 = 3$. Therefore, the solution is

$$x(t) = 9 + 3\sin 4t - 4\cos 4t$$

The plot of this function is shown in Figure 3.1.4. The response is a constant-amplitude oscillation about the value $x = 9$. The radian frequency of oscillation is 4, which gives a period of $2\pi/4 = \pi/2$.

Several examples in Chapter 2 resulted in equations of motion of the following form:

$$\ddot{\theta} + b\sin\theta = T(t)$$

If $|\theta|$ is small, then equations of this type can be solved by using the following approximation to obtain a linear model: $\sin\theta \approx \theta$. The resulting model is

$$\ddot{\theta} + b\theta = T(t)$$

and its characteristic roots are $s = \pm j\sqrt{b}$. Therefore the solution will oscillate with a radian frequency of $\sqrt{b}$.

Motion of a Robot-Arm Link | **EXAMPLE 3.1.5**

■ **Problem**

In Example 2.3.5 in Chapter 2 we derived the equation of motion of a single link in a robot arm, shown again in Figure 3.1.5. The equation is

$$(0.23 + 0.5mL^2)\ddot{\theta} = T_m - 4.9mL\sin\theta$$
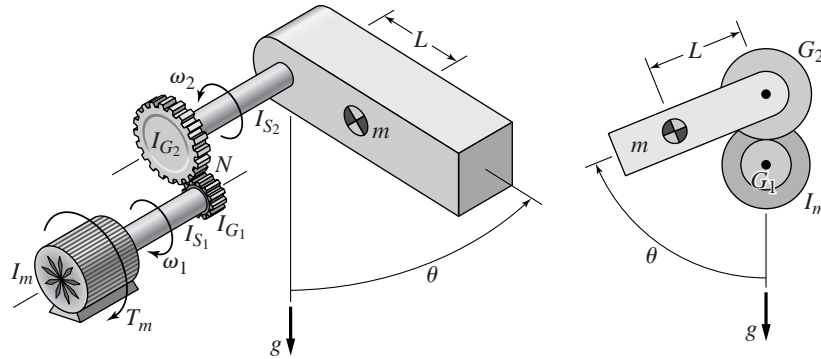
Solve this equation for the case where $T_m = 0.5$ N·m, $m = 10$ kg, and $L = 0.3$ m. Assume that
the system starts from rest at $\theta = 0$ and that the angle θ remains small.

■ **Solution**
If $|\theta|$ is small, then $\sin \theta \approx \theta$ and the equation of motion becomes

$$\left(0.23 + 0.5mL^2\right)\ddot{\theta} = T_m - 4.9mL\theta \tag{1}$$

For the given values, the equation is

$$0.68\ddot{\theta} = 0.5 - 14.7\theta$$

The characteristic equation is $0.68s^2 + 14.7 = 0$ and the roots are $s = \pm 4.65j$. Following the
method of Example 3.1.4, we obtain the following solution:

$$\theta(t) = \frac{0.5}{14.7}(1 - \cos 4.65t) = 0.034(1 - \cos 4.65t) \tag{2}$$

Thus, the model predicts that the arm oscillates with a frequency of 4.65 rad/s about $\theta = 0.034$
with an amplitude of 0.034 rad. To obtain the linear model (1), we used the approximation
$\sin \theta \approx \theta$. Therefore, before we accept the solution (2), we should check the validity of this
assumption. The maximum value that θ will reach is twice the amplitude, or $2(0.034) = 0.068$ rad.
Therefore, because $\sin 0.068 = 0.068$ to three decimal places, our assumption that $\sin \theta \approx \theta$ is
justified.

Example 3.1.4 shows that when the roots are imaginary, the trial solution should
contain a sine and a cosine function. When the roots are complex, the sine and cosine
should be multiplied by an exponential. This is illustrated in Example 3.1.6.

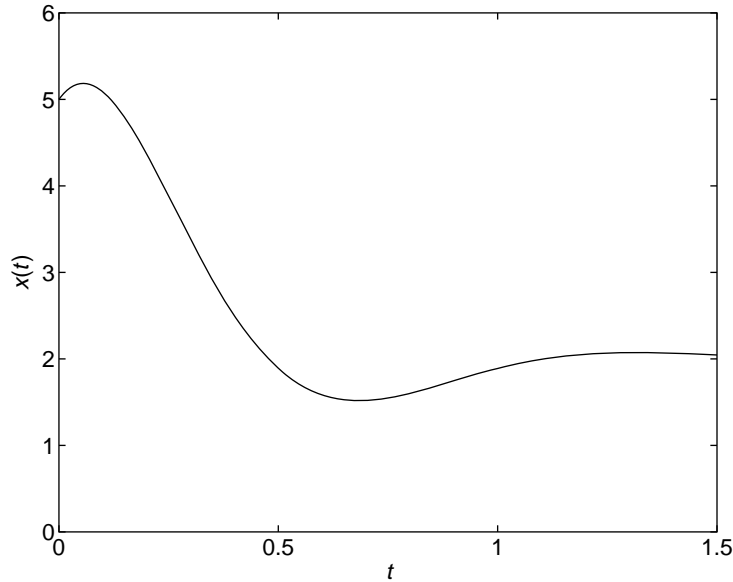
EXAMPLE 3.1.6 | Two Complex Roots

■ **Problem**
Use the trial-solution method to solve the following problem:

$$\ddot{x} + 6\dot{x} + 34x = 68 \qquad x(0) = 5 \qquad \dot{x}(0) = 7$$

■ **Solution**
Substituting $x(t) = C + De^{st}$ into the ODE gives

$$\left(s^2 + 6s + 34\right)De^{st} + 34C = 68$$

Figure 3.1.6 Response for Example 3.1.6.

Thus $C = 68/34 = 2$ and $s^2 + 6s + 34 = 0$, so the roots are $s = -3 \pm 5j$. Because two initial conditions must be satisfied, from Example 3.1.4 we know that the solution form is

$$x(t) = C + D_1 e^{-3+5jt} + D_2 e^{-3-5jt} = C + e^{-3t}\left(D_1 e^{5jt} + D_2 e^{-5jt}\right)$$

From Example 3.1.4, we now know that the complex exponential terms within the parentheses produce the following form:

$$x(t) = C + e^{-3t}(B_1 \cos 5t + B_2 \sin 5t)$$

From the initial conditions,

$$x(0) = C + B_1 = 5 \qquad \dot{x}(0) = -3B_1 + 5B_2 = 7$$

or $B_1 = 5 - C = 3$ and $B_2 = 16/5$. The solution is

$$x(t) = 2 + e^{-3t}\left(3 \cos 5t + \frac{16}{5} \sin 5t\right)$$

The plot of this function is shown in Figure 3.1.6. Note that although the solution contains a sine and a cosine function, which oscillate, we do not observe many oscillations in the plot. This is because the exponential e^{-3t}, and thus the amplitude of the oscillations, decays to a very small number before more than one oscillation can occur ($e^{-3t} < 0.02$ for $t > 4/3 = 1.33$ and the oscillation period is $2\pi/5 = 1.26$).

SUMMARY OF THE TRIAL-SOLUTION METHOD

The examples of the trial-solution method were chosen to illustrate the solutions to the most common differential equations encountered in system dynamics, namely, linear, constant-coefficient equations, each with a *constant* on the right-hand side. We treated two categories:

1. First-order equation: $\dot{x} + ax = b \quad a \neq 0$
2. Second-order equation: $\ddot{x} + a\dot{x} + bx = c \quad b \neq 0$

Table 3.1.3 Solution forms.

Equation	Solution form
First order: $\dot{x} + ax = b \quad a \neq 0$	$x(t) = \dfrac{b}{a} + Ce^{-at}$
Second order: $\ddot{x} + a\dot{x} + bx = c \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \dfrac{c}{b}$
2. $(a^2 = 4b)$ repeated, real roots: s_1, s_1	$x(t) = (C_1 + tC_2)e^{s_1 t} + \dfrac{c}{b}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$, $\omega = \sqrt{b}$	$x(t) = C_1 \sin \omega t + C_2 \cos \omega t + \dfrac{c}{b}$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = \sigma \pm j\omega$, $\sigma = -a/2, \omega = \sqrt{4b - a^2}/2$	$x(t) = e^{\sigma t}(C_1 \sin \omega t + C_2 \cos \omega t) + \dfrac{c}{b}$

The solution form for each case is given in Table 3.1.3. Note that the solution for the second-order case can have one of four possible forms, depending on the nature of the two roots. The case with imaginary roots is actually a special case of complex roots where the real part is zero. Table 3.1.3 does not give formulas for the undetermined constants in the solution, because often all we require is the general form of the solution. To determine the values of these constants, you must be given the initial conditions.

ASSESSMENT OF SOLUTION BEHAVIOR

Note that the characteristic equation can be quickly identified from the ODE by replacing $\dot{x}$ with s, $\ddot{x}$ with s^2, and so forth. For example, $3\ddot{x} + 30\dot{x} + 222x = 148$ has the characteristic equation $3s^2 + 30s + 222 = 0$ and the roots $s = -5 \pm 7j$. This is case 4 in Table 3.1.3, and the solution form is (since $148/222 = 2/3$)

$$x(t) = e^{-5t}(C_1 \sin 7t + C_2 \cos 7t) + \frac{2}{3}$$

From this form we can tell that the solution will oscillate with a radian frequency of 7. The oscillations will eventually disappear because of the exponential term e^{-5t}, which is less than 0.02 for $t > 4/5$. Thus, as $t \to \infty$, $x(t) \to 2/3$. Sometimes this is all the information we need about the solution, and if so, we need not evaluate the constants C_1 and C_2.

　　The trial-solution method works well when the right-hand side of the equation is constant. However, as we will discover, there are important applications where the right-hand side is a function of time, such as t, t^2, $\sin t$, and so forth. For such applications, a set of trial-solution forms can be developed, but a more systematic and general method is provided by the Laplace transform, which is the subject of Section 3.3.

3.2 RESPONSE TYPES AND STABILITY

The solution of $\dot{x} + ax = b$ given by (3.1.3) is

$$x(t) = \underbrace{\frac{b}{a}}_{\text{steady state}} + \underbrace{\left[x(0) - \frac{b}{a}\right]e^{-at}}_{\text{transient}} \tag{3.2.1}$$

Note that the solution consists of two parts, one that disappears with time due to the e^{-at} term, and one that remains. These terms are called the *transient* and the *steady-state* responses, respectively. Both responses need not occur; it is possible to have one without

the other. For example, there is no transient response in the preceding solution if $x(0) = b/a$, and there is no steady-state response if $b = 0$. The *complete* or *total* response is the sum of the transient and steady-state responses. This property of linear differential equations, called *superposition,* is a useful property, as we will see in our later studies.

The solution (3.1.3) can also be rearranged as follows:

$$x(t) = \underbrace{x(0)e^{-at}}_{\text{free}} + \underbrace{\frac{b}{a}\left(1 - e^{-at}\right)}_{\text{forced}} \tag{3.2.2}$$

The first part of the solution depends on the initial condition $x(0)$ and is called the *free* response, because it describes the system's behavior when it is "free" of the externally applied input. The second part depends on the input b and is called the *forced* response. Both responses need not occur; it is possible to have one without the other. For example, there is no free response if the initial conditions are zero, and there is no forced response if there is no input.

It is extremely useful to distinguish between the free and the forced responses because this separation enables us to focus on the effects of the input by temporarily setting the initial conditions to zero and concentrating on the forced response. When we have finished analyzing the forced response we can obtain the complete response by adding the free response to the forced response. This separation is possible because of the superposition property.

THE TIME CONSTANT

The free response of the first-order model may be written in the form

$$x(t) = x(0)e^{-at} = x(0)e^{-t/\tau} \tag{3.2.3}$$

where we have introduced a new parameter τ with the definition

$$\tau = \frac{1}{a} \quad \text{if} \quad a > 0 \tag{3.2.4}$$

The equation $\dot{x} + ax = b$ may be expressed in terms of τ by replacing a with $1/\tau$ as follows.

$$\tau\dot{x} + x = b\tau \tag{3.2.5}$$

The new parameter τ has units of time and is the model's *time constant*. It gives a convenient measure of the exponential decay curve and an estimate of how long it will take for the transient response to disappear, leaving only the steady-state response. The free response and the meaning of the time constant are illustrated in Figure 3.2.1. After a time equal to one time constant has elapsed, x has decayed to 37% of its initial value. We can also say that x has decayed *by* 63%. At $t = 4\tau$, $x(t)$ has decayed to 2% of its initial value. At $t = 5\tau$, $x(t)$ has decayed to 1% of its initial value. The time constant is defined only when $a > 0$. If $a \leq 0$ the free response does not decay to 0 and thus the time constant has no meaning.

The time constant is useful also for analyzing the response when the forcing function is a constant. We can express the total response given by form (3.2.1) in terms of τ by substituting $a = 1/\tau$ as follows.
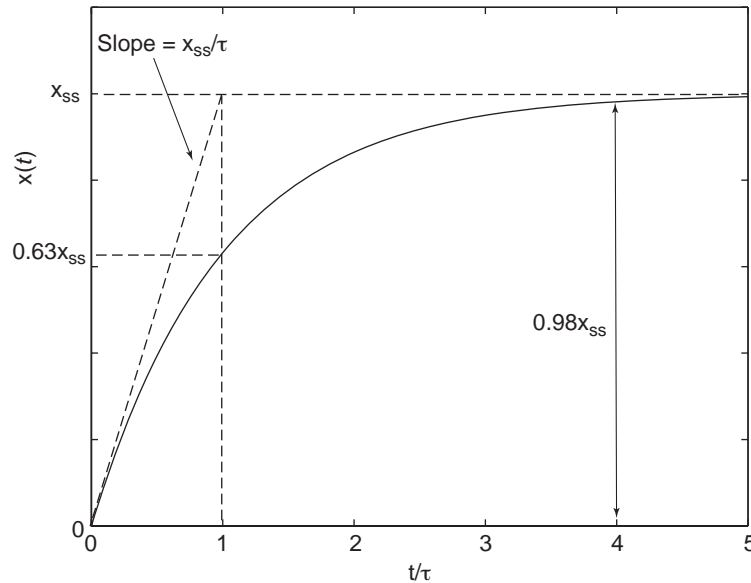
$$x(t) = \underbrace{b\tau}_{\text{steady state}} + \underbrace{[x(0) - b\tau]\,e^{-t/\tau}}_{\text{transient}} = x_{ss} + [x(0) - x_{ss}]\,e^{-t/\tau} \tag{3.2.6}$$

Figure 3.2.1 The free
response $x(t) = x(0)e^{-t/\tau}$.
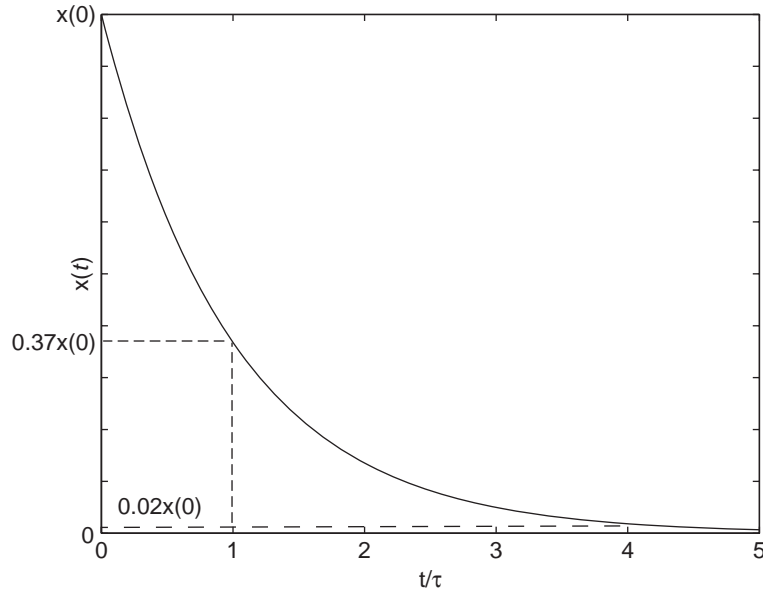
Figure 3.2.2 The step
response $x(t) = x_{ss}(1 - e^{-t/\tau})$.

where $x_{ss} = b/a = b\tau$, which is the steady-state response. The response approaches
the constant value $b\tau$ as $t \to \infty$. The forced response (for which $x(0) = 0$) is plotted
in Figure 3.2.2. At $t = \tau$, the response is 63% of the steady-state value. At $t = 4\tau$,
the response is 98% of the steady-state value, and at $t = 5\tau$, it is 99% of steady-state.
Thus, because the difference between 98% and 99% is so small, for most engineering
purposes we may say that $x(t)$ reaches steady-state at $t = 4\tau$, although mathematically,
steady-state is not reached until $t = \infty$.

If $x(0) \neq 0$, the response is shifted by $x(0)e^{-t/\tau}$. At $t = \tau$, 37% of the difference
between the initial value $x(0)$ and the steady-state value remains. At $t = 4\tau$, only 2%
of this difference remains.

■ Problem

Identify the transient, steady-state, free, and forced responses of the following equation, whose characteristic roots are -2 and -5:

$$\ddot{x} + 7\dot{x} + 10x = c$$

■ Solution

Following the procedure used in Example 3.1.2, we see that the solution form is

$$x(t) = \underbrace{\frac{c}{10}}_{\text{steady state}} + \underbrace{D_1 e^{-2t} + D_2 e^{-5t}}_{\text{transient}} \tag{1}$$

where we have identified the transient and steady-state solutions. Because of the e^{-2t} and e^{-5t} terms, the transient response eventually disappears as t increases. The coefficients D_1 and D_2 can be found in terms of arbitrary initial conditions in the usual way:

$$D_1 = \frac{5}{3}x(0) + \frac{1}{3}\dot{x}(0) - \frac{c}{6} \qquad D_2 = -\frac{2}{3}x(0) - \frac{1}{3}\dot{x}(0) + \frac{c}{15}$$

Substituting these expressions into equation (1) and isolating the initial conditions gives

$$x(t) = \underbrace{\left[\frac{5}{3}x(0) + \frac{1}{3}\dot{x}(0)\right]e^{-2t} + \left[-\frac{2}{3}x(0) - \frac{1}{3}\dot{x}(0)\right]e^{-5t}}_{\text{free}} + \underbrace{c\left(\frac{1}{10} - \frac{1}{6}e^{-2t} + \frac{1}{15}e^{-5t}\right)}_{\text{forced}}$$

Note that if the initial conditions are zero, the solution consists of only the forced response. If the input c is zero, the solution consists of only the free response.

THE DOMINANT-ROOT APPROXIMATION

The time constant concept is not limited to first-order models. It can also be used to estimate the response time of higher-order models. In Example 3.2.1, the two roots are -2 and -5. They generate the exponentials e^{-2t} and e^{-5t} in the response. Because e^{-5t} decays to 0 faster than e^{-2t}, the term $D_2 e^{-5t}$ will disappear faster than the term $D_1 e^{-2t}$, provided that $|D_2|$ is not much greater than $|D_1|$. The time constant of the root -5 is $\tau_1 = 1/5$, and thus the term $D_2 e^{-5t}$ will be essentially 0 for $t > 4\tau_1 = 4/5$. Thus, for $t > 4/5$, the response from equation (1) of the example is essentially given by $x(t) = c/10 + D_1 e^{-2t}$. The root -2 is said to be the *dominant root* because it dominates the response relative to the term $D_2 e^{-5t}$, and the time constant $\tau_2 = 1/2$ is said to be the *dominant time constant*. We can estimate how long it will take for the transient response to disappear by multiplying the dominant time constant by 4. Here the answer is $t = 4/2 = 2$.

We cannot make exact predictions based on the dominant root because the initial conditions that determine the values of D_1 and D_2 can be such that $|D_2| >> |D_1|$, so that the second exponential influences the response for longer than expected. The dominant root approximation, however, is often used to obtain a quick estimate of response time. The farther away the dominant root is from the other roots (the "secondary" roots), the better the approximation.

The dominant-root approximation can be applied to higher-order models. If all the roots have negative real parts (some roots may be complex), the dominant root is the one having the largest time constant, and the time it will take the transient response to disappear can be estimated by multiplying the dominant time constant by 4.

EXAMPLE 3.2.2

Responses for Second-Order, Complex Roots

■ Problem
Identify the transient, steady-state, free, and forced responses of the following equation, whose characteristic roots are $-3 \pm 5j$:

$$\ddot{x} + 6\dot{x} + 34x = c$$

■ Solution
Following the procedure used in Example 3.1.6, we see that the solution form is

$$x(t) = \underbrace{\frac{c}{34}}_{\text{steady state}} + \underbrace{e^{-3t}\left(B_1 \cos 5t + B_2 \sin 5t\right)}_{\text{transient}} \qquad (1)$$

where we have identified the transient and steady-state solutions. Because of the e^{-3t} term, the transient response, which is oscillatory, eventually disappears as t increases. The coefficients B_1 and B_2 can be found in terms of arbitrary initial conditions in the usual way.

$$B_1 = x(0) - \frac{c}{34} \qquad B_2 = \frac{34\dot{x}(0) + 102x(0) - 3c}{170}$$

Substituting these expressions into equation (1) and isolating the initial conditions gives

$$x(t) = \underbrace{x(0)e^{-3t}\cos 5t + \frac{\dot{x}(0) + 3x(0)}{5}e^{-3t}\sin 5t}_{\text{free}} + \underbrace{\frac{c}{34}\left(1 - e^{-3t}\cos 5t - \frac{3}{5}e^{-3t}\sin 5t\right)}_{\text{forced}}$$

If the initial conditions are zero, the solution consists of only the forced response. If the input c is zero, the solution consists of only the free response.

TIME CONSTANTS AND COMPLEX ROOTS

The model in Example 3.2.2 has complex roots: $-3 \pm 5j$. These lead to the term e^{-3t} in the response. Thus we may apply the concept of a time constant to complex roots by computing the time constant from the negative inverse of the real part of the roots. Here the model's time constant is $\tau = 1/3$, and thus the response is essentially at steady state for $t > 4\tau = 4/3$. Since complex roots occur only in conjugate pairs, each pair has the same time constant.

EXAMPLE 3.2.3

Responses for Second-Order, Imaginary Roots

■ Problem
Identify the transient, steady-state, free, and forced responses of the following equation, whose characteristic roots are $\pm 4j$:

$$\ddot{x} + 16x = c$$

■ **Solution**

Following the procedure used in Example 3.1.6, we see that the solution form is

$$x(t) = \underbrace{\frac{c}{16} + B_1 \cos 4t + B_2 \sin 4t}_{\text{steady state}}$$

Because there are no terms that disappear as $t \to \infty$, there is no transient response. Note that part of the steady-state response is oscillatory. This example shows that the steady-state response need not be constant.

To identify the free and forced responses, we must obtain the expressions for B_1 and B_2 as functions of the initial conditions. These are $B_1 = x(0) - c/16$ and $B_2 = \dot{x}(0)/4$. Thus the solution can be expressed as

$$x(t) = \underbrace{x(0) \cos 4t + \frac{\dot{x}(0)}{4} \sin 4t}_{\text{free}} + \underbrace{\frac{c}{16} (1 - \cos 4t)}_{\text{forced}}$$

Since the roots here have no real part, no time constant is defined for this model. This makes sense because there is no transient response here.

To summarize, we can separate the total response as follows:

1. *Transient Response* The part of the response that disappears with time.
2. *Steady-State Response* The part of the response that remains with time.
3. *Free Response* The part of the response that depends on the initial conditions.
4. *Forced Response* The part of the response due to the forcing function.

STABILITY

We have seen free responses that approach 0 as $t \to \infty$. We have also seen free responses that approach a constant-amplitude oscillation as $t \to \infty$. The free response may also approach ∞. For example, the model $\dot{x} = 2x$ has the free response $x(t) = x(0)e^{2t}$ and thus $x \to \infty$ as $t \to \infty$.

Let us define some terms.

Unstable   A model whose free response approaches ∞ as $t \to \infty$ is said to be *unstable*.

Stable   If the free response approaches 0, the model is *stable*.

Neutral Stability   The borderline case between stable and unstable. Neutral stability describes a situation where the free response does not approach ∞ but does not approach 0.

The stability properties of a linear model are determined from its characteristic roots. To understand the relationship between stability and the characteristic roots, consider some simple examples.

The first-order model $\dot{x} + ax = f(t)$ has the free response $x(t) = x(0)e^{-at}$, which approaches 0 as $t \to \infty$ if the characteristic root $s = -a$ is *negative*. The model is unstable if the root is *positive* because $x(t) \to \infty$ as $t \to \infty$.

A borderline case, called neutral stability, occurs if the root is 0. In this case $x(t)$ remains at $x(0)$. Neutral stability describes a situation where the free response does not approach ∞ but does not approach 0.

Now consider the following second-order models, which have the same initial conditions: $x(0) = 1$ and $\dot{x}(0) = 0$. Their responses are shown in Figure 3.2.3.

Figure 3.2.3 Examples of
unstable and neutrally stable
models.
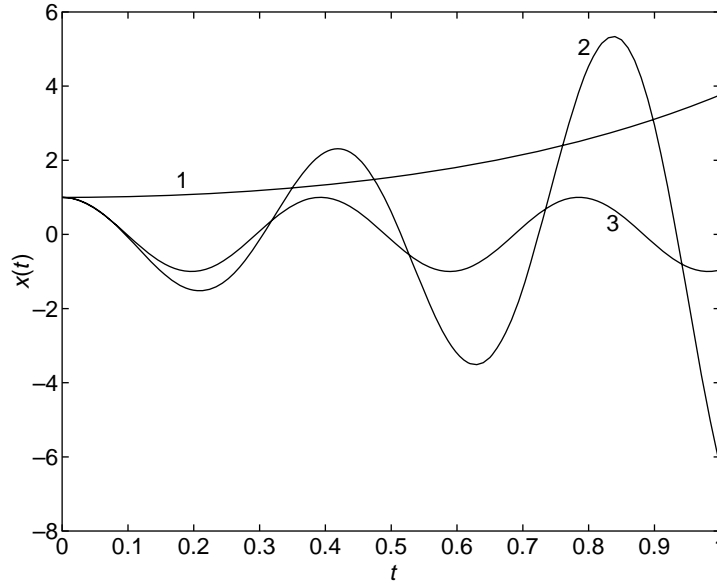
1. The model $\ddot{x} - 4x = f(t)$ has the roots $s = \pm 2$, and the free response:

$$x(t) = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)$$

2. The model $\ddot{x} - 4\dot{x} + 229x = f(t)$ has the roots $s = 2 \pm 15j$, and the free response:

$$x(t) = e^{2t}\left(\cos 15t - \frac{2}{15}\sin 15t\right)$$

3. The model $\ddot{x} + 256x = f(t)$ has the roots $s = \pm 16j$, and the free response:

$$x(t) = \cos 16t$$

From the plots we can see that none of the three models displays stable behavior. The first and second models are unstable, while the third is neutrally stable. Thus, the free response of a neutrally stable model can either approach a nonzero constant or settle down to a constant amplitude oscillation.

The effect of the real part of the characteristic roots can be seen from the free response form for the complex roots $\sigma \pm \omega j$ (entry 4 in Table 3.1.3, with $c = 0$):

$$x(t) = e^{\sigma t}(C_1 \sin \omega t + C_2 \cos \omega t)$$

Clearly, if the real part is positive (that is, if $\sigma > 0$), then the exponential $e^{\sigma t}$ will grow with time and so will the amplitude of the oscillations. This is an unstable case.

If the real part is 0 (that is, $\sigma = 0$), then the exponential becomes $e^{0t} = 1$, and the amplitude of the oscillations remains constant. This is the neutrally stable case. The imaginary part of the root is the frequency of oscillation; it has no effect on stability.

Model 1 is unstable because of the exponential e^{2t}, which is due to the positive root $s = +2$. The negative root $s = -2$ does not cause instability because its exponential disappears in time.

If we realize that a real number is simply a special case of a complex number whose imaginary part is 0, then these examples show that a linear model is unstable if at least one root has a positive real part. We will see that the free response of any linear,

constant-coefficient model, of any order, consists of a sum of terms, each multiplied by an exponential. Each exponential will approach ∞ as $t \to \infty$, if its corresponding root has a positive real part. Thus we can make the following statement about linear models.

Stability Test for Linear Constant-Coefficient Models

A constant-coefficient linear model is *stable* if and only if all of its characteristic roots have negative real parts.

The model is *neutrally stable* if one or more roots have a zero real part, and the remaining roots have negative real parts.

The model is *unstable* if *any* root has a positive real part.

If a linear model is stable, then it not possible to find a set of initial conditions for which the free response approaches ∞ as $t \to \infty$. However, if the model is unstable, there might still be certain initial conditions that result in a response that disappears in time. For example, the model $\ddot{x} - 4x = 0$ has the roots $s = \pm 2$, and thus is unstable. However, if the initial conditions are $x(0) = 1$ and $\dot{x}(0) = -2$, then the free response is $x(t) = e^{-2t}$, which approaches 0 as $t \to \infty$. Note that the exponential e^{2t} corresponding to the root at $s = +2$ does not appear in the response because of the special nature of these initial conditions.

Because the time constant is a measure of how long it takes for the exponential terms in the response to disappear, we see that the time constant is not defined for neutrally stable and unstable cases.

A PHYSICAL EXAMPLE

A physical example illustrating the meaning of stability is shown in Figure 3.2.4. Suppose that there is some slight viscous friction in the pivot point of the pendulum. In part (a) the pendulum is hanging and is at rest at $\theta = 0$. If something disturbs it slightly, it will oscillate about $\theta = 0$ and eventually return to rest at $\theta = 0$. We thus see that the system is *stable*. As we will see in Chapter 4, its equation of motion for small θ is $mL^2\ddot{\theta} + c\dot{\theta} + mgL\theta = 0$. Its roots are

$$s = \frac{-c \pm \sqrt{c^2 - 4m^2L^3g}}{2mL^2}$$

If c is positive and small but nonzero, these roots will be complex with a negative real part, and the pendulum will oscillate before coming to rest. If c is large enough, these roots will be real and negative, and the pendulum will not oscillate before coming
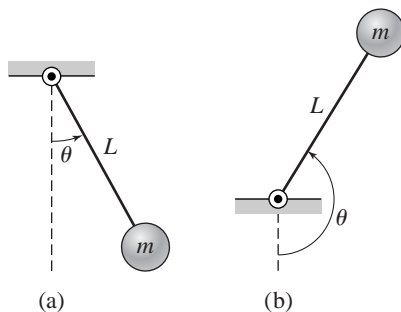
Figure 3.2.4 Pendulum illustration of stability properties.

to rest. In both cases, however, the system is stable because the pendulum eventually returns to its rest position at $\theta = 0$.

Now suppose that $c = 0$. The roots will be imaginary, and the pendulum therefore will oscillate forever about $\theta = 0$ if disturbed. This is an example of neutral stability.

If the pendulum is perfectly balanced at $\theta = \pi$, as in Figure 3.2.4b, it will never return to $\theta = \pi$ if it is disturbed. So, this is a case of an unstable system. The equation of motion for this case, for θ near π, is $mL^2\ddot{\theta} + c\dot{\theta} - mgL\theta = 0$. Its roots are

$$s = \frac{-c \pm \sqrt{c^2 + 4m^2L^3g}}{2mL^2}$$

Because of the $+$ sign under the square root, we can see that the roots will be real and positive. Thus the pendulum will not oscillate about $\theta = \pi$ but will continue to fall away if disturbed. In this case, the system is unstable. Because the model is based on the assumption that θ is close to π, we cannot draw any conclusions from the model regarding the behavior when θ is not near π.

THE ROUTH-HURWITZ CONDITION

The characteristic equation of many systems has the form $ms^2 + cs + k = 0$. A simple criterion exists for quickly determining the stability of such a system. This is proved in homework problem 3.6. The condition states that the second-order system whose characteristic polynomial is $ms^2 + cs + k$ is stable if and only if m, c, and k have the same sign. This requirement is called the Routh-Hurwitz condition.

STABILITY AND EQUILIBRIUM

An *equilibrium* is a state of no change. The pendulum in Figure 3.2.4 is in equilibrium at $\theta = 0$ and when perfectly balanced at $\theta = \pi$. The equilibrium at $\theta = 0$ is stable, while the equilibrium at $\theta = \pi$ is unstable. From this we see that the same physical system can have different stability characteristics at different equilibria. So we see that stability is not a property of the system alone, but is a property of a specific equilibrium of the system. When we speak of the stability properties of a model, we are actually speaking of the stability properties of the specific equilibrium on which the model is based.

Figure 3.2.5 shows a ball on a surface that has a valley and a hill. The bottom of the valley is an equilibrium, and if the ball is displaced slightly from this position, it will oscillate forever about the bottom if there is no friction. In this case, the equilibrium is neutrally stable. The ball, however, will return to the bottom if friction is present, and the equilibrium is stable in this case.

If we displace the ball so much to the left that it lies outside the valley, it will never return. Thus, if friction is present, we say that the valley equilibrium is *locally stable* but *globally unstable*. An equilibrium is globally stable only if the system returns to it for *any* initial displacement.

Figure 3.2.5 Surface illustration of stability properties.

Hill equilibrium

Valley equilibrium
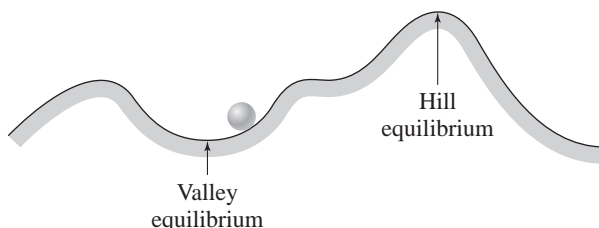
The equilibrium on the hilltop is globally unstable because, if displaced, the ball will continue to roll down the hill.

For linear models, stability analysis using the characteristic roots gives global stability information. However, for nonlinear models, linearization about an equilibrium gives us only local stability information.

3.3 THE LAPLACE TRANSFORM METHOD

The Laplace transform provides a systematic and general method for solving linear ODEs, and is especially useful either for nonhomogeneous equations whose right-hand side is a function of time or for sets of equations. Another advantage is that the transform converts linear differential equations into algebraic relations that can be handled easily.

Although named after Laplace, the transform actually is based on the work of Léonard Euler, from 1763, for solving second-order linear ODEs. The Laplace transform $\mathcal{L}[x(t)]$ of a function $x(t)$ is defined as follows.

$$\mathcal{L}[x(t)] = \lim_{T \to \infty} \left[\int_0^T x(t)e^{-st}\, dt \right] \tag{3.3.1}$$

but is usually expressed more compactly as

$$\mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}\, dt \tag{3.3.2}$$

The variable of integration, t, is arbitrary, and the transform is a function of the parameter s, which is a complex number. This definition is the so-called *one-sided* transform, which is used in applications where the variable $x(t)$ is zero for $t < 0$. In our applications we will always assume that this is true.

A simpler notation for the transform uses the uppercase symbol to represent the transform of the corresponding lowercase symbol; that is,

$$X(s) = \mathcal{L}[x(t)] \tag{3.3.3}$$

The process of determining the time function $x(t)$ whose transform is $X(s)$ is denoted by

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

where the symbol $\mathcal{L}^{-1}$ denotes the *inverse* transform.

When the limit in (3.3.1) is not finite, there is no Laplace transform defined for $x(t)$. For many common functions, we can choose s to obtain a finite limit. In practice, however, when solving ODEs, we need not be concerned with the choice for s, because the transforms of common functions that have transforms have been derived and tabulated. Table 3.3.1 is a table of commonly needed transforms.

For some relatively simple functions either the Laplace transform does not exist (such as for e^{t^2} and $1/t$), or it cannot be represented as an algebraic expression (such as for $1/(t + a)$). In the latter case, the integral must be represented as an infinite series, which is not very useful for our purposes.

The Laplace transform of a function $x(t)$ exists for all $s > \gamma$ if $x(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies the relation $|x(t)| \leq Me^{\gamma t}$

Table 3.3.1 Table of Laplace transform pairs.

$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\dfrac{1}{s}$	$u_s(t)$, unit step
3. $\dfrac{c}{s}$	constant, c
4. $\dfrac{e^{-sD}}{s}$	$u_s(t - D)$, shifted unit step
5. $\dfrac{n!}{s^{n+1}}$	t^n
6. $\dfrac{1}{s + a}$	e^{-at}
7. $\dfrac{1}{(s + a)^n}$	$\dfrac{1}{(n - 1)!}t^{n-1}e^{-at}$
8. $\dfrac{b}{s^2 + b^2}$	$\sin bt$
9. $\dfrac{s}{s^2 + b^2}$	$\cos bt$
10. $\dfrac{b}{(s + a)^2 + b^2}$	$e^{-at}\sin bt$
11. $\dfrac{s + a}{(s + a)^2 + b^2}$	$e^{-at}\cos bt$
12. $\dfrac{a}{s(s + a)}$	$1 - e^{-at}$
13. $\dfrac{1}{(s + a)(s + b)}$	$\dfrac{1}{b - a}\left(e^{-at} - e^{-bt}\right)$
14. $\dfrac{s + p}{(s + a)(s + b)}$	$\dfrac{1}{b - a}\left[(p - a)e^{-at} - (p - b)e^{-bt}\right]$
15. $\dfrac{1}{(s + a)(s + b)(s + c)}$	$\dfrac{e^{-at}}{(b - a)(c - a)} + \dfrac{e^{-bt}}{(c - b)(a - b)} + \dfrac{e^{-ct}}{(a - c)(b - c)}$
16. $\dfrac{s + p}{(s + a)(s + b)(s + c)}$	$\dfrac{(p - a)e^{-at}}{(b - a)(c - a)} + \dfrac{(p - b)e^{-bt}}{(c - b)(a - b)} + \dfrac{(p - c)e^{-ct}}{(a - c)(b - c)}$

for all $t \geq 0$ and for some constants γ and M [Kreyzig, 2006]. These conditions are sufficient but not necessary. For example, the function $1/\sqrt{t}$ is infinite at $t = 0$ but it has a transform, which is $\sqrt{\pi/s}$.

The *inverse* Laplace transform $\mathcal{L}^{-1}[X(s)]$ is that time function $x(t)$ whose transform is $X(s)$; that is, $x(t) = \mathcal{L}^{-1}[X(s)]$. It can be shown that if two continuous functions have the same transform, then the functions are identical. This is of practical significance because we will need to find the function $x(t)$, given $X(s)$.

TRANSFORMS OF COMMON FUNCTIONS

We now derive a few transforms to develop an understanding of the method. Remember that because the lower limit in the transform definition is $t = 0$, the behavior of the function $x(t)$ for $t < 0$ is irrelevant to finding the transform.

| Transform of a Constant | **EXAMPLE 3.3.1** |

■ **Problem**

Suppose $x(t) = c$, a constant, for $t \geq 0$. Determine its Laplace transform.

■ **Solution**

From the transform definition, we have

$$\mathcal{L}[x(t)] = \lim_{T \to \infty} \left(\int_0^T c e^{-st} \, dt \right) = c \lim_{T \to \infty} \left(\int_0^T e^{-st} \, dt \right)$$

or

$$\mathcal{L}(c) = c \lim_{T \to \infty} \left(\frac{1}{-s} e^{-st} \Big|_0^T \right) = c \lim_{T \to \infty} \left(\frac{1}{-s} e^{-sT} + \frac{1}{s} e^0 \right) = \frac{c}{s}$$

where we have assumed that the real part of s is greater than zero, so that the limit of e^{-sT} exists as $T \to \infty$. Thus the transform of a constant c is c/s.

The Step Function   The *step function* is so named because its shape resembles a stair step (see Figure 3.3.1). If height of the step is 1, the function is called the *unit-step function,* denoted by $u_s(t)$. It is defined as

$$u_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

and is undefined and discontinuous at $t = 0$.

Suppose the function $x(t)$ is zero for $t < 0$ and a nonzero constant, say M, for $t > 0$. Then we can express it as $x(t) = M u_s(t)$. The value of M is the *magnitude* of the step function; if $M = 1$, the function is the unit step function. From the results of Example 3.3.1, we can easily see that the Laplace transform of $x(t) = M u_s(t)$ is M/s.

Figure 3.3.1 The unit-step function.

$u_s(t)$

1

t
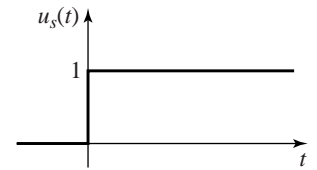
| The Exponential Function | **EXAMPLE 3.3.2** |

■ **Problem**

Derive the Laplace transform of the exponential function $x(t) = e^{-at}$, where a is a constant.

■ **Solution**

From the transform definition, we have

$$\mathcal{L}\left(e^{-at}\right) = \lim_{T \to \infty} \left(\int_0^T e^{-at} e^{-st} \, dt \right) = \lim_{T \to \infty} \left(\int_0^T e^{-(s+a)t} \, dt \right)$$

or

$$\mathcal{L}\left(e^{-at}\right) = \lim_{T \to \infty} \left[\frac{1}{-(s+a)} e^{-(s+a)t} \Big|_0^T \right] = \frac{1}{s+a}$$

THE LINEARITY PROPERTY

The Laplace transform is a definite integral, and thus it has the properties of such integrals (Table 3.3.2). For example, a multiplicative constant can be factored out of the integral, and the integral of a sum equals the sum of the integrals. These facts lead to the

Table 3.3.2 Properties of the Laplace transform.

$x(t)$	$X(s) = \int_0^\infty f(t)e^{-st}\, dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2 X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^n x}{dt^n}$	$s^n X(s) - \displaystyle\sum_{k=1}^{n} s^{n-k} g_{k-1}$
	$g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right|_{t=0}$
5. $\displaystyle\int_0^t x(t)\, dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$
	$g(0) = \left.\displaystyle\int x(t)\, dt\right|_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \ge D \end{cases}$	
$= u_s(t-D)g(t-D)$	$X(s) = e^{-sD} G(s)$
7. $e^{-at} x(t)$	$X(s+a)$
8. $tx(t)$	$-\dfrac{dX(s)}{ds}$
9. $x(\infty) = \lim\limits_{s \to 0} sX(s)$	
10. $x(0+) = \lim\limits_{s \to \infty} sX(s)$	

linearity property of the transform; namely, for the functions $f(t)$ and $g(t)$, if a and b are constants, then

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$$

One use of the linearity property is to determine transforms of functions that are linear combinations of functions whose transforms are already known. For example, if $x(t) = 6 + 4e^{-3t}$, its transform is

$$X(s) = \frac{6}{s} + \frac{4}{s+3} = \frac{10s + 18}{s(s+3)}$$

The inverse transform also has the linearity property, so that

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)] = af(t) + bg(t)$$

We can often avoid the integration operations by using the linearity property and suitable identities, as shown in Example 3.3.3.

EXAMPLE 3.3.3   The Sine and Cosine Functions

■ **Problem**
Derive the Laplace transforms of the exponentially decaying sine and cosine functions, $e^{-at} \sin \omega t$ and $e^{-at} \cos \omega t$, where a and ω are constants.

■ Solution

Note that from the Euler identity, $e^{j\theta} = \cos\theta + j\sin\theta$, with $\theta = \omega t$, we have

$$e^{-at}(\cos\omega t + j\sin\omega t) = e^{-at}e^{j\omega t} = e^{-(a-j\omega)t} \tag{1}$$

Thus the real part of $e^{-(a-j\omega)t}$ is $e^{-at}\cos\omega t$ and the imaginary part is $e^{-at}\sin\omega t$. However, from the result of Example 3.3.2, with a replaced by $a - j\omega$, we have

$$\mathcal{L}\left[e^{-(a-j\omega)t}\right] = \frac{1}{s+a-j\omega} \tag{2}$$

In this form, we cannot identify the real and imaginary parts. To do so we multiply the numerator and denominator by the complex conjugate of the denominator and use the fact that $(x - jy)(x + jy) = x^2 + y^2$ (see Table 3.1.1); that is,

$$\frac{1}{x - jy} = \frac{x + jy}{(x - jy)(x + jy)} = \frac{x + jy}{x^2 + y^2}$$

Thus with $x = s + a$ and $y = \omega$, equation (2) becomes

$$\mathcal{L}\left[e^{-(a-j\omega)t}\right] = \frac{1}{s+a-j\omega} = \frac{s+a+j\omega}{(s+a-j\omega)(s+a+j\omega)}$$

$$= \frac{s+a+j\omega}{(s+a)^2+\omega^2} = \frac{s+a}{(s+a)^2+\omega^2} + j\frac{\omega}{(s+a)^2+\omega^2}$$

From equation (1) we see that the real part of this expression is the transform of $e^{-at}\cos\omega t$ and the imaginary part is the transform of $e^{-at}\sin\omega t$. Therefore,

$$\mathcal{L}\left(e^{-at}\cos\omega t\right) = \frac{s+a}{(s+a)^2+\omega^2}$$

and

$$\mathcal{L}\left(e^{-at}\sin\omega t\right) = \frac{\omega}{(s+a)^2+\omega^2}$$

Note that the transforms of the sine and cosine can be obtained by letting $a = 0$. Thus

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \qquad \text{and} \qquad \mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

Another property of the Laplace transform is called *shifting along the s-axis* or *multiplication by an exponential*. This property states that

$$\mathcal{L}\left[e^{-at}x(t)\right] = X(s+a) \tag{3.3.4}$$

To derive this property, note that

$$\mathcal{L}\left[e^{-at}x(t)\right] = \int_0^\infty e^{-at}x(t)e^{-st}\,dt = \int_0^\infty x(t)e^{-(s+a)t}\,dt$$

$$= X(s+a)$$

EXAMPLE 3.3.4 | The Function te^{-at}

■ **Problem**

Derive the Laplace transform of the function te^{-at}.

■ **Solution**

Here the function $x(t)$ is t, $X(s) = 1/s^2$, and thus from (3.3.4),

$$\mathcal{L}\left[e^{-at}x(t)\right] = \mathcal{L}\left(te^{-at}\right) = \left.\frac{1}{s^2}\right|_{s \to s+a} = \frac{1}{(s+a)^2}$$

Another property is *multiplication by t*. It states that

$$\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds} \tag{3.3.5}$$

To derive this property, note that

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left[\int_0^\infty x(t)e^{-st}\,dt\right] = -\int_0^\infty tx(t)e^{-st}\,dt = -\mathcal{L}[tx(t)]$$

EXAMPLE 3.3.5 | The Function $t\cos\omega t$

■ **Problem**

Derive the Laplace transform of the function $t\cos\omega t$.

■ **Solution**

Here the function $x(t)$ is $\cos\omega t$, $X(s) = s/(s^2 + \omega^2)$, and thus from (3.3.5),

$$\mathcal{L}[tx(t)] = \mathcal{L}(t\cos\omega t) = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

EXAMPLE 3.3.6 | The Shifted Step Function

■ **Problem**

If the discontinuity in the unit-step function occurs at $t = D$, Figure 3.3.2, then the function $x(t) = Mu_s(t - D)$ is 0 for $t < D$ and M for $t > D$. The function $u_s(t - D)$ is called the *shifted step function*. Determine $X(s)$.

■ **Solution**

From the transform definition, we have

$$\mathcal{L}[x(t)] = \lim_{T \to \infty}\left[\int_0^T Mu_s(t-D)e^{-st}\,dt\right] = \lim_{T \to \infty}\left(\int_0^D 0e^{-st}\,dt + \int_D^T Me^{-st}\,dt\right)$$

Figure 3.3.2 Shifted step function.
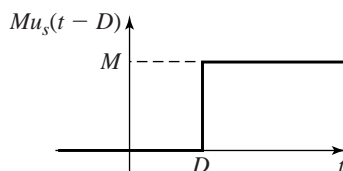
or

$$\mathcal{L}[x(t)] = 0 + M \lim_{T \to \infty} \left(\frac{1}{-s} e^{-st} \Big|_D^T \right) = M \lim_{T \to \infty} \left(\frac{1}{-s} e^{-sT} + \frac{1}{s} e^{-sD} \right) = \frac{M}{s} e^{-sD}$$

Thus the transform of the shifted unit-step function $u_s(t - D)$ is e^{-sD}/s.

Example 3.3.6 introduces a property of the transform called *shifting along the t-axis*. From this example, we see that the effect of the time shift D is to multiply the transform of the unshifted function by e^{-sD}. This illustrates the time-shifting property, which states that if

$$x(t) = \begin{cases} 0 & t < D \\ g(t - D) & t > D \end{cases}$$

then $X(s) = e^{-sD} G(s)$.

The Rectangular Pulse Function | **EXAMPLE 3.3.7**

■ **Problem**

The rectangular pulse function $P(t)$ is shown in Figure 3.3.3a. Derive the Laplace transform of this function (a) from the basic definition of the transform and (b) from the time-shifting property.

■ **Solution**

a. From the definition of the transform,

$$\mathcal{L}[P(t)] = \int_0^\infty P(t) e^{-st} \, dt = \int_0^D 1 e^{-st} \, dt + \int_D^\infty 0 e^{-st} \, dt = \int_0^D 1 e^{-st} \, dt$$

$$= \frac{e^{-st}}{-s} \Big|_0^D = \frac{1}{s} \left(1 - e^{-sD} \right)$$

b. Figure 3.3.3b shows that the pulse can be considered to be composed of the sum of a unit-step function and a shifted, negative unit-step function. Thus, $P(t) = u_s(t) - u_s(t - D)$ and from the time-shifting property,

$$P(s) = \mathcal{L}[u_s(t)] - \mathcal{L}[u_s(t - D)] = \frac{1}{s} - e^{-sD} \frac{1}{s} = \frac{1}{s} \left(1 - e^{-sD} \right)$$

which is the same result obtained in part (a)

(a)   (b)
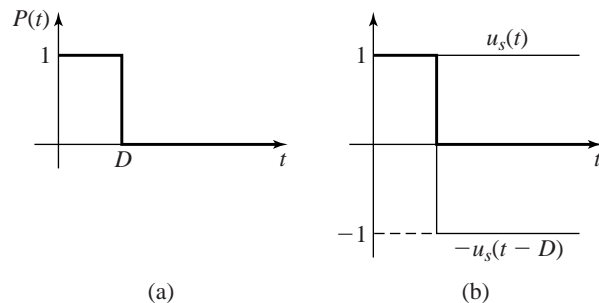
Figure 3.3.3 (a) Rectangular pulse function. (b) Rectangular pulse composed of two step functions.

THE DERIVATIVE PROPERTY

To use the Laplace transform to solve differential equations, we will need to obtain the transforms of derivatives. Applying integration by parts to the definition of the transform, we obtain

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^\infty \frac{dx}{dt} e^{-st}\, dt = x(t)e^{-st}\big|_0^\infty + s \int_0^\infty x(t)e^{-st}\, dt$$

$$= s\mathcal{L}[x(t)] - x(0) = sX(s) - x(0) \tag{3.3.6}$$

This procedure can be extended to higher derivatives. For example, the result for the second derivative is

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2 X(s) - sx(0) - \dot{x}(0) \tag{3.3.7}$$

The general result for any order derivative is given in Table 3.3.2.

THE INITIAL VALUE THEOREM

Sometimes we will need to find the value of the function $x(t)$ at $t = 0+$ (a time infinitesimally greater than 0), given the transform $X(s)$. The answer can be obtained with the *initial value theorem,* which states that

$$x(0+) = \lim_{t \to 0+} x(t) = \lim_{s \to \infty}[sX(s)] \tag{3.3.8}$$

The conditions for which the theorem is valid are that the latter limit exists and that the transforms of $x(t)$ and dx/dt exist. If $X(s)$ is a rational function and if the degree of the numerator of $X(s)$ is less than the degree of the denominator, then the theorem will give a finite value for $x(0+)$. If the degrees are equal, then the initial value is undefined and the initial value theorem is invalid (see Section 3.6 for a discussion of this case). The proof of the theorem is obtained in Problem 3.43.

For the transform

$$X(s) = \frac{7s + 2}{s(s + 6)}$$

the theorem gives

$$x(0+) = \lim_{s \to \infty} \frac{7s + 2}{s + 6} = 7$$

This is confirmed by evaluating the inverse transform, $x(t) = 1/3 + (20/3)e^{-6t}$. (We state the inverse transform here for illustrative purposes only; normally we apply the theorem in situations where the inverse transform is not convenient to obtain.)

We will see important applications of this theorem in Section 3.6.

THE FINAL VALUE THEOREM

To find the limit of the function $x(t)$ as $t \to \infty$, we can use the *final value theorem*. The theorem states that

$$f(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} sF(s) \tag{3.3.9}$$

The theorem is true if the following conditions are satisfied. The functions $x(t)$ and dx/dt must possess Laplace transforms, and $x(t)$ must approach a constant value as $t \to \infty$. The latter condition will be satisfied if all the roots of the denominator of $sX(s)$ have *negative* real parts. The proof of the theorem is obtained in Problem 3.44.

For example, if $X(s) = 1/s$, which corresponds to $x(t) = 1$, then

$$\lim_{s \to 0} sX(s) = \lim_{s \to 0} s\frac{1}{s} = 1$$

which is correct. (We state the inverse transform here for illustrative purposes only; normally we apply the theorem in situations where the inverse transform is not convenient to obtain.)

As another example, if

$$X(s) = \frac{7}{(s + 4)^2 + 49}$$

then

$$\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{7s}{(s + 4)^2 + 49} = 0$$

which is correct [$X(s)$ has the inverse transform $x(t) = e^{-4t} \sin 7t$].

The function $x(t)$ will approach a constant value if all the roots of the denominator of $sX(s)$ have negative real parts. Thus a common situation in which the theorem does not apply is a periodic function. For example, if $x(t) = \sin 5t$, then $X(s) = 5/(s^2 + 25)$, and

$$\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{5s}{s^2 + 25} = 0$$

Thus the limit exists but the result is incorrect, because $x(t)$ continually oscillates and therefore does not approach a constant value.

The theorem is not applicable to the transform

$$X(s) = \frac{9s + 2}{s(s - 8)}$$

because, after the s terms are canceled, the denominator of $sX(s)$ is $s - 8$, which has the positive root $s = 8$. Therefore $x(t)$ does not approach a constant value as $t \to \infty$ [this can be observed from the inverse transform, which is $x(t) = -1/4 + (37/4)e^{8t}$].

SOLVING EQUATIONS WITH THE LAPLACE TRANSFORM

We now show how to solve differential equations by using the Laplace transform. Consider the linear first-order equation

$$\dot{x} + ax = f(t) \tag{3.3.10}$$

where $f(t)$ is the input and a is a constant. If we multiply both sides of the equation by e^{-st} and then integrate over time from $t = 0$ to $t = \infty$, we obtain

$$\int_0^\infty (\dot{x} + ax) e^{-st} \, dt = \int_0^\infty f(t) e^{-st} \, dt$$

or

$$\mathcal{L}(\dot{x} + ax) = \mathcal{L}[f(t)]$$

Using the linearity property, this becomes

$$\mathcal{L}(\dot{x}) + a\mathcal{L}(x) = \mathcal{L}[f(t)]$$

Using the derivative property and the alternative transform notation, the above equation can be written as

$$sX(s) - x(0) + aX(s) = F(s) \tag{3.3.11}$$

This equation is an algebraic equation that can be solved for $X(s)$ in terms of $F(s)$ and $x(0)$. Its solution is

$$X(s) = \frac{x(0)}{s + a} + \frac{1}{s + a}F(s) \tag{3.3.12}$$

The denominator of the first term on the right side is $s + a$, which is the characteristic polynomial of the equation.

The inverse operation gives

$$x(t) = \mathcal{L}^{-1}\left[\frac{x(0)}{s + a}\right] + \mathcal{L}^{-1}\left[\frac{1}{s + a}F(s)\right] \tag{3.3.13}$$

This equation shows that the solution consists of the sum of two terms. The first term depends on the initial condition $x(0)$ but not on the forcing function $f(t)$. This part of the solution is the free response. From Table 3.3.1, entry 6, it is seen that the free response is $x(0)e^{-at}$.

The second term on the right side of (3.3.13) depends on the forcing function $f(t)$ but not on the initial condition $x(0)$. This part of the solution is forced response. It cannot be evaluated until $F(s)$ is available.

EXAMPLE 3.3.8

Step Response of a First-Order Equation

■ Problem
When the input is a step function, the response is sometimes called the *step response*. Suppose that the input $f(t)$ of the equation $\dot{x} + ax = f(t)$ is a step function of magnitude M whose transform is $F(s) = M/s$. Obtain the expression for the complete response.

■ Solution
From (3.3.13) the forced response is obtained from

$$\mathcal{L}^{-1}\left[\frac{1}{s + a}F(s)\right] = \mathcal{L}^{-1}\left(\frac{1}{s + a}\frac{M}{s}\right)$$

This transform can be converted into a sum of simple transforms as follows.

$$\frac{1}{s + a}\frac{M}{s} = \frac{C_1}{s} + \frac{C_2}{s + a} \tag{1}$$

To determine C_1 and C_2 we can use the *least common denominator (LCD)* $s(s + a)$ and write the expression as

$$\frac{1}{s + a}\frac{M}{s} = \frac{C_1(s + a) + C_2 s}{s(s + a)}$$

Comparing the numerators on the left and right sides, we see that this equation is true only if

$$M = C_1(s + a) + C_2 s = (C_1 + C_2)s + aC_1$$

for arbitrary values of s. This requires that $C_1 + C_2 = 0$ and $aC_1 = M$. Thus, $C_1 = M/a$ and $C_2 = -M/a$, and equation (1) becomes

$$\frac{1}{s+a}\frac{M}{s} = \frac{M}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

Thus, using entries 2 and 6 in Table 3.3.1, we see that the forced response is

$$\frac{M}{a}\left(1 - e^{-at}\right)$$

The addition of the free and the forced responses gives the complete response:

$$x(t) = x(0)e^{-at} + \frac{M}{a}\left(1 - e^{-at}\right) \tag{2}$$

The principle of superposition for a linear equation implies that the complete response is the sum of the free and the forced responses. With the Laplace transform approach the initial conditions are automatically accounted for and we can treat the free response separately from the forced response.

To solve a differential equation by using the Laplace transform, we must be able to obtain a function $x(t)$ from its transform $X(s)$. This process is called *inverting* the transform. In Example 3.3.8 we inverted the transform by expressing it as a sum of simple transforms that appear in our transform table. The algebra required to find the coefficients C_1 and C_2 is rather straightforward. This sum is called a *partial-fraction expansion*. The form of a partial-fraction expansion depends on the roots of the transform's denominator. These roots consist of the characteristic roots plus any roots introduced by the transform of the forcing function. When there are only a few roots, using the LCD method quickly produces a solution for the expansion's coefficients. The coefficients can also be determined by the general-purpose formulas, which are advantageous for some problems. These are discussed in Section 3.5.

A case requiring special attention occurs when there are repeated factors in the denominator of the transform. Example 3.3.9 shows how this case is handled.

Ramp Response of a First-Order Equation | **EXAMPLE 3.3.9**

■ **Problem**

Determine the complete response of the following model, which has a ramp input:

$$\dot{x} + 3x = 5t \qquad x(0) = 10$$

■ **Solution**

Applying the transform to the equation we obtain

$$sX(s) - x(0) + 3X(s) = \frac{5}{s^2}$$

Solve for $X(s)$.

$$X(s) = \frac{x(0)}{s+3} + \frac{5}{s^2(s+3)} = \frac{10}{s+3} + \frac{5}{s^2(s+3)}$$

[We could also have obtained this result directly by using (3.3.12) with $a = 3$ and $F(s) = 5/s^2$.] The free response is given by the first term on the right-hand side and is $10e^{-3t}$. To find the

Figure 3.3.4 Response for Example 3.3.9.
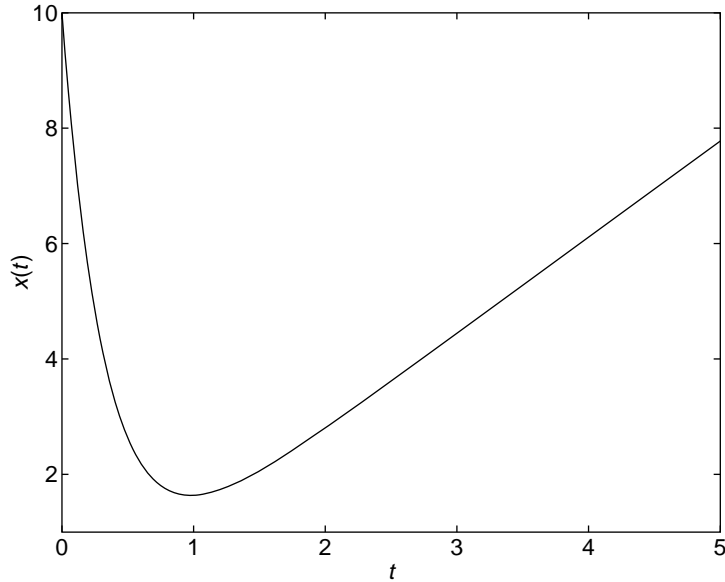
forced response, we express the second term on the right as

$$\frac{5}{s^2(s+3)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s+3}$$

As a general rule, the partial-fraction expansion must include one term for each distinct factor of the denominator; here, s and $s+3$. However, when there are repeated factors (here, the extra factor s), we must include additional terms, one for each extra factor that is repeated. We may now use the LCD method to obtain the coefficients C_1, C_2, and C_3.

$$\frac{5}{s^2(s+3)} = \frac{C_1(s+3) + C_2s(s+3) + C_3s^2}{s^2(s+3)} = \frac{(C_2+C_3)s^2 + (C_1+3C_2)s + 3C_1}{s^2(s+3)}$$

Comparing the numerators we see that $C_2 + C_3 = 0$, $C_1 + 3C_2 = 0$, and $3C_1 = 5$. Thus, $C_1 = 5/3$, $C_2 = -C_1/3 = -5/9$, and $C_3 = -C_2 = 5/9$. The forced response is

$$C_1t + C_2 + C_3e^{-3t} = \frac{5}{3}t - \frac{5}{9} + \frac{5}{9}e^{-3t}$$

The complete response is the sum of the free and the forced response, and is

$$x(t) = 10e^{-3t} + \frac{5}{3}t - \frac{5}{9} + \frac{5}{9}e^{-3t}$$

The plot of the response is shown in Figure 3.3.4. Because $e^{-3t} < 0.02$ for $t > 4/3$, the response for $t > 4/3$ is approximately given by $x(t) = 5t/3 - 5/9$, which is the equation of a straight line with slope $5/3$ and intercept $-5/9$.

Complex factors in the denominator of the transform can be often handled easily by using the fact that the complex conjugates $s = -a \pm bj$ correspond to the quadratic factor $(s+a)^2 + b^2$.

| Transform Inversion for Complex Factors | **EXAMPLE 3.3.10** |

■ Problem

Invert the following transform by representing it as the sum of terms that appear in Table 3.3.1:

$$X(s) = \frac{8s + 13}{s^2 + 4s + 53}$$

■ Solution

The roots of the denominator are $s = -2 \pm 7j$ and so the transform can be expressed as

$$X(s) = \frac{8s + 13}{(s + 2)^2 + 49}$$

We can express $X(s)$ as a sum of terms similar to entries 10 and 11 in Table 3.3.1, as follows (note that $a = 2$ and $b = 7$):

$$X(s) = \frac{8s + 13}{(s + 2)^2 + 49} = C_1 \frac{s + 2}{(s + 2)^2 + 49} + C_2 \frac{7}{(s + 2)^2 + 49} = \frac{C_1(s + 2) + 7C_2}{(s + 2)^2 + 49}$$

Comparing numerators, we see that

$$8s + 13 = C_1(s + 2) + 7C_2 = C_1 s + 2C_1 + 7C_2$$

This is true only if $C_1 = 8$ and $2C_1 + 7C_2 = 13$, or $C_2 = -3/7$. Thus,

$$x(t) = C_1 e^{-2t} \cos 7t + C_2 e^{-2t} \sin 7t = 8e^{-2t} \cos 7t - \frac{3}{7} e^{-2t} \sin 7t$$

| Step Response of a Second-Order Equation | **EXAMPLE 3.3.11** |

■ Problem

Obtain the complete response of the following model:

$$\ddot{x} + 4\dot{x} + 53x = 15u_s(t) \qquad x(0) = 8 \qquad \dot{x}(0) = -19$$

■ Solution

Transforming the equation gives

$$s^2 X(s) - sx(0) - \dot{x}(0) + 4[sX(s) - x(0)] + 53X(s) = \frac{15}{s}$$

Solve for $X(s)$ using the given initial conditions.

$$X(s) = \frac{x(0)s + \dot{x}(0) + 4x(0)}{s^2 + 4s + 53} + \frac{15}{s(s^2 + 4s + 53)}$$

$$= \frac{8s + 13}{s^2 + 4s + 53} + \frac{15}{s(s^2 + 4s + 53)} \qquad (1)$$

The first term on the right of equation (1) corresponds to the free response and is the same transform inverted in Example 3.3.10. Thus the free response is

$$8e^{-2t} \cos 7t - \frac{3}{7} e^{-2t} \sin 7t \qquad (2)$$

The second term on the right of equation (1) corresponds to the forced response. It can be expressed as follows:

$$\frac{15}{s[(s+2)^2 + 7^2]} = \frac{C_1}{s} + C_2 \frac{s+2}{(s+2)^2 + 7^2} + C_3 \frac{7}{(s+2)^2 + 7^2}$$

$$= \frac{C_1[(s+2)^2 + 7^2] + C_2 s(s+2) + 7C_3 s}{s[(s+2)^2 + 7^2]}$$

$$= \frac{(C_1 + C_2)s^2 + (4C_1 + 2C_2 + 7C_3)s + 53C_1}{s[(s+2)^2 + 7^2]}$$

Comparing numerators on the left and right sides, we see that $C_1 + C_2 = 0$, $4C_1 + 2C_2 + 7C_3 = 0$, and $53C_1 = 15$. Thus, $C_1 = 15/53$, $C_2 = -15/53$, and $C_3 = -30/371$, and the forced response is

$$\frac{15}{53} - \frac{15}{53} e^{-2t} \cos 7t - \frac{30}{371} e^{-2t} \sin 7t \tag{3}$$

The complete response is the sum of the free and forced responses given by equations (2) and (3):

$$x(t) = \frac{15}{53} + \frac{409}{53} e^{-2t} \cos 7t - \frac{27}{53} e^{-2t} \sin 7t \tag{4}$$

The response is plotted in Figure 3.3.5. The radian frequency of the oscillations is 7, which corresponds to a period of $2\pi/7 = 0.897$. The oscillations are difficult to see for $t > 2$ because $e^{-2t} < 0.02$ for $t > 2$. So for most practical purposes we may say that the response is essentially constant with a value $15/53$ for $t > 2$.

Alternatively, we could have combined the terms on the right side of equation (1) into a single term as follows:

$$X(s) = \frac{x(0)s^2 + [\dot{x}(0) + 4x(0)]s + 15}{s(s^2 + 4s + 53)} = \frac{8s^2 + 13s + 15}{s[(s+2)^2 + 7^2]}$$

Figure 3.3.5 Response for Example 3.3.11.
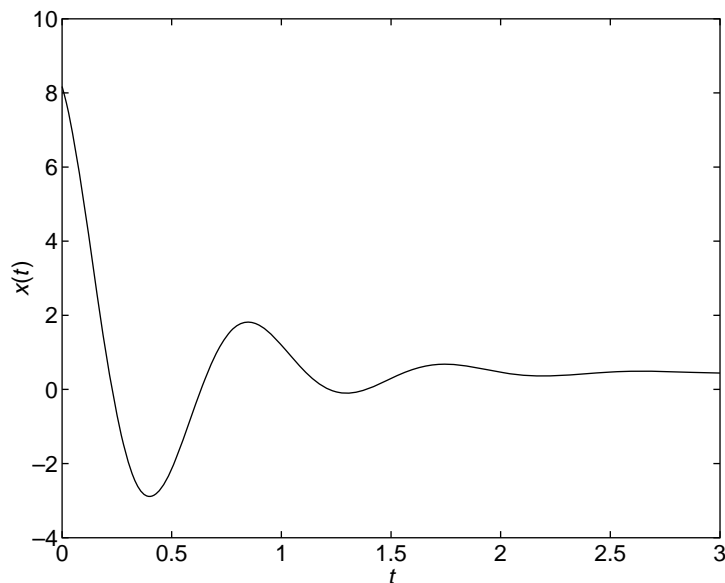
Inversion of this transform gives the complete response directly.

$$X(s) = \frac{8s^2 + 13s + 15}{s(s^2 + 4s + 53)} = \frac{C_1}{s} + C_2 \frac{s+2}{(s+2)^2 + 7^2} + C_3 \frac{7}{(s+2)^2 + 7^2}$$

$$= \frac{(C_1 + C_2)s^2 + (4C_1 + 2C_2 + 7C_3)s + 53C_1}{s[(s+2)^2 + 7^2]}$$

Comparing numerators on the left and right sides, we see that $C_1 + C_2 = 8$, $4C_1 + 2C_2 + 7C_3 = 13$, and $53C_1 = 15$. Thus, $C_1 = 15/53$, $C_2 = 409/53$, and $C_3 = -27/53$, and the corresponding response is identical to that given by equation (4), as it should be.

As Example 3.3.11 shows, the step response of a second-order equation with complex roots results in a transform of the following form:

$$\frac{As^2 + Bs + C}{s[(s+a)^2 + b^2]} = \frac{C_1}{s} + C_2 \frac{s+a}{(s+a)^2 + b^2} + C_3 \frac{b}{(s+a)^2 + b^2} \tag{3.3.14}$$

Using the same procedure employed in Example 3.3.11, we can show that the resulting coefficients are as follows:

$$C_1 = \frac{C}{a^2 + b^2} \qquad C_2 = A - C_1 \qquad C_3 = \frac{B - aA - aC_1}{b} \tag{3.3.15}$$

The response is

$$x(t) = C_1 + C_2 e^{-at} \cos bt + C_3 e^{-at} \sin bt \tag{3.3.16}$$

3.4 TRANSFER FUNCTIONS

The complete response of a linear ODE is the sum of the free and the forced responses. For zero initial conditions the free response is zero, and the complete response is the same as the forced response. Thus we can focus our analysis on the effects of the input only by taking the initial conditions to be zero temporarily. When we have finished analyzing the effects of the input, we can add to the result the free response due to any nonzero initial conditions.

The concept of the *transfer function* is useful for analyzing the effects of the input. Consider the model

$$\dot{x} + ax = f(t) \tag{3.4.1}$$

and assume that $x(0) = 0$. Transforming both sides of the equation gives

$$sX(s) + aX(s) = F(s)$$

Then solve for the ratio $X(s)/F(s)$ and denote it by $T(s)$:

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{s+a}$$

The function $T(s)$ is called the *transfer function* of (3.4.1).

The transfer function is the transform of the forced response divided by the transform of the input. It can be used as a multiplier to obtain the forced response transform from the input transform; that is, $X(s) = T(s)F(s)$. The transfer function is a property of the system model only. The transfer function is independent of the input function and the initial conditions.

The transfer function concept is extremely useful for several reasons.

1. *Transfer Functions and Software.* Software packages such as MATLAB do not accept system descriptions expressed as single, higher-order differential equations. Such software, however, does accept a description based on the transfer function. In Section 3.9 we will see how MATLAB does this. In Chapter 9 we will see that the transfer function is the basis for a graphical system description called the *block diagram,* and block diagrams are used to program the Simulink dynamic simulation software. So the transfer function is an important means of describing dynamic systems.

2. *ODE Equivalence.* It is important to realize that the transfer function is equivalent to the ODE. If we are given the transfer function we can reconstruct the corresponding ODE. For example, the transfer function

$$\frac{X(s)}{F(s)} = \frac{5}{s^2 + 7s + 10}$$

corresponds to the equation $\ddot{x} + 7\dot{x} + 10x = 5f(t)$. You should develop the ability to obtain transfer functions from ODEs and ODEs from transfer functions. This process is easily done because the initial conditions are assumed to be zero when working with transfer functions. From the derivative property, this means that to work with a transfer function you can use the relations $\mathcal{L}(\dot{x}) = sX(s)$, $\mathcal{L}(\ddot{x}) = s^2X(s)$, and so forth. Examples 3.4.1 and 3.4.2 will show how straightforward this process is.

3. *The Transfer Function and Characteristic Roots.* Note that the denominator of the transfer function is the characteristic polynomial, and thus the transfer function tells us something about the intrinsic behavior of the model, apart from the effects of the input and specific values of the initial conditions. In the previous equation, the characteristic polynomial is $s^2 + 7s + 10$ and the roots are -2 and -5. The roots are real, and this tells us that the free response does not oscillate and that the forced response does not oscillate unless the input is oscillatory. Because the roots are negative, the model is stable and its free response disappears with time.

MULTIPLE INPUTS AND OUTPUTS

Obtaining a transfer function from a single ODE is straightforward, as we have seen. Sometimes, however, models have more than one input or occur as sets of equations with more than one dependent variable. It is important to realize that there is one transfer function for each input-output pair. If a model has more than one input, a particular transfer function is the ratio of the output transform over the input transform, with all the remaining inputs ignored (set to zero temporarily).

EXAMPLE 3.4.1

Two Inputs and One Output

■ **Problem**

Obtain the transfer functions $X(s)/F(s)$ and $X(s)/G(s)$ for the following equation.

$$5\ddot{x} + 30\dot{x} + 40x = 6f(t) - 20g(t)$$

■ **Solution**

Using the derivative property with zero initial conditions, we can immediately write the equation as

$$5s^2 X(s) + 30s X(s) + 40X(s) = 6F(s) - 20G(s)$$

Solve for $X(s)$.

$$X(s) = \frac{6}{5s^2 + 30s + 40} F(s) - \frac{20}{5s^2 + 30s + 40} G(s)$$

When there is more than one input, the transfer function for a specific input can be obtained by temporarily setting the other inputs equal to zero (this is another aspect of the superposition property of linear equations). Thus, we obtain

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40} \qquad \frac{X(s)}{G(s)} = -\frac{20}{5s^2 + 30s + 40}$$

Note that the denominators of both transfer functions have the same roots: $s = -2$ and $s = -4$.

We can obtain transfer functions from systems of equations by first transforming the equations and then algebraically eliminating all variables except for the specified input and output. This technique is especially useful when we want to obtain the response of one or more of the dependent variables in the system of equations.

A System of Equations | **EXAMPLE 3.4.2**

■ **Problem**

a. Obtain the transfer functions $X(s)/V(s)$ and $Y(s)/V(s)$ of the following system of equations:

$$\dot{x} = -3x + 2y$$
$$\dot{y} = -9y - 4x + 3v(t)$$

b. Obtain the forced response for $x(t)$ and $y(t)$ if the input is $v(t) = 5u_s(t)$.

■ **Solution**

a. Here two outputs are specified, x and y, with one input, v. Thus there are two transfer functions. To obtain them, transform both sides of each equation, assuming zero initial conditions.

$$sX(s) = -3X(s) + 2Y(s)$$
$$sY(s) = -9Y(s) - 4X(s) + 3V(s)$$

These are two algebraic equations in the two unknowns, $X(s)$ and $Y(s)$. Solve the first equation for $Y(s)$:

$$Y(s) = \frac{s+3}{2} X(s) \qquad\qquad (1)$$

Substitute this into the second equation.

$$s\frac{s+3}{2} X(s) = -9\frac{s+3}{2} X(s) - 4X(s) + 3V(s)$$

Then solve for $X(s)/V(s)$ to obtain

$$\frac{X(s)}{V(s)} = \frac{6}{s^2 + 12s + 35} \tag{2}$$

Now substitute this into equation (1) to obtain

$$\frac{Y(s)}{V(s)} = \frac{s+3}{2}\frac{X(s)}{V(s)} = \frac{s+3}{2}\frac{6}{s^2 + 12s + 35} = \frac{3(s+3)}{s^2 + 12s + 35} \tag{3}$$

The desired transfer functions are given by equations (2) and (3). Note that denominators of both transfer functions have the same factors, $s = -5$ and $s = -7$, which are the roots of the characteristic equation: $s^2 + 12s + 35$.

b. From equation (2),

$$X(s) = \frac{6}{s^2 + 12s + 35}V(s) = \frac{6}{s^2 + 12s + 35}\frac{5}{s} = \frac{30}{s(s^2 + 12s + 35)}$$

The denominator factors are $s = 0$, $s = -5$, and $s = -7$, and thus the partial-fraction expansion is

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+5} + \frac{C_3}{s+7}$$

where $C_1 = 6/7$, $C_2 = -3$, and $C_3 = 15/7$. The forced response is

$$x(t) = \frac{6}{7} - 3e^{-5t} + \frac{15}{7}e^{-7t} \tag{4}$$

From (1) we have $y = (\dot{x} + 3x)/2$. From (4) we obtain

$$y(t) = \frac{9}{7} + 3e^{-5t} - \frac{30}{7}e^{-7t}$$

3.5 PARTIAL-FRACTION EXPANSION

To solve a differential equation by using the Laplace transform, we must be able to obtain a function $x(t)$ from its transform $X(s)$. This process is called *inverting* the transform. Unless the transform is a simple one appearing in the transform table, it will have to be represented as a combination of simple transforms.

The expansions in Section 3.3 are simple examples of partial-fraction expansions. In practice, however, we might encounter higher-order system models or complicated inputs. Both situations require a general approach to obtaining the expansion, and this section develops such an approach.

Most transforms occur in the form of a ratio of two polynomials, such as

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{3.5.1}$$

In all of our examples, $m \leq n$. If $X(s)$ is of the form (3.5.1), the method of partial-fraction expansion can be used. Note that we assume that the coefficient a_n is unity. If not, divide the numerator and denominator by a_n. The first step is to solve for the n roots of the denominator. If the a_i coefficients are real (as they will be for all our applications), any complex roots will occur in conjugate pairs.

There are two cases to be considered. The first is where all the roots are distinct; the second is where two or more roots are identical (repeated).

DISTINCT ROOTS CASE

If all the roots are distinct, we can express $X(s)$ in (3.5.1) in factored form as follows:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2) \cdots (s + r_n)} \tag{3.5.2}$$

where the roots are $s = -r_1, -r_2, \ldots, -r_n$. This form can be expanded as

$$X(s) = \frac{C_1}{s + r_1} + \frac{C_2}{s + r_2} + \cdots + \frac{C_n}{s + r_n} \tag{3.5.3}$$

where

$$C_i = \lim_{s \to -r_i} [X(s)(s + r_i)] \tag{3.5.4}$$

Multiplying by the factor $(s + r_i)$ cancels that term in the denominator before the limit is taken. This is a good way of remembering (3.5.4). Each factor corresponds to an exponential function of time, and the inverse transform is

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \cdots + C_n e^{-r_n t} \tag{3.5.5}$$

One Zero Root and One Negative Root | **EXAMPLE 3.5.1**

■ Problem
Obtain the inverse Laplace transform of

$$X(s) = \frac{5}{s(s + 3)}$$

■ Solution
The denominator roots are $s = 0$ and $s = -3$, which are distinct and real. Thus the partial-fraction expansion has the form

$$X(s) = \frac{5}{s(s + 3)} = \frac{C_1}{s} + \frac{C_2}{s + 3}$$

Using the coefficient formula (3.5.4), we obtain

$$C_1 = \lim_{s \to 0} \left[s \frac{5}{s(s + 3)} \right] = \lim_{s \to 0} \left[\frac{5}{(s + 3)} \right] = \frac{5}{3}$$

$$C_2 = \lim_{s \to -3} \left[(s + 3) \frac{5}{s(s + 3)} \right] = \lim_{s \to -3} \left(\frac{5}{s} \right) = -\frac{5}{3}$$

The inverse transform is

$$x(t) = C_1 + C_2 e^{-3t} = \frac{5}{3} - \frac{5}{3} e^{-3t}$$

A Third-Order Equation | **EXAMPLE 3.5.2**

■ Problem
Use two methods to obtain the solution of the following problem:

$$10 \frac{d^3 x}{dt^3} + 100 \frac{d^2 x}{dt^2} + 310 \frac{dx}{dt} + 300x = 750 u_s(t)$$

$$x(0) = 2 \qquad \dot{x}(0) = 4 \qquad \ddot{x}(0) = 3$$

■ Solution

a.   Using the Laplace transform method, we have

$$10\left[s^3 X(s) - \ddot{x}(0) - s\dot{x}(0) - s^2 x(0)\right] + 100\left[s^2 X(s) - \dot{x}(0) - sx(0)\right]$$

$$+ 310[sX(s) - x(0)] + 300X(s) = \frac{750}{s}$$

Solving for $X(s)$ using the given initial values we obtain

$$X(s) = \frac{2s^3 + 24s^2 + 105s + 75}{s(s^3 + 10s^2 + 31s + 30)} = \frac{2s^3 + 24s^2 + 105s + 75}{s(s+2)(s+3)(s+5)}$$

Since the roots of the cubic are $s = -2, -3$, and -5, the partial-fraction expansion is

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+3} + \frac{C_4}{s+5}$$

For this problem the LCD method requires a lot of algebra, and the coefficients can be more easily obtained from the formula (3.5.4). They are

$$C_1 = \lim_{s \to 0} sX(s) = \frac{5}{2}$$

$$C_2 = \lim_{s \to -2} (s+2)X(s) = \frac{55}{6}$$

$$C_3 = \lim_{s \to -3} (s+3)X(s) = -13$$

$$C_4 = \lim_{s \to -5} (s+5)X(s) = \frac{10}{3}$$

Thus, the answer is

$$x(t) = \frac{5}{2} + \frac{55}{6}e^{-2t} - 13e^{-3t} + \frac{10}{3}e^{-5t} \tag{1}$$

The plot of the response is shown in Figure 3.5.1. The response contains three exponentials. The terms e^{-3t} and e^{-5t} die out faster than e^{-2t}, so for $t > 4/3$, the response is approximately given by $x(t) = 5/2 + (55/6)e^{-2t}$. For $t > 2$, the response is

Figure 3.5.1 Response for Example 3.5.2.
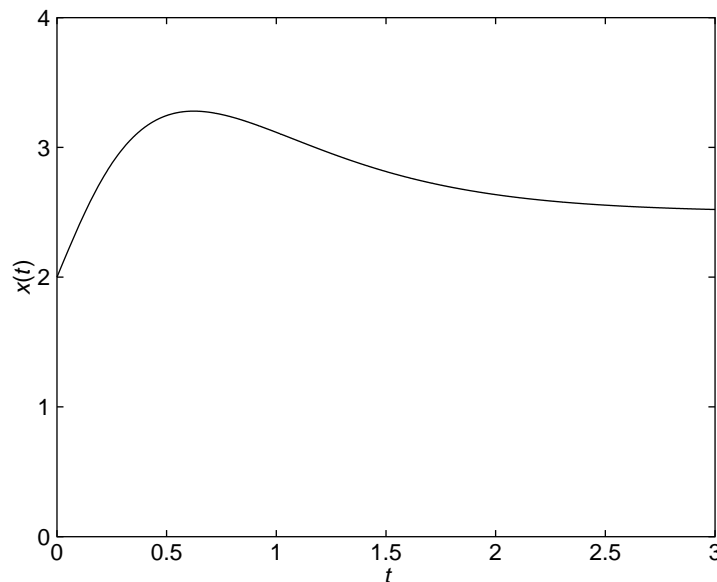
approximately constant at $x = 5/2$. The "hump" in the response is produced by the positive values of $\dot{x}(0)$ and $\ddot{x}(0)$.

b. From the ODE we can immediately obtain the characteristic equation $10s^3 + 100s^2 + 310s + 300 = 0$, which has the roots $s = -2, -3$, and -5. These roots will generate in the solution negative exponential functions that will approach zero as $t \to \infty$, leaving only a constant term produced by the step input. Therefore the form of the solution is

$$x(t) = C_1 + C_2 e^{-2t} + C_3 e^{-3t} + C_4 e^{-5t} \tag{2}$$

Because the time derivatives of x will approach zero as $t \to \infty$, the value of the constant term C_1 can be found by setting the derivatives equal to zero in the ODE and solving for x. This gives $x = 750/300 = 5/2$ and therefore $C_1 = 5/2$.

The values of the remaining coefficients can be found with the initial conditions by differentiating (2) to obtain

$$x(0) = C_1 + C_2 + C_3 + C_4 = \frac{5}{2} + C_2 + C_3 + C_4 = 2$$

$$\dot{x}(0) = -2C_2 - 3C_3 - 5C_4 = 4$$

$$\ddot{x}(0) = 4C_2 + 9C_3 + 25C_4 = 3$$

These are three equations in three unknowns. Their solution is $C_2 = 55/6$, $C_3 = -13$, and $C_4 = 10/3$. Thus the solution is given by equation (1).

The preferred method is a personal choice. The Laplace method requires some algebra to handle the initial conditions, whereas the second method requires that three simultaneous equations be solved.

Calculator Tip

Sometimes it is convenient to obtain the coefficients of a partial-fraction expansion by solving simultaneous linear algebraic equations. So now is a good time to learn how to use your calculator to solve such equations. Remember to arrange the unknowns in the equations in consistent order. For example, the three equations from the previous example are

$$C_2 + C_3 + C_4 = -0.5$$

$$-2C_2 - 3C_3 - 5C_4 = 4$$

$$4C_2 + 9C_3 + 25C_4 = 3$$

Determine how to enter the equation coefficients; usually for these equations you would first enter the coefficients of the first equation as 1, 1, 1, −0.5. Then enter the coefficients of the second equation as −2, −3, −5, 4, and enter the coefficients of the third equation in a similar way. Solve the preceding three equations as a test; the answer to four decimal places is $C_2 = 9.1666$, $C_3 = -13$, and $C_4 = 3.3333$, which is equivalent to the hand solution.

REPEATED-ROOTS CASE

Suppose that p of the roots have the same value $s = -r_1$, and the remaining $(n - p)$ roots are distinct and real. Then $X(s)$ is of the form

$$X(s) = \frac{N(s)}{(s + r_1)^p (s + r_{p+1})(s + r_{p+2}) \cdots (s + r_n)} \tag{3.5.6}$$

The expansion is

$$X(s) = \frac{C_1}{(s + r_1)^p} + \frac{C_2}{(s + r_1)^{p-1}} + \cdots + \frac{C_p}{s + r_1} + \cdots$$
$$+ \frac{C_{p+1}}{s + r_{p+1}} + \cdots + \frac{C_n}{s + r_n} \tag{3.5.7}$$

The coefficients for the repeated roots are found from

$$C_1 = \lim_{s \to -r_1} \left[X(s)(s + r_1)^p \right] \tag{3.5.8}$$

$$C_2 = \lim_{s \to -r_1} \left\{ \frac{d}{ds} \left[X(s)(s + r_1)^p \right] \right\} \tag{3.5.9}$$

$$\vdots$$

$$C_i = \lim_{s \to -r_1} \left\{ \frac{1}{(i - 1)!} \frac{d^{i-1}}{ds^{i-1}} \left[X(s)(s + r_1)^p \right] \right\} \qquad i = 1, 2, \ldots, p \tag{3.5.10}$$

The coefficients for the distinct roots are found from (3.5.4). The solution for the time function is

$$f(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \cdots + C_p e^{-r_1 t} + \cdots$$
$$+ C_{p+1} e^{-r_{p+1} t} + \cdots + C_n e^{-r_n t} \tag{3.5.11}$$

EXAMPLE 3.5.3

One Negative Root and Two Zero Roots

■ Problem
Compare two methods for obtaining the inverse Laplace transform of

$$X(s) = \frac{5}{s^2(3s + 12)}$$

■ Solution
The denominator roots are $s = -12/3 = -4, s = 0$, and $s = 0$. Thus the partial-fraction expansion has the form

$$X(s) = \frac{5}{s^2(3s + 12)} = \frac{1}{3} \frac{5}{s^2(s + 4)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s + 4}$$

Using the coefficient formulas (3.5.4), (3.5.8), and (3.5.9) with $p = 2$ and $r_1 = 0$, we obtain

$$C_1 = \lim_{s \to 0} \left[s^2 \frac{5}{3s^2(s + 4)} \right] = \lim_{s \to 0} \left[\frac{5}{3(s + 4)} \right] = \frac{5}{12}$$

$$C_2 = \lim_{s \to 0} \frac{d}{ds} \left[s^2 \frac{5}{3s^2(s + 4)} \right] = \lim_{s \to 0} \frac{d}{ds} \left[\frac{5}{3(s + 4)} \right] = \lim_{s \to 0} \left[-\frac{5}{3} \frac{1}{(s + 4)^2} \right] = -\frac{5}{48}$$

$$C_3 = \lim_{s \to -4} \left[(s + 4) \frac{5}{3s^2(s + 4)} \right] = \lim_{s \to -4} \left(\frac{5}{3s^2} \right) = \frac{5}{48}$$

With the LCD method we have

$$\frac{1}{3}\left[\frac{5}{s^2(s+4)}\right] = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s+4} = \frac{C_1(s+4) + C_2 s(s+4) + C_3 s^2}{s^2(s+4)}$$

$$= \frac{(C_2 + C_3)s^2 + (C_1 + 4C_2)s + 4C_1}{s^2(s+4)}$$

Comparing numerators, we see that $C_2 + C_3 = 0$, $C_1 + 4C_2 = 0$, and $4C_1 = 5/3$, which give $C_1 = 5/12$, $C_2 = -5/48$, and $C_3 = 5/48$.

The inverse transform is

$$x(t) = C_1 t + C_2 + C_3 e^{-4t} = \frac{5}{12}t - \frac{5}{48} + \frac{5}{48}e^{-4t}$$

In terms of effort required, for this example the two methods are roughly equivalent. For repeated roots, coefficient formula (3.5.9) requires that we obtain the derivative of a ratio of functions, but the LCD method requires three equations to be solved for three unknowns, although here the third equation is easily solved because it contains only one unknown.

| Ramp Response of a First-Order Model | **EXAMPLE 3.5.4** |

■ Problem
Use the Laplace transform to solve the following problem:

$$3\dot{x} + 12x = 5t \qquad x(0) = 0$$

■ Solution
Taking the transform of both sides of the equation, we obtain

$$3[sX(s) - x(0)] + 12X(s) = \frac{5}{s^2}$$

Solve for $X(s)$ using the given value of $x(0)$.

$$X(s) = \frac{5}{s^2(3s + 12)}$$

The partial-fraction expansion was obtained in Example 3.5.3. It is

$$X(s) = \frac{5}{12}\frac{1}{s^2} - \frac{5}{48}\frac{1}{s} + \frac{5}{48}\frac{1}{s+4}$$

and the inverse transform is

$$x(t) = \frac{5}{12}t - \frac{5}{48} + \frac{5}{48}e^{-4t}$$

| Two Repeated Roots and One Distinct Root | **EXAMPLE 3.5.5** |

■ Problem
Obtain the inverse Laplace transform of

$$X(s) = \frac{7}{(s+3)^2(s+5)}$$

■ Solution

The denominator roots are $s = -5$, $s = -3$, and $s = -3$. Thus, the partial-fraction expansion has the form

$$X(s) = \frac{7}{(s+3)^2(s+5)} = \frac{C_1}{(s+3)^2} + \frac{C_2}{s+3} + \frac{C_3}{s+5}$$

where

$$C_1 = \lim_{s \to -3} \left[(s+3)^2 \frac{7}{(s+3)^2(s+5)} \right] = \lim_{s \to -3} \left(\frac{7}{s+5} \right) = \frac{7}{2}$$

$$C_2 = \lim_{s \to -3} \frac{d}{ds} \left[(s+3)^2 \frac{7}{(s+3)^2(s+5)} \right] = \lim_{s \to -3} \frac{d}{ds} \left(\frac{7}{s+5} \right)$$

$$= \lim_{s \to -3} \left[\frac{-7}{(s+5)^2} \right] = -\frac{7}{4}$$

$$C_3 = \lim_{s \to -5} \left[(s+5) \frac{7}{(s+3)^2(s+5)} \right] = \lim_{s \to -5} \left[\frac{7}{(s+3)^2} \right] = \frac{7}{4}$$

The inverse transform is

$$x(t) = C_1 t e^{-3t} + C_2 e^{-3t} + C_3 e^{-5t} = \frac{7}{2} t e^{-3t} - \frac{7}{4} e^{-3t} + \frac{7}{4} e^{-5t}$$

For this example the LCD method would be the more difficult method because it requires more algebra and the solution of the following equations: $C_2 + C_3 = 0$, $C_1 + 8C_2 + 6C_3 = 0$, and $5C_1 + 15C_2 + 9C_3 = 7$.

EXAMPLE 3.5.6

Exponential Response of a First-Order Model

■ Problem

Use the Laplace transform to solve the following problem.

$$\dot{x} + 5x = 7te^{-3t} \qquad x(0) = 0$$

■ Solution

Taking the transform of both sides of the equation, we obtain

$$sX(s) - x(0) + 5X(s) = \frac{7}{(s+3)^2}$$

Solve for $X(s)$ using the given value of $x(0)$.

$$X(s) = \frac{7}{(s+3)^2(s+5)}$$

The partial-fraction expansion was obtained in Example 3.5.5. It is

$$X(s) = \frac{7}{2(s+3)^2} - \frac{7}{4(s+3)} + \frac{7}{4(s+5)}$$

and the inverse transform is

$$x(t) = \frac{7}{2} t e^{-3t} - \frac{7}{4} e^{-3t} + \frac{7}{4} e^{-5t}$$

The plot of the response is shown in Figure 3.5.2. The "hump" in the response is caused by the multiplicative factor of t in the input $7te^{-3t}$.
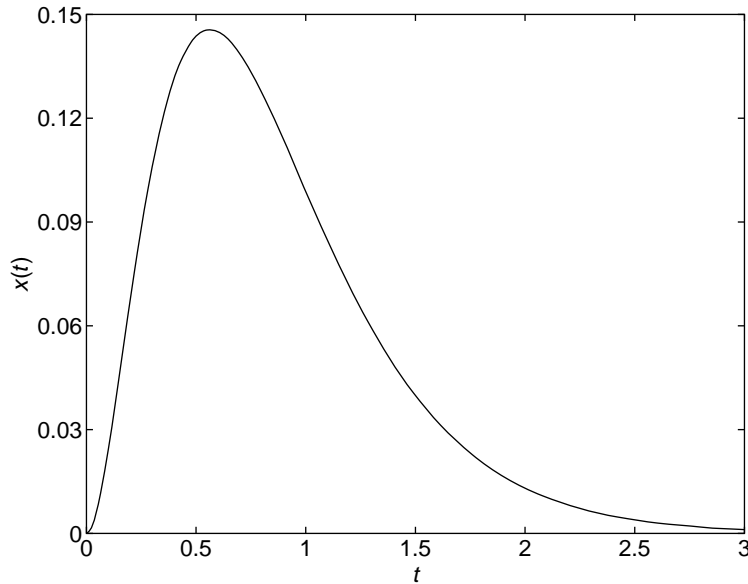
Figure 3.5.2 Response for Example 3.5.6.

Four Repeated Roots | **EXAMPLE 3.5.7**

■ **Problem**

Choose the most convenient method for obtaining the inverse transform of

$$X(s) = \frac{s^2 + 2}{s^4(s + 1)}$$

■ **Solution**

There are four repeated roots ($s = 0$) and one distinct root, so the expansion is

$$X(s) = \frac{C_1}{s^4} + \frac{C_2}{s^3} + \frac{C_3}{s^2} + \frac{C_4}{s} + \frac{C_5}{s + 1}$$

Because there are four repeated roots, use of (3.5.10) to find the coefficients would require taking the first, second, and third derivatives of the ratio $(s^2 + 2)/(s + 1)$. Therefore the LCD method is easier to use for this problem. Using the LCD we obtain

$$X(s) = \frac{C_1(s + 1) + C_2 s(s + 1) + C_3 s^2(s + 1) + C_4 s^3(s + 1) + C_5 s^4}{s^4(s + 1)}$$

$$= \frac{(C_5 + C_4)s^4 + (C_4 + C_3)s^3 + (C_3 + C_2)s^2 + (C_2 + C_1)s + C_1}{s^4(s + 1)}$$

Comparing numerators we see that

$$s^2 + 2 = (C_5 + C_4)s^4 + (C_4 + C_3)s^3 + (C_3 + C_2)s^2 + (C_2 + C_1)s + C_1$$

and thus $C_1 = 2$, $C_2 + C_1 = 0$, $C_3 + C_2 = 1$, $C_4 + C_3 = 0$, and $C_5 + C_4 = 0$. These give $C_1 = 2$, $C_2 = -2$, $C_3 = 3$, $C_4 = -3$, and $C_5 = 3$. So the expansion is

$$X(s) = \frac{2}{s^4} - \frac{2}{s^3} + \frac{3}{s^2} - \frac{3}{s} + \frac{3}{s + 1}$$

The inverse transform is

$$x(t) = \frac{1}{3}t^3 - t^2 + 3t - 3 + 3e^{-t}$$

COMPLEX ROOTS

When some of the roots of the transform denominator are complex, the expansion has the same form as (3.5.3), because the roots are in fact distinct. Thus, the coefficients C_i can be found from (3.5.4). However, these coefficients will be complex numbers, and the form of the inverse transform given by (3.5.5) will not be convenient to use. We now demonstrate two methods that can be used; the choice depends on whether or not you want to avoid the use of complex-valued coefficients.

EXAMPLE 3.5.8

Two Complex Roots

■ Problem
Use two methods to obtain the inverse Laplace transform of

$$X(s) = \frac{3s + 7}{4s^2 + 24s + 136} = \frac{3s + 7}{4(s^2 + 6s + 34)}$$

■ Solution
a.   The denominator roots are $s = -3 \pm 5j$. To avoid complex-valued coefficients, we note that the denominator of $X(s)$ can be written as $(s + 3)^2 + 5^2$, and we can express $X(s)$ as follows:

$$X(s) = \frac{1}{4}\left[\frac{3s + 7}{(s + 3)^2 + 5^2}\right] \tag{1}$$

which can be expressed as the sum of two terms that are proportional to entries 10 and 11 in Table 3.3.1.

$$X(s) = \frac{1}{4}\left[C_1\frac{5}{(s + 3)^2 + 5^2} + C_2\frac{s + 3}{(s + 3)^2 + 5^2}\right]$$

We can obtain the coefficients by noting that

$$C_1\frac{5}{(s + 3)^2 + 5^2} + C_2\frac{s + 3}{(s + 3)^2 + 5^2} = \frac{5C_1 + C_2(s + 3)}{(s + 3)^2 + 5^2} \tag{2}$$

Comparing the numerators of equations (1) and (2), we see that

$$5C_1 + C_2(s + 3) = C_2 s + 5C_1 + 3C_2 = 3s + 7$$

which gives $C_2 = 3$ and $5C_1 + 3C_2 = 7$. Thus $C_1 = -2/5$. The inverse transform is

$$x(t) = \frac{1}{4}C_1 e^{-3t}\sin 5t + \frac{1}{4}C_2 e^{-3t}\cos 5t = -\frac{1}{10}e^{-3t}\sin 5t + \frac{3}{4}e^{-3t}\cos 5t$$

b.   The denominator roots are distinct and the expansion (3.5.4) gives

$$X(s) = \frac{3s + 7}{4s^2 + 24s + 136} = \frac{3s + 7}{4(s + 3 - 5j)(s + 3 + 5j)}$$

$$= \frac{C_1}{s + 3 - 5j} + \frac{C_2}{s + 3 + 5j}$$

where, from (3.5.4),

$$C_1 = \lim_{s \to -3+5j} (s + 3 - 5j)X(s) = \lim_{s \to -3+5j} \frac{3s + 7}{4(s + 3 + 5j)}$$

$$= \frac{-2 + 15j}{40j} = \frac{15 + 2j}{40}$$

This can be expressed in complex exponential form as follows (see Table 3.3.2):

$$C_1 = |C_1|e^{j\phi} = \left| \frac{15 + 2j}{40} \right| e^{j\phi} = \frac{\sqrt{229}}{40} e^{j\phi}$$

where $\phi = \tan^{-1}(2/15) = 0.1326$ rad.

The second coefficient is

$$C_2 = \lim_{s \to -3-5j} (s + 3 + 5j)X(s) = \lim_{s \to -3-5j} \frac{3s + 7}{4(s + 3 - 5j)}$$

$$= \frac{2 + 15j}{40j} = \frac{15 - 2j}{40}$$

Note that C_1 and C_2 are complex conjugates. This will always be the case for coefficients of complex-conjugate roots in a partial-fraction expansion. Thus, $C_2 = |C_1|e^{-j\phi} = \sqrt{229}e^{-0.1326j}/40$.

The inverse transform gives

$$x(t) = C_1 e^{(-3+5j)t} + C_2 e^{(-3-5j)t} = C_1 e^{-3t} e^{5jt} + C_2 e^{-3t} e^{-5jt}$$

$$= |C_1|e^{-3t} \left[e^{(5t+\phi)j} + e^{-(5t+\phi)j} \right] = 2|C_1|e^{-3t} \cos(5t + \phi)$$

where we have used the relation $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, which can be derived from the Euler identity (Table 3.3.2). Thus

$$x(t) = \frac{\sqrt{229}}{20} e^{-3t} \cos(5t + 0.1326)$$

This answer is equivalent to that found in part (a), as can be seen by applying the trigonometric identity $\cos(5t + \phi) = \cos 5t \cos \phi - \sin 5t \sin \phi$.

Free Response of a Second-Order Model with Complex Roots	**EXAMPLE 3.5.9**

■ **Problem**

Use the Laplace transform to solve the following problem:

$$4\ddot{x} + 24\dot{x} + 136x = 0 \qquad x(0) = \frac{7}{4} \qquad \dot{x}(0) = -\frac{35}{4}$$

■ **Solution**

Taking the transform of both sides of the equation, we obtain

$$4[s^2 X(s) - x(0)s - \dot{x}(0)] + 24[sX(s) - x(0)] + 136X(s) = 0$$

Solve for $X(s)$ using the given values of $x(0)$ and $\dot{x}(0)$.

$$X(s) = \frac{4[x(0)s + \dot{x}(0)] + 24x(0)}{4s^2 + 24s + 136} = \frac{3s + 7}{4(s^2 + 6s + 34)}$$

The expansion was obtained in Example 3.5.8. It is

$$X(s) = -\frac{1}{10}\left[\frac{5}{(s+3)^2 + 5^2}\right] + \frac{3}{4}\left[\frac{s+3}{(s+3)^2 + 5^2}\right]$$

and the inverse transform is

$$f(t) = -\frac{1}{10}e^{-3t}\sin 5t + \frac{3}{4}e^{-3t}\cos 5t$$

3.6 THE IMPULSE AND NUMERATOR DYNAMICS

In our development of the Laplace transform and its associated methods, we have assumed that the process under study starts at time $t = 0$. Thus the given initial conditions, say $x(0), \dot{x}(0), \ldots$, represent the situation at the start of the process and are the result of any inputs applied prior to $t = 0$. That is, we need not know what the inputs were before $t = 0$ because their effects are contained in the initial conditions.

The effects of any inputs starting at $t = 0$ are not felt by the system until an infinitesimal time later, at $t = 0+$. For the models we have seen thus far, the dependent variable $x(t)$ and its derivatives do not change between $t = 0$ and $t = 0+$, and thus the solution $x(t)$ obtained from the differential equation will match the given initial conditions when $x(t)$ and its derivatives are evaluated at $t = 0$. The results obtained from the initial value theorem will also match the given initial conditions.

However, we will now investigate the behavior of some systems for which $x(0) \neq x(0+)$, or $\dot{x}(0) \neq \dot{x}(0+)$, and so forth for higher derivatives. The initial value theorem gives the value at $t = 0+$, which for some models is not necessarily equal to the value at $t = 0$. In these cases the solution of the differential equation is correct only for $t > 0$. This phenomenon occurs in models having impulse inputs and in models containing derivatives of a discontinuous input, such as a step function.

THE IMPULSE

An input that changes at a constant rate is modeled by the ramp function. The step function models an input that rapidly reaches a constant value, while the rectangular pulse function models a constant input that is suddenly removed. The *impulse* is similar to the pulse function, but it models an input that is suddenly applied and removed after a *very short* time. The impulse, which is a mathematical function only and has no physical counterpart, has an infinite magnitude for an infinitesimal time.

Consider the rectangular pulse function shown in Figure 3.6.1a. Its transform is $M(1 - e^{-sD})/s$. The area A under the pulse is $A = MD$ and is called the *strength* of the pulse. If we let this area remain constant at the value A and let the pulse duration D approach zero, we obtain the impulse, represented in Figure 3.6.1b. Because $M = A/D$,

Figure 3.6.1 (a) The rectangular pulse. (b) The impulse.
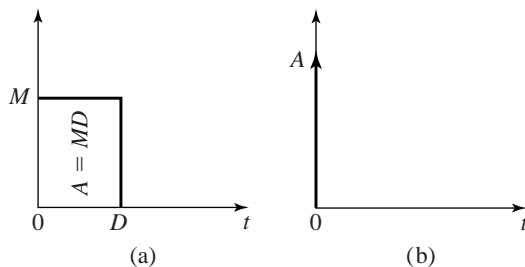
the transform $F(s)$ is

$$F(s) = \lim_{D \to 0} \frac{A}{D} \frac{1 - e^{-sD}}{s} = \lim_{D \to 0} \frac{Ase^{-sD}}{s} = A$$

after using L'Hopital's limit rule. If the strength $A = 1$, the function is called a *unit impulse*.

The unit impulse, called the *Dirac delta function* $\delta(t)$ in mathematics literature, often appears in the analysis of dynamic systems. It is an analytically convenient approximation of an input applied for only a very short time, such as when a high-speed object strikes a stationary object. The impulse is also useful for estimating the system's parameters experimentally and for analyzing the effect of differentiating a step or any other discontinuous input function.

In keeping with our interpretation of the initial conditions, we consider the impulse $\delta(t)$ to start at time $t = 0$ and finish at $t = 0+$, with its effects first felt at $t = 0+$.

<div style="text-align:center">Impulse Response of a Simple First-Order Model</div> | **EXAMPLE 3.6.1**

■ **Problem**
Obtain the unit-impulse response of the following model in two ways: (a) by separation of variables and (b) with the Laplace transform. The initial condition is $x(0) = 3$. What is the value of $x(0+)$?

$$\dot{x} = \delta(t)$$

■ **Solution**
a. Integrate both sides of the equation to obtain

$$\int_{x(0)}^{x(t)} \dot{x}(t)\, dx = \int_0^t \delta(t)\, dt = 1$$

because the area under a unit impulse is 1. This gives

$$x(t) - x(0) = 1 \qquad \text{or} \qquad x(t) = x(0) + 1 = 3 + 1 = 4$$

This is the solution for $t > 0$ but not for $t = 0$. Thus, $x(0+) = 4$ but $x(0) = 3$, so the impulse has changed $x(t)$ instantaneously from 3 to 4.

b. The transformed equation is

$$sX(s) - x(0) = 1 \qquad \text{or} \qquad X(s) = \frac{1 + x(0)}{s}$$

which gives the solution $x(t) = 1 + x(0) = 4$. Note that the initial value used with the derivative property is the value of x at $t = 0$.

The initial value theorem gives

$$x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s \frac{1 + x(0)}{s} = 1 + 3 = 4$$

which is correct.

| EXAMPLE 3.6.2 | Impulse Response of a First-Order Model |

■ Problem
Obtain the unit-impulse response of the following model. The initial condition is $x(0) = 0$. What is the value of $x(0+)$?

$$\frac{X(s)}{F(s)} = \frac{1}{s+5}$$

■ Solution
Because $f(t) = \delta(t)$, $F(s) = 1$, and the response is obtained from

$$X(s) = \frac{1}{s+5}F(s) = \frac{1}{s+5}$$

The response is $x(t) = e^{-5t}$ for $t > 0$. This gives

$$x(0+) = \lim_{t \to 0+} x(t) = \lim_{t \to 0+} e^{-5t} = 1$$

So the impulse input has changed x from 0 at $t = 0$ to 1 at $t = 0+$. This same result could have been obtained from the initial value theorem:

$$x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\frac{1}{s+5} = 1$$

| EXAMPLE 3.6.3 | Impulse Response of a Simple Second-Order Model |

■ Problem
Obtain the unit-impulse response of the following model in two ways: (a) by separation of variables and (b) with the Laplace transform. The initial conditions are $x(0) = 5$ and $\dot{x}(0) = 10$. What are the values of $x(0+)$ and $\dot{x}(0+)$?

$$\ddot{x} = \delta(t) \tag{1}$$

■ Solution
a. Let $v(t) = \dot{x}(t)$. Then the equation (1) becomes $\dot{v} = \delta(t)$, which can be integrated to obtain $v(t) = v(0) + 1 = 10 + 1 = 11$. Thus $\dot{x}(0+) = 11$ and is not equal to $\dot{x}(0)$.

Now integrate $\dot{x} = v = 11$ to obtain $x(t) = x(0) + 11t = 5 + 11t$. Thus, $x(0+) = 5$ which is the same as $x(0)$.

So for this model the unit-impulse input changes $\dot{x}$ from $t = 0$ to $t = 0+$ but does not change x.

b. The transformed equation is

$$s^2 X(s) - sx(0) - \dot{x}(0) = 1$$

or

$$X(s) = \frac{sx(0) + \dot{x}(0) + 1}{s^2} = \frac{5s + 11}{s^2} = \frac{5}{s} + \frac{11}{s^2}$$

which gives the solution $x(t) = 5 + 11t$ and $\dot{x}(t) = 11$. Note that the initial values used with the derivative property are the values at $t = 0$.

The initial value theorem gives

$$x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\frac{5s + 11}{s^2} = 5$$

and because $\mathcal{L}(\dot{x}) = sX(s) - x(0)$,

$$\dot{x}(0+) = \lim_{s \to \infty} s[sX(s) - x(0)] = \lim_{s \to \infty} s\left(\frac{5s + 11}{s} - 5\right) = 11$$

as we found in part (a).

| Impulse Response of a Second-Order Model | **EXAMPLE 3.6.4** |

■ **Problem**

Obtain the unit-impulse response of the following model. The initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$. What are the values of $x(0+)$ and $\dot{x}(0+)$?

$$\frac{X(s)}{F(s)} = \frac{1}{2s^2 + 14s + 20}$$

■ **Solution**

Because $F(s) = 1$, the response is obtained from

$$X(s) = \frac{1}{2s^2 + 14s + 20}F(s) = \frac{1}{2s^2 + 14s + 20} = \frac{1}{6}\frac{1}{s + 2} - \frac{1}{6}\frac{1}{s + 5}$$

and the response is $x(t) = (e^{-2t} - e^{-5t})/6$. This gives

$$x(0+) = \lim_{t \to 0+} x(t) = \lim_{t \to 0+}\left(\frac{e^{-2t} - e^{-5t}}{6}\right) = 0$$

and

$$\dot{x}(0+) = \lim_{t \to 0+} \dot{x}(t) = \lim_{t \to 0+}\left(\frac{-2e^{-2t} + 5e^{-5t}}{6}\right) = \frac{1}{2}$$

So the impulse input has not changed x between $t = 0$ and $t = 0+$ but has changed $\dot{x}$ from 0 to $1/2$. These results could have been obtained from the initial value theorem:

$$x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\frac{1}{2s^2 + 14s + 20} = 0$$

and, noting that $x(0) = 0$,

$$\dot{x}(0+) = \lim_{s \to \infty} s[sX(s) - x(0)] = \lim_{s \to \infty} s\frac{s}{2s^2 + 14s + 20} = \frac{1}{2}$$

In summary, be aware that the solution $x(t)$ and its derivatives $\dot{x}(t)$, $\ddot{x}(t)$, ... will match the given initial conditions at $t = 0$ only if there are no impulse inputs and no derivatives of inputs that are discontinuous at $t = 0$.

If $X(s)$ is a rational function and if the degree of the numerator of $X(s)$ is less than the degree of the denominator, then the initial value theorem will give a finite value for $x(0+)$. If the degrees are equal, then initial value is undefined and the initial value theorem is invalid. The latter situation corresponds to an impulse in $x(t)$ at $t = 0$ and therefore $x(0+)$ is undefined. When the degrees are equal the transform can be expressed as a constant plus a partial-fraction expansion. For example, consider the transform

$$X(s) = \frac{9s + 4}{s + 3} = 9 - \frac{23}{s + 3}$$

The inverse transform is $x(t) = 9\delta(t) - 23e^{-3t}$ and therefore $x(0+)$ is undefined.

NUMERATOR DYNAMICS

The following model contains a derivative of the input $g(t)$:

$$5\dot{x} + 10x = 2\dot{g}(t) + 10g(t)$$

Its transfer function is

$$\frac{X(s)}{G(s)} = \frac{2s + 10}{5s + 10}$$

Note that the input derivative $\dot{g}(t)$ results in an s term in the numerator of the transfer function, and such a model is said to have *numerator dynamics*. So a model with input derivatives has numerator dynamics, and vice versa, and thus the two terms describe the same condition.

With such models we must proceed carefully if the input is discontinuous, as is the case with the step function, because the input derivative produces an impulse when acting on a discontinuous input. To help you understand this, we state without rigorous proof that the unit impulse $\delta(t)$ is the time-derivative of the unit-step function $u_s(t)$; that is,

$$\delta(t) = \frac{d}{dt} u_s(t) \tag{3.6.1}$$

This result does not contradict common sense, because the step function changes from 0 at $t = 0$ to 1 at $t = 0+$ in an infinitesimal amount of time. Therefore its derivative should be infinite during this time. To further indicate the correctness of this relation, we integrate both sides and note that the area under the unit impulse is unity. Thus,

$$\int_0^{0+} \delta(t)\, dt = \int_0^{0+} \frac{d}{dt} u_s(t)\, dt = u_s(0+) - u_s(0) = 1 - 0 = 1$$

which gives $1 = 1$.

Thus an input derivative will create an impulse in response to a step input. For example, consider the model

$$5\dot{x} + 10x = 2\dot{g}(t) + 10g(t)$$

If the input $g(t) = u_s(t)$, the model is equivalent to

$$5\dot{x} + 10x = 2\delta(t) + 10u_s(t)$$

which has an impulse input.

Numerator dynamics can significantly alter the response, and the Laplace transform is a convenient and powerful tool for analyzing models having numerator dynamics.

EXAMPLE 3.6.5 | A First-Order Model with Numerator Dynamics

■ **Problem**

Obtain the transfer function and investigate the response of the following model in terms of the parameter a. The input $g(t)$ is a unit-step function.

$$5\dot{x} + 10x = a\dot{g}(t) + 10g(t) \qquad x(0) = 0$$
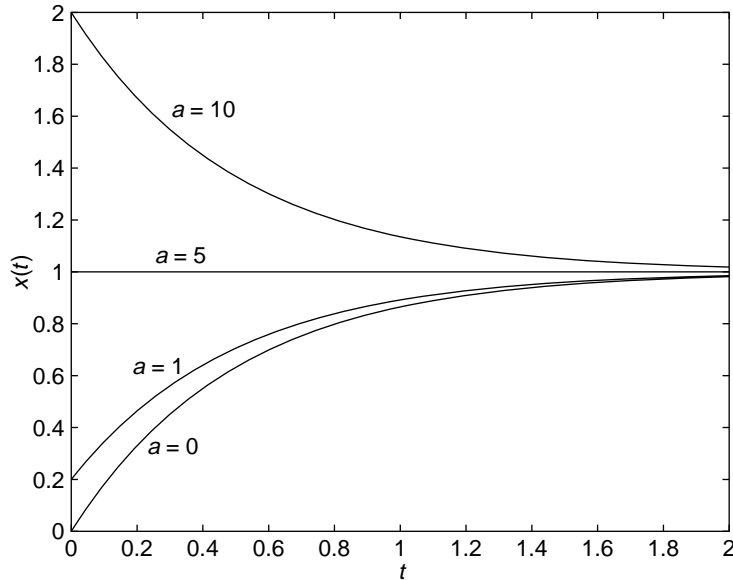
Figure 3.6.2 Plot of the response for Example 3.6.5.

■ **Solution**

Transforming the equation with $x(0) = 0$ and solving for the ratio $X(s)/G(s)$ gives the transfer function:

$$\frac{X(s)}{G(s)} = \frac{as + 10}{5s + 10}$$

Note that the model has numerator dynamics if $a \neq 0$.

For a unit-step input, $G(s) = 1/s$ and

$$X(s) = \frac{as + 10}{s(5s + 10)} = \frac{1}{s} + \frac{a - 5}{5} \frac{1}{s + 2}$$

Thus the response is

$$x(t) = 1 + \frac{a - 5}{5} e^{-2t}$$

From this solution or the initial-value theorem we find that $x(0+) = a/5$, which is not equal to $x(0)$ unless $a = 0$ (which corresponds to the absence of numerator dynamics). The plot of the response is given in Figure 3.6.2 for several values of a. The initial condition is different for each case, but for all cases the response is essentially constant for $t > 2$ because of the term e^{-2t}.

A Second-Order Model with Numerator Dynamics | **EXAMPLE 3.6.6**

■ **Problem**

Obtain the transfer function and investigate the response of the following model in terms of the parameter a. The input $g(t)$ is a unit-step function.

$$3\ddot{x} + 18\dot{x} + 24x = a\dot{g}(t) + 6g(t) \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

■ **Solution**

Transforming the equation with zero initial conditions and solving for the ratio $X(s)/G(s)$ gives the transfer function:

$$\frac{X(s)}{G(s)} = \frac{as + 6}{3s^2 + 18s + 24}$$

Note that the model has numerator dynamics if $a \neq 0$.

Figure 3.6.3 Plot of the
response for Example 3.6.6.
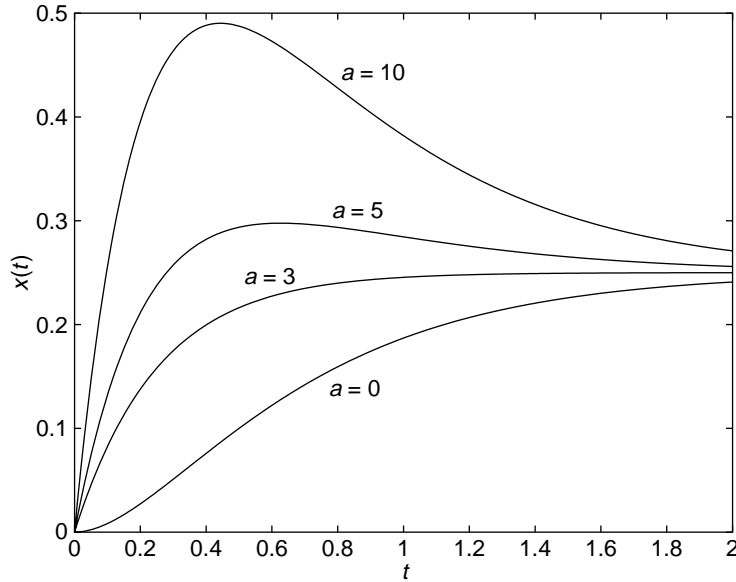
For a unit-step input, $G(s) = 1/s$ and

$$X(s) = \frac{as + 6}{s(3s^2 + 18s + 24)} = \frac{1}{4}\frac{1}{s} + \frac{a-3}{6}\frac{1}{s+2} + \frac{3-2a}{12}\frac{1}{s+4}$$

Thus the response is

$$x(t) = \frac{1}{4} + \frac{a-3}{6}e^{-2t} + \frac{3-2a}{12}e^{-4t}$$

From this solution or the initial-value theorem we find that $x(0+) = 0$, which is equal to $x(0)$, and that $\dot{x}(0+) = a/3$, which is not equal to $\dot{x}(0)$ unless $a = 0$ (which corresponds to the absence of numerator dynamics). The plot of the response is given in Figure 3.6.3 for several values of a. Notice that a "hump" in the response (called an "overshoot") does not occur for smaller values of a and the height of the hump increases as a increases. However, the value of a does not affect the steady-state response.

3.7 ADDITIONAL EXAMPLES

This section contains additional examples of solving dynamic models.

EXAMPLE 3.7.1

Transform of $A\sin(\omega t + \phi)$

■ Problem
(a) Derive the Laplace transform of the function $A\sin(\omega t + \phi)$. (b) Generalize the answer from part (a) to find the Laplace transform of $Ae^{-at}\sin(\omega t + \phi)$.

■ Solution
a.   Because we already have the transforms of $\sin\omega t$ and $\cos\omega t$, we can use the following trigonometric identity to obtain our answer:

$$\sin(\omega t + \phi) = \sin\omega t \cos\phi + \cos\omega t \sin\phi$$

From the linearity property of the transform,

$$\mathcal{L}[A \sin(\omega t + \phi)] = A \int_0^\infty \sin \omega t \cos \phi e^{-st}\, dt + A \int_0^\infty \cos \omega t \sin \phi e^{-st}\, dt$$

$$= A \cos \phi \mathcal{L}(\sin \omega t) + A \sin \phi \mathcal{L}(\cos \omega t)$$

$$= A \cos \phi \frac{\omega}{s^2 + \omega^2} + A \sin \phi \frac{s}{s^2 + \omega^2}$$

Combining these terms gives the answer:

$$\mathcal{L}[A \sin(\omega t + \phi)] = A \frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2}$$

b. Following the same procedure and using the fact that

$$\mathcal{L}\left(e^{-at} \sin \omega t\right) = \frac{\omega}{(s + a)^2 + \omega^2} \qquad \mathcal{L}\left(e^{-at} \cos \omega t\right) = \frac{s + a}{(s + a)^2 + \omega^2}$$

we see that

$$\mathcal{L}\left[A e^{-at} \sin(\omega t + \phi)\right] = A \cos \phi \mathcal{L}\left(e^{-at} \sin \omega t\right) + A \sin \phi \mathcal{L}\left(e^{-at} \cos \omega t\right)$$

$$= A \cos \phi \frac{\omega}{(s + a)^2 + \omega^2} + A \sin \phi \frac{s + a}{(s + a)^2 + \omega^2}$$

Combining these terms gives the answer:

$$\mathcal{L}\left[A e^{-at} \sin(\omega t + \phi)\right] = A \frac{s \sin \phi + a \sin \phi + \omega \cos \phi}{(s + a)^2 + \omega^2}$$

Response in the Form $A \sin(\omega t + \phi)$ | **EXAMPLE 3.7.2**

■ **Problem**

Given that

$$F(s) = \frac{3s + 10}{s^2 + 16}$$

obtain $f(t)$ in the form $f(t) = A \sin(\omega t + \phi)$, where $A > 0$.

■ **Solution**

From the results of part (a) of Example 3.7.1, we see that the form $f(t) = A \sin(\omega t + \phi)$ has the transform

$$A \frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2}$$

Comparing this with $F(s)$ we see that $\omega = 4$ and $A(s \sin \phi + \omega \cos \phi) = 3s + 10$. Therefore, $A \sin \phi = 3$ and $A\omega \cos \phi = 4A \cos \phi = 10$, or

$$\sin \phi = \frac{3}{A} \qquad \cos \phi = \frac{10}{4A} \qquad (1)$$

Because $A > 0$, these equations reveal that $\sin \phi > 0$ and $\cos \phi > 0$. Therefore, ϕ is in the first quadrant ($0 \le \phi \le \pi/2$ rad) and can be computed as follows:

$$\phi = \tan^{-1} \frac{\sin \phi}{\cos \phi} = \tan^{-1} \frac{3/A}{10/4A} = \tan^{-1} \frac{3}{10/4} = 0.876 \text{ rad}$$

To find A, we use the fact that $\sin^2 \phi + \cos^2 \phi = 1$ for *any* angle ϕ. From equations (1) we have

$$\sin^2 \phi + \cos^2 \phi = \left(\frac{3}{A}\right)^2 + \left(\frac{10}{4A}\right)^2 = 1$$

This gives $A^2 = 9 + 100/16 = 244/16$. Because A was specified to be positive, we take the positive square root to obtain $A = \sqrt{61}/2$. The solution is

$$f(t) = \frac{1}{2}\sqrt{61}\sin(4t + 0.876)$$

EXAMPLE 3.7.3

Sine Form of the Response

■ Problem
Obtain the solution to the following problem in the form of a sine function with a phase angle:

$$3\ddot{x} + 12\dot{x} + 87x = 5 \qquad x(0) = 2 \qquad \dot{x}(0) = 7$$

■ Solution
Applying the Laplace transform we obtain

$$3\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + 12[s X(s) - x(0)] + 87X(s) = \frac{5}{s}$$

which gives

$$X(s) = \frac{6s^2 + 45s + 5}{3s(s^2 + 4s + 29)} = \frac{2s^2 + 15s + 5/3}{s\left[(s + 2)^2 + 25\right]} \tag{1}$$

Because the denominator roots are $s = 0$ and $s = -2 \pm 5j$, we know that the form of the solution is

$$x(t) = C_1 + C_2 e^{-2t}\sin(5t + \phi)$$

Using the results of Example 3.7.1 with $a = 2$ and $\omega = 5$, we can write the transform of $x(t)$ as

$$X(s) = \frac{C_1}{s} + C_2\frac{s\sin\phi + 2\sin\phi + 5\cos\phi}{(s + 2)^2 + 25}$$

This can be expressed as a single fraction as follows:

$$X(s) = \frac{C_1\left[(s + 2)^2 + 25\right] + C_2 s(s\sin\phi + 2\sin\phi + 5\cos\phi)}{s\left[(s + 2)^2 + 25\right]}$$

$$= \frac{(C_1 + C_2\sin\phi)s^2 + (4C_1 + 2C_2\sin\phi + 5C_2\cos\phi)s + 29C_1}{s\left[(s + 2)^2 + 25\right]} \tag{2}$$

Comparing the numerators of equations (1) and (2) we see that

$$C_1 + C_2\sin\phi = 2 \qquad 4C_1 + 2C_2\sin\phi + 5C_2\cos\phi = 15 \qquad 29C_1 = \frac{5}{3}$$

The third equation gives $C_1 = 5/87 = 0.0575$, and the first equation gives

$$C_2\sin\phi = 2 - C_1 = \frac{169}{87}$$

From the second equation, we have $C_2 \cos \phi = 947/435$. Taking C_2 to be positive, we see that both $\sin \phi$ and $\cos \phi$ are positive, and thus ϕ is in the first quadrant. Therefore

$$\phi = \tan^{-1} \left(\frac{\sin \phi}{\cos \phi} \right) = \tan^{-1} \left(\frac{169/87}{947/435} \right) = 0.729 \text{ rad}$$

Because $\sin^2 \phi + \cos^2 \phi = 1$,

$$\sin^2 \phi + \cos^2 \phi = \left(\frac{169}{87C_2} \right)^2 + \left(\frac{947}{435C_2} \right)^2 = 1$$

which gives $C_2 = 2.918$. Thus the solution is

$$x(t) = 0.0575 + 2.918e^{-2t} \sin(5t + 0.729)$$

The shifting property, entry 6 in Table 3.3.2, can be used to invert transforms containing the function e^{-Ds}. This enables us to obtain the response to inputs composed of shifted step or ramp functions.

| Pulse Response of a First-Order Model | **EXAMPLE 3.7.4** |

■ **Problem**

Suppose a rectangular pulse $P(t)$ of unit height and duration 2 is applied to the first-order model $\dot{x} + 4x = P(t)$ with a zero initial condition. Use the Laplace transform to determine the response.

■ **Solution**

The problem to be solved is

$$\dot{x} + 4x = P(t) \qquad x(0) = 0$$

Taking the transform of both sides of the equation and noting that the initial condition is zero, we obtain

$$sX(s) + 4X(s) = P(s) = \frac{1 - e^{-2s}}{s}$$

Solve for $X(s)$.

$$X(s) = \frac{P(s)}{s + 4} = \frac{1 - e^{-2s}}{s(s + 4)}$$

Let

$$Y(s) = \frac{1}{s(s + 4)}$$

Then

$$X(s) = \left(1 - e^{-2s} \right) Y(s)$$

and from the shifting property,

$$x(t) = y(t)u_s(t) - y(t - 2)u_s(t - 2) \tag{1}$$

Figure 3.7.1 Plot of the
solution for Example 3.7.4.
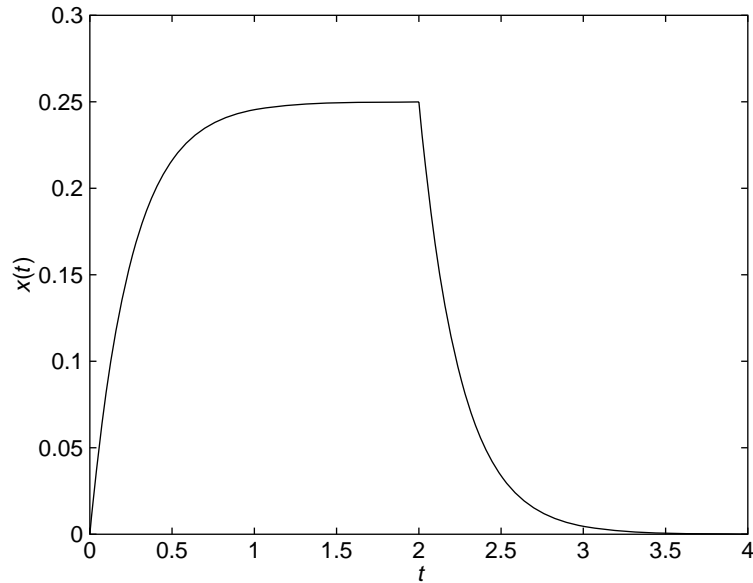
To find $y(t)$, note that the denominator roots of $Y(s)$ are $s = 0$ and $s = -4$. Thus we can express
$Y(s)$ as follows:

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s + 4} = \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s + 4}\right)$$

Therefore,

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}$$

and from equation (1),

$$x(t) = \left(\frac{1}{4} - \frac{1}{4}e^{-4t}\right) - \left[\frac{1}{4} - \frac{1}{4}e^{-4(t-2)}\right]u_s(t - 2)$$

So for $0 \le t \le 2$,

$$x(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}$$

and for $t \ge 2$,

$$x(t) = \frac{1}{4} - \frac{1}{4}e^{-4t} - \frac{1}{4} + \frac{1}{4}e^{-4(t-2)} = \frac{1}{4}\left[e^{-4(t-2)} - e^{-4t}\right]$$

The function's graph is shown in Figure 3.7.1.

EXAMPLE 3.7.5   Series Solution Method

■ **Problem**
Obtain an approximate, closed-form solution of the following problem for $0 \le t \le 0.5$:

$$\dot{x} + x = \tan t \qquad x(0) = 0 \tag{1}$$

■ **Solution**

If we attempt to use separation of variables to solve this problem we obtain

$$\frac{dx}{\tan t - x} = dt$$

so the variables do not separate. In general, when the input is a function of time, the equation $\dot{x} + g(x) = f(t)$ does not separate. The Laplace transform method cannot be used when the Laplace transform or inverse transform either does not exist or cannot be found easily. In this example, the equation cannot be solved by the Laplace transform method, because the transform of $\tan t$ does not exist.

An approximate solution of the equation $\dot{x} + x = \tan t$ can be obtained by replacing $\tan t$ with a series approximation. The number of terms used in the series determines the accuracy of the resulting solution for $x(t)$. The Taylor series expansion for $\tan t$ is

$$\tan t = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \cdots \qquad |t| < \frac{\pi}{2}$$

The more terms we retain, the more accurate is the series. Also, the series becomes less accurate as the absolute value of t increases. To demonstrate the series solution method, let us use a series with two terms: $\tan t = t + t^3/3$. At the largest value of t, $t = 0.5$, the two-term series gives 0.5417 versus 0.5463 for the true value of $\tan 0.5$. So the two-term series is accurate to at least two decimal places over the range of t we are interested in $(0 \leq t \leq 0.5)$.

Using the two-term series we need to solve the following problem:

$$\dot{x} + x = t + \frac{t^3}{3} \qquad x(0) = 0$$

Using the Laplace transform we obtain

$$sX(s) + X(s) = \frac{1}{s^2} + \frac{1}{3}\frac{3!}{s^4}$$

or

$$X(s) = \frac{s^2 + 2}{s^4(s + 1)}$$

The inverse transform was obtained in Example 3.5.7 and is

$$x(t) = \frac{1}{3}t^3 - t^2 + 3t - 3 + 3e^{-t}$$

We can expect this approximate solution of equation (1) to be accurate to at least two decimal places for $0 \leq t \leq 0.5$. Of course, greater accuracy can be achieved by retaining more terms in the Taylor series for $\tan t$.

3.8 COMPUTING EXPANSION COEFFICIENTS WITH MATLAB

You can use MATLAB to easily compute the coefficients in the partial-fraction expansion. The appropriate MATLAB function is `residue`. Let $X(s)$ denote the transform. In the terminology of the `residue` function, the expansion coefficients are called the *residues* and the factors of the denominator of $X(s)$ are called the *poles*. The poles include the characteristic roots of the model and any denominator roots introduced by the input function. If the order m of the numerator of $X(s)$ is greater than the order n

of the denominator, the transform can be represented by a polynomial $K(s)$, called the *direct term,* plus a ratio of two polynomials where the denominator degree is greater than the numerator degree. For example,

$$X(s) = \frac{6s^3 + 57s^2 + 120s + 80}{s^2 + 9s + 14} = 6s + 3 + \frac{9s + 38}{s^2 + 9s + 14} \qquad (3.8.1)$$

or

$$X(s) = 6s + 3 + 5\frac{1}{s + 7} + 4\frac{1}{s + 2} \qquad (3.8.2)$$

Here the direct term is the polynomial $K(s) = 6s + 3$.

SYNTAX OF THE `residue` FUNCTION

The syntax of the `residue` function is as follows:

```
[r,p,K] = residue(num,den)
```

where `num` and `den` are arrays containing the coefficients of the numerator and denominator of $X(s)$. The output of the function consists of the array `r`, which contains the residues, the array `p`, which contains the poles, and the array `K`, which contains the coefficients of the direct term $K(s)$ in polynomial form.

Using (3.8.1) as an example, you would type

```
[r,p,K] = residue([6,57,120,80],[1,9,14])
```

The answer given by MATLAB is `r = [5, 4]`, `p = [-7, -2]`, and `K = [6, 3]`, which corresponds to (3.8.2). Note that the order in which the residues are displayed corresponds to the order in which the poles are displayed.

REPEATED POLES

Repeated poles are handled as follows. Consider the equation $\ddot{x} + 9\dot{x} + 14x = 3\dot{g} + 2g$, where $g(t) = 4e^{-7t}$. If the initial conditions are zero, the transform of the response is

$$X(s) = \frac{3s + 2}{s^2 + 9s + 14}\frac{4}{s + 7} = \frac{12s + 8}{(s + 2)(s + 7)^2}$$
$$= \frac{12s + 8}{s^3 + 16s^2 + 77s + 98}$$

The repeated poles are $s = -7, -7$; one of them is a characteristic root and the other is due to the input. To obtain the expansion, type

```
[r,p,K] = residue([12, 8],[1, 16, 77, 98])
```

The answer given by MATLAB is `r = [0.64, 15.2, -0.64]`, `p = [-7, -7, -2]`, and `K = [ ]`. This corresponds to the expansion

$$X(s) = 0.64\frac{1}{s + 7} + 15.2\frac{1}{(s + 7)^2} - 0.64\frac{1}{s + 2}$$

Note that for the residues due to repeated poles, the residue corresponding to the *highest* power is displayed as the *last* of those residues. The response here is

$$x(t) = 0.64e^{-7t} + 15.2te^{-7t} - 0.64e^{-2t}$$

COMPLEX POLES

Complex poles are handled as follows. Consider the equation $\ddot{x} + 6\dot{x} + 34x = 4\dot{g} + g$, where $g(t)$ is a unit-step function and the initial conditions are zero. The transform of the response is

$$X(s) = \frac{4s + 1}{(s^2 + 6s + 34)s} = \frac{4s + 1}{s^3 + 6s^2 + 34s}$$

To obtain the expansion, type

```
[r,p,K] = residue([4, 1],[1, 6, 34, 0])
```

Observe that the last coefficient in the denominator is 0. The answer given by MATLAB is r = [-0.0147-0.3912i, -0.0147+0.3912i, 0.0294], p = [-3+5i, -3-5i, 0], and K = []. (Note that MATLAB uses the symbol i to represent the imaginary number $\sqrt{-1}$, whereas we have been using the symbol j.) The MATLAB results correspond to the expression

$$X(s) = \frac{-0.0147 - 0.3912j}{s + 3 - 5j} + \frac{-0.0147 + 0.3912j}{s + 3 + 5j} + \frac{0.0294}{s}$$

The response is

$$x(t) = (-0.0147 - 0.3912j)e^{(-3+5j)t} + (-0.0147 + 0.3912j)e^{(-3-5j)t} + 0.0294 \tag{3.8.3}$$

This form is not very useful because of its complex coefficients, but we can convert it to a more useful form by noting that the first two terms in the expansion have the form

$$\frac{C + jD}{s + a - jb} + \frac{C - jD}{s + a + jb} \tag{3.8.4}$$

which corresponds to the time function

$$(C + jD)e^{(-a+jb)t} + (C - jD)e^{(-a-jb)t}$$

Using Euler's identities: $e^{\pm jbt} = \cos bt \pm j \sin bt$, the previous form can be written as

$$2e^{-at}(C \cos bt - D \sin bt) \tag{3.8.5}$$

Using this identity with $C = -0.0147$ and $D = -0.3912$, we can write the response (3.8.3) as

$$x(t) = 2e^{-3t}(-0.0147 \cos 5t + 0.3912 \sin 5t) + 0.0294$$

3.9 TRANSFER-FUNCTION ANALYSIS IN MATLAB

Some of the functions from the MATLAB Control System Toolbox can be used to solve linear, time-invariant (constant-coefficient) differential equations. They are called ODE solvers and are powerful tools for studying dynamic systems.

THE `tf` AND `tfdata` FUNCTIONS

The ODE solvers in the Control System Toolbox can accept various descriptions of the equations to be solved. Here we will focus on the solvers that accept a transfer function model of the system. An *LTI object* describes a linear, time-invariant model, or sets of equations, here referred to as the *system*. An LTI object can be created from different descriptions of the system, it can be analyzed with several functions, and it can be accessed to provide alternative descriptions of the system. For example, the equation

$$5\ddot{x} + 9\dot{x} + 4x = f(t)$$

is the reduced-form description of a particular system. From the reduced form we can immediately obtain the transfer function description of the model. It is

$$\frac{X(s)}{F(s)} = \frac{1}{5s^2 + 9s + 4} \tag{3.9.1}$$

To create an LTI object from a transfer function, you use the `tf(num,den)` function, where the array `num` is the array of coefficients of the numerator of the transfer function, arranged in order of descending powers of s, and `den` is the array of coefficients of the denominator of the transfer function, also arranged in descending order. The result is the LTI object that describes the system in the transfer function form. For equation (3.9.1), the session is

```
≫sys1 = tf(1, [5, 9, 4]);
```

Here is another example. The LTI object `sys2` in transfer function form for the equation

$$5\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 3x = 6\frac{d^2f}{dt^2} + 9\frac{df}{dt} + 2f \tag{3.9.2}$$

is created with the session:

```
≫sys2 = tf([6, 9, 2],[5, 4, 7, 3]);
```

As we will see in Chapter 5, we can also create an LTI object from descriptions of the system other than its transfer function. If the LTI object already exists, we can extract the coefficients of the numerator and denominator of the transfer function model by using the `tfdata` function. Its syntax is `[num, den] = tfdata(sys)`.

ODE SOLVERS

The Control System Toolbox provides several solvers for linear models. These solvers are categorized by the type of input function they can accept: some of these are a step input, an impulse input, and a general input function.

In their basic form, each of the following functions automatically puts a title and axis labels on the plot. You can change these by activating the Plot Editor or by right-clicking on the plot. This brings up a menu that includes the Properties as a choice. Selecting Properties enables you to change the labels as well as other features such as limits, units, and style.

The menu obtained by right-clicking on the plot also contains Characteristics as a choice. The contents of the subsequent menu depend on the particular function. When the `step` function is used, the Characteristics menu includes Peak Response, Rise Time, Settling Time, and Steady State. When you select Peak Response, for example, MATLAB identifies the peak value of the response curve and marks its location with a dot and dashed lines. Moving the cursor over the dot displays the numerical values of the peak response and the time at which it occurs. The Rise Time is the time required for the response to go from 10% to 90% of its steady-state value. The Settling Time is the time required for the response to settle within 2% of its steady-state value. You can change these percents by selecting the Characteristics tab under the Properties menu. The rise time and settling time are frequently used as measures of system performance and are discussed in Chapter 9.

The `step` Function The `step` function plots the unit-step response, assuming that the initial conditions are zero. The basic syntax is `step(sys)`, where `sys` is the LTI object. The time span and number of solution points are chosen automatically. To specify the final time `tfinal`, use the syntax `step(sys,tfinal)`. To specify a vector of times of the form `t = (0:dt:tfinal)`, at which to obtain the solution, use the syntax `step(sys,t)`. When called with left-hand arguments, as `[y, t] = step(sys,...)`, the function returns the output response `y`, and the time array `t` used for the simulation. No plot is drawn. The array `y` is $p \times q \times m$, where p is `length(t)`, q is the number of outputs, and m is the number of inputs.

The syntax `step(sys1,sys2,...,t)` plots the step response of multiple LTI systems on a single plot. The time vector `t` is optional. You can specify line color, line style, and marker for each system; for example, `step(sys1,'r', sys2, 'y--',sys3,'gx')`. The steady-state response and the time to reach that state are automatically determined. The steady-state response is indicated by a horizontal dotted line.

Step Response of a Second-Order Model | **EXAMPLE 3.9.1**

■ **Problem**

Consider the following model:

$$\frac{X(s)}{F(s)} = \frac{cs + 5}{10s^2 + cs + 5}$$

a. Plot the unit-step response for $c = 3$ using the time span selected by MATLAB.
b. Plot the unit-step response for $c = 3$ over the range $0 \le t \le 15$.
c. Plot the unit-step responses for $c = 3$ and $c = 8$ over the range $0 \le t \le 15$. Put the plots on the same graph.
d. Plot the step response for $c = 3$, where the magnitude of the step input is 20. Use the time span selected by MATLAB.

■ **Solution**

The MATLAB programs are shown below for each case.

a. This illustrates the use of the `step` function in its most basic form.

```
sys1 = tf([3,5],[10,3,5]);
step(sys1)
```

The plot is shown in Figure 3.9.1.
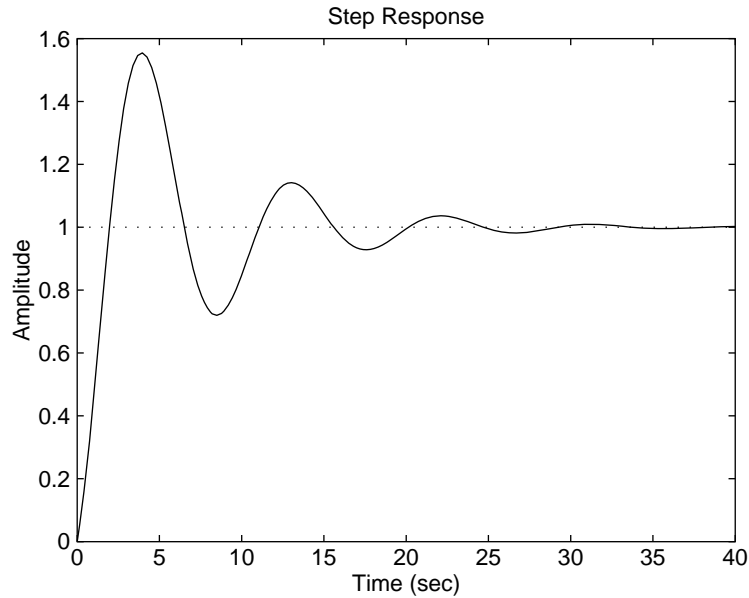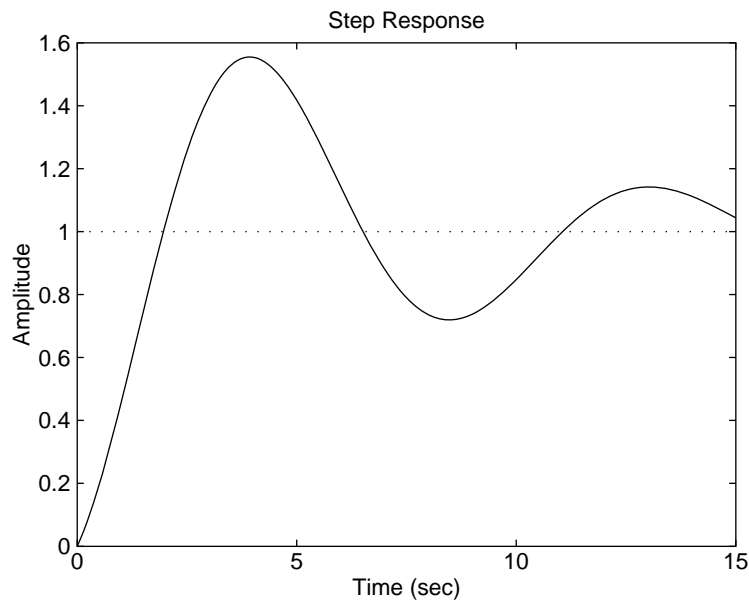
b.   This illustrates how to use a user-selected time span and spacing.

```
sys1 = tf([3,5],[10,3,5]);
t = (0:0.01:15);
step(sys1,t)
```

The plot is shown in Figure 3.9.2.

c.   This illustrates how to plot two responses on the same graph.

```
sys1 = tf([3,5],[10,3,5]);
sys2 = tf([8,5],[10,8,5]);
t = (0:0.01:15);
step(sys1,' ',sys2,'--',t)
```

The plot is shown in Figure 3.9.3.
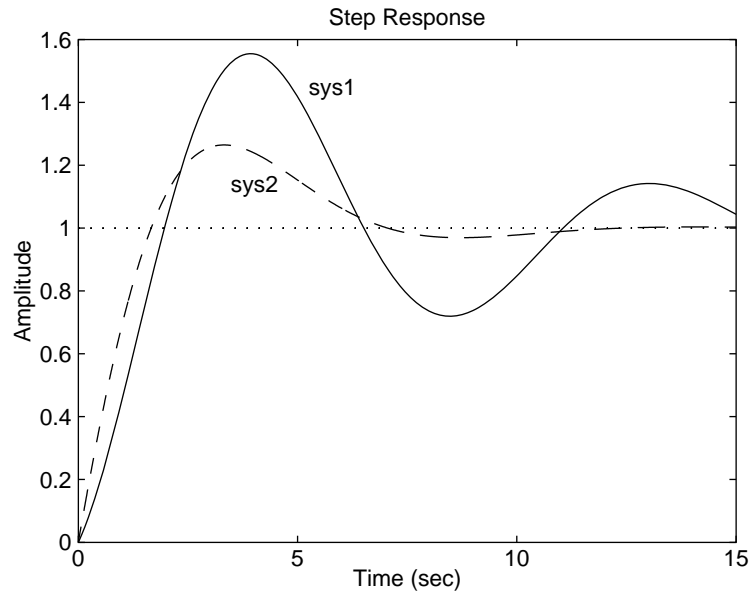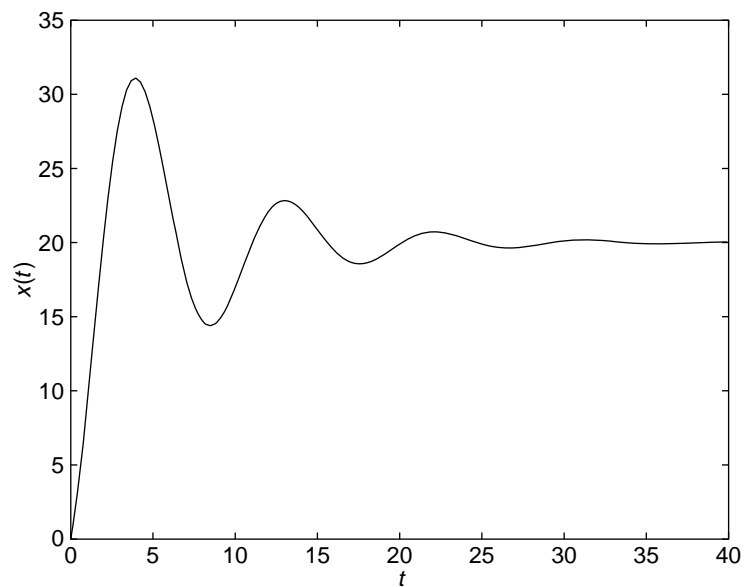
Step Response

d. This illustrates how to obtain the response when the magnitude of the step input is not unity. The output of the `step(sys1)` function is for a unit-step input, and so it must be multiplied by 20. This multiplication can be performed within the `plot` function.

```
sys1 = tf([3,5],[10,3,5]);
[y, t] = step(sys1);
plot(t,20*y),xlabel('t'),ylabel('x(t)')
```

The plot is shown in Figure 3.9.4.

Note that when the `step` function is used without an assignment on the left-hand side of the equal sign, it automatically computes and plots the steady-state response, and puts a title and axis labels on the plot, with the assumption that the unit of time is seconds. When the form `[y, t] = step(sys)` is used, however, the steady-state response is not computed, and you must put the labels on the plot.

The `impulse` Function   The `impulse` function plots the unit-impulse response, assuming that the initial conditions are zero. The basic syntax is `impulse(sys)`, where `sys` is the LTI object. The time span and number of solution points are chosen automatically. For example, the impulse response of (3.9.1) is found as follows:

```
≫sys1 = tf(1,[5, 9, 4]);
≫impulse(sys1)
```

The extended syntax of the `impulse` function is identical to that of the `step` function. The characteristics available with the `impulse` function by right-clicking on the plot are the Peak Response and the Settling Time.

EXAMPLE 3.9.2

Impulse Response of Second-Order Models

■ Problem
In Example 3.6.4 we obtained the response of the following model to a unit impulse:

$$\frac{X(s)}{F(s)} = \frac{1}{2s^2 + 14s + 20}$$

Our analysis showed that if $x(0) = \dot{x}(0) = 0$, then $x(0+) = 0$ and $\dot{x}(0+) = 1/2$. Use the `impulse` function to verify these results.

■ Solution
The session is shown here.

```
≫sys1 = tf(1,[2, 14, 20]);
≫impulse(sys1)
```

The plot is shown in Figure 3.9.5. From it we see that $x(0+) = 0$ and that $\dot{x}(0+)$ is positive as predicted. We are unable to determine the exact value of $\dot{x}(0+)$ from this plot, so we multiply the transfer function by s to obtain the transfer function for $v = \dot{x}$.

$$\frac{V(s)}{F(s)} = \frac{s}{2s^2 + 14s + 20}$$

We now use the `impulse` function on this transfer function.

```
≫sys2 = tf([1, 0],[2, 14, 20]);
≫impulse(sys2)
```

The plot is shown in Figure 3.9.6. From it we see that $\dot{x}(0+) = 0.5$ as predicted.
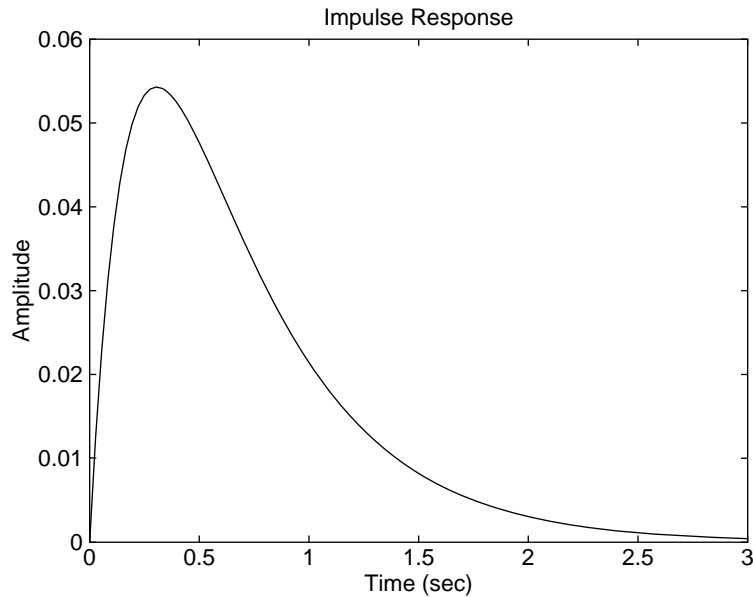
Impulse Response
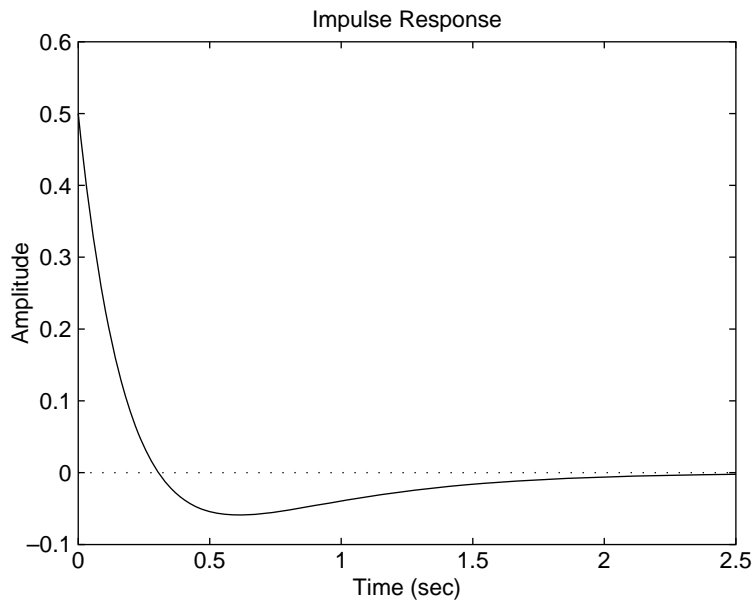
Impulse Response

Figure 3.9.6 Impulse response of $v = \dot{x}(t)$ for Example 3.9.2.

The `lsim` Function The `lsim` function plots the response of the system to an arbitrary input. The basic syntax for zero-initial conditions is `lsim(sys,u,t)`, where `sys` is the LTI object, `t` is a time array having regular spacing, as `t = (0:dt:tf)`, and `u` is a matrix with as many columns as inputs, and whose ith row specifies the value of the input at time `t(i)`.

The extended syntax of the `lsim` function is the same as that of the `step` function. When `lsim` is used without the left-hand arguments, the Peak Response is available from the Characteristics menu.

EXAMPLE 3.9.3

Ramp Response with the `lsim` Function

■ **Problem**

Plot the forced response of

$$\ddot{x} + 3\dot{x} + 5x = 10f(t)$$

to a *ramp* input, $u(t) = 1.5t$, over the time interval $0 \le t \le 2$.

■ **Solution**

We choose to generate the plot with 300 points. The MATLAB session is the following.

```
≫t = linspace(0,2,300);
≫f = 1.5*t;
≫sys = tf(10, [1, 3, 5]);
≫[y, t] = lsim(sys,f,t);
≫plot(t,y,t,f),xlabel('t'),ylabel('x(t) and f(t)')
```

The plot is shown in Figure 3.9.7.

Figure 3.9.7 Ramp response for Example 3.9.3.
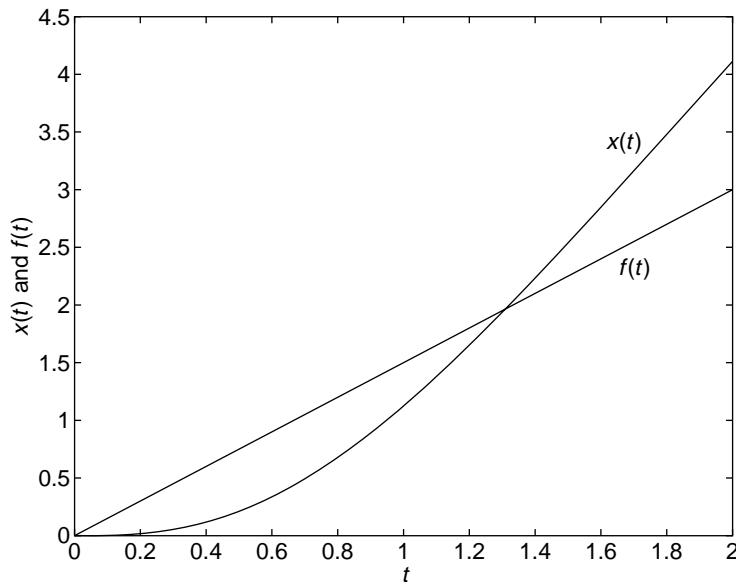
Sine Response with the `lsim` Command As another example, to find the response to $f(t) = 15\sin(3t)$, replace the second line in the previous program with `f = 15*sin(3*t);`.

3.10 CHAPTER REVIEW

Chapter 3 covers methods for solving ordinary differential equations. Now that you have finished this chapter you should be able to do the following:

1. Choose and apply the separation-of-variables method, the trial-solution method, or the Laplace transform method to obtain the solution of a differential equation model.

2. When applying the Laplace transform method, be able to perform the appropriate expansion and apply the appropriate transform properties to obtain the inverse transform.
3. Identify the free, forced, transient, and steady-state components of the complete response.
4. Evaluate the effects of impulse inputs and input derivatives on the response.
5. Obtain transfer functions from models expressed as single equations or as sets of equations.
6. Use MATLAB to obtain partial-fraction expansions and the forced response.

Perspective on Solving Differential Equations

In this chapter we have seen several methods for solving differential equations, and it is useful to understand the advantages of each method.

■ The trial solution method using exponential or harmonic functions is sometimes the easiest method, especially for first- and second-order linear equations with constant inputs.
■ Separation of variables can be used to solve some linear and nonlinear ODEs, but the technique is limited to rather simple equations. For example, the method does not work for the equation $\dot{x} + g(x) = f(t)$, which does not separate.
■ For some equations, the separation of variables method can be used to obtain a formal solution, but the resulting integral cannot be evaluated in terms of known elementary functions. An example is the equation $\dot{x} = \sin t^2$, for which the integral must be evaluated numerically.
■ The Laplace transform method is especially advantageous when the mathematical model is linear, has constant coefficients, and either

  1. Is of order higher than three,
  2. Has a nonconstant input,
  3. Has input derivatives, or
  4. Consists of coupled differential equations.

■ However, the Laplace transform method cannot be used when the Laplace transform or inverse transform either does not exist or cannot be found easily. For example, the equation $\dot{x} = \tan t$ can be solved by separation of variables but not by the Laplace transform method, because the transform of $\tan t$ does not exist.
■ As shown in Example 3.7.6, an approximate closed-form solution can sometimes be obtained by using a series approximation. The number of terms used in the series determines the accuracy of the resulting solution.

When we cannot obtain a closed-form solution we must solve the differential equation model using numerical methods, which are the subject of Chapter 5.

REFERENCE

[Kreyzig, 2006] Kreyzig, E., *Advanced Engineering Mathematics,* 9th ed., John Wiley and Sons, New York, 2006.

PROBLEMS

Section 3.1 Differential Equations

3.1   Determine whether or not the following equations are linear or nonlinear, and state the reason for your answer.

    a. $y\ddot{y} + 5\dot{y} + y = 0$

    b. $\dot{y} + \sin y = 0$

    c. $\dot{y} + \sqrt{y} = 0$

3.2   Solve each of the following problems by separation of variables:

    a. $4\dot{x} = 3t \qquad x(0) = 2$

    b. $5\dot{x} = 2e^{-4t} \qquad x(0) = 3$

    c. $3\ddot{x} = 5t \qquad x(0) = 2 \qquad \dot{x}(0) = 7$

    d. $4\ddot{x} = 7e^{-2t} \qquad x(0) = 4 \qquad \dot{x}(0) = 2$

3.3   Solve each of the following problems by separation of variables:

    a. $\dot{x} + 5x^2 = 25 \qquad x(0) = 3$

    b. $\dot{x} - 4x^2 = 36 \qquad x(0) = 10$

    c. $x\dot{x} - 5x = 25 \qquad x(0) = 4$

    d. $\dot{x} + 2e^{-4t}x = 0 \qquad x(0) = 5$

Section 3.2 Response Types and Stability

3.4   Solve each of the following problems with the trial solution method. Identify the free, forced, transient, and steady-state responses.

    a. $\ddot{x} + 8\dot{x} + 15x = 30 \qquad x(0) = 10 \qquad \dot{x}(0) = 4$

    b. $\ddot{x} + 10\dot{x} + 25x = 75 \qquad x(0) = 10 \qquad \dot{x}(0) = 4$

    c. $\ddot{x} + 25x = 100 \qquad x(0) = 10 \qquad \dot{x}(0) = 4$

    d. $\ddot{x} + 8\dot{x} + 65x = 130 \qquad x(0) = 10 \qquad \dot{x}(0) = 4$

3.5   Determine whether the following models are stable, unstable, or neutrally stable:

    a. $3\dot{x} - 5x = 12$ b. $\ddot{x} - 3\dot{x} - 10x = 50$

    c. $\ddot{x} - 6\dot{x} + 34x = 68$ d. $\dot{x} = 3$

    e. $\ddot{x} + 4x = 5$ f. $\ddot{x} + 5\dot{x} = 7$

3.6   (a) Prove that the second-order system whose characteristic polynomial is $ms^2 + cs + k$ is stable if and only if m, c, and k have the same sign. (b) Derive the conditions for neutral stability.

Section 3.3 The Laplace Transform Method

Note: see also the problems for Section 3.7, Additional Examples.

3.7   Derive the Laplace transform of the ramp function $x(t) = mt$, whose slope is the constant m.

3.8   Extend the results of Problem 3.7 to obtain the Laplace transform of t^2.

3.9   Obtain the Laplace transform of the following functions:

    a. $x(t) = 10 + t^2$

    b. $x(t) = 6te^{-5t} + e^{-3t}$

c. $x(t) = te^{-3t} \sin 5t$

d. $x(t) = \begin{cases} 0 & t < 5 \\ t - 5 & t > 5 \end{cases}$

3.10 Obtain the Laplace transform of the function shown in Figure P3.10.
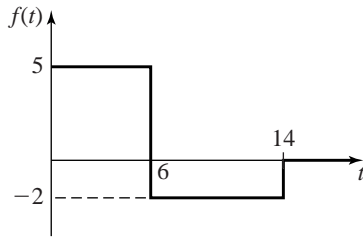
3.11 Use the initial and final value theorems to determine $x(0+)$ and $x(\infty)$ for the following transforms:

a. $X(s) = \dfrac{5}{3s + 7}$

b. $X(s) = \dfrac{10}{3s^2 + 7s + 4}$

3.12 Obtain the inverse Laplace transform $x(t)$ for each of the following transforms:

a. $X(s) = \dfrac{6}{s(s + 4)}$

b. $X(s) = \dfrac{12s + 5}{s(s + 3)}$

c. $X(s) = \dfrac{4s + 7}{(s + 2)(s + 5)}$

d. $X(s) = \dfrac{5}{s^2(2s + 8)}$

e. $X(s) = \dfrac{3s + 2}{s^2(s + 5)}$

f. $X(s) = \dfrac{12s + 5}{(s + 3)^2(s + 7)}$

3.13 Obtain the inverse Laplace transform $x(t)$ for each of the following transforms:

a. $X(s) = \dfrac{7s + 2}{s^2 + 6s + 34}$

b. $X(s) = \dfrac{4s + 3}{s(s^2 + 6s + 34)}$

c. $X(s) = \dfrac{4s + 9}{(s^2 + 6s + 34)(s^2 + 4s + 20)}$

d. $X(s) = \dfrac{5s^2 + 3s + 7}{s^3 + 12s^2 + 44s + 48}$

3.14 Solve the following problems:

a. $5\dot{x} = 7t$ 　　 $x(0) = 3$

b. $4\dot{x} = 3e^{-5t}$ 　　 $x(0) = 4$

c. $7\ddot{x} = 4t$ 　　 $x(0) = 3$ 　　 $\dot{x}(0) = 5$

d. $3\ddot{x} = 8e^{-4t}$ 　　 $x(0) = 3$ 　　 $\dot{x}(0) = 5$

3.15 Solve the following problems:

a. $5\dot{x} + 7x = 0$ 　　 $x(0) = 4$

b. $5\dot{x} + 7x = 15$ 　　 $x(0) = 0$

c. $5\dot{x} + 7x = 15$ 　　 $x(0) = 4$

d. $\dot{x} + 7x = 4t$ 　　 $x(0) = 5$

3.16 Solve the following problems:

    a. $\ddot{x} + 10\dot{x} + 21x = 0$ $x(0) = 4$ $\dot{x}(0) = -3$

    b. $\ddot{x} + 14\dot{x} + 49x = 0$ $x(0) = 1$ $\dot{x}(0) = 3$

    c. $\ddot{x} + 14\dot{x} + 58x = 0$ $x(0) = 4$ $\dot{x}(0) = -8$

3.17 Solve the following problems:

    a. $3\ddot{x} + 30\dot{x} + 63x = 5$ $x(0) = \dot{x}(0) = 0$

    b. $\ddot{x} + 14\dot{x} + 49x = 98$ $x(0) = \dot{x}(0) = 0$

    c. $\ddot{x} + 14\dot{x} + 58x = 174$ $x(0) = \dot{x}(0) = 0$

Section 3.4 Transfer Functions

3.18 For each of the following equations, determine the transfer function $X(s)/F(s)$ and compute the characteristic roots:

    a. $5\dot{x} + 7x = 15f(t)$

    b. $3\ddot{x} + 30\dot{x} + 63x = 5f(t)$

    c. $\ddot{x} + 10\dot{x} + 21x = 4f(t)$

    d. $\ddot{x} + 14\dot{x} + 49x = 7f(t)$

    e. $\ddot{x} + 14\dot{x} + 58x = 6\dot{f}(t) + 4f(t)$

    f. $5\dot{x} + 7x = 4\dot{f}(t) + 15f(t)$

3.19 Obtain the transfer functions $X(s)/F(s)$ and $Y(s)/F(s)$ for the following model:

$$3\dot{x} = y \qquad \dot{y} = f(t) - 3y - 15x$$

3.20 Obtain the transfer functions $X(s)/F(s)$ and $Y(s)/F(s)$ for the following model:

$$\dot{x} = -2x + 5y \qquad \dot{y} = f(t) - 6y - 4x$$

Section 3.5 Partial-Fraction Expansion

Note: see also the problems for Section 3.7, Additional Examples.

3.21 Invert the following transforms:

    a. $\dfrac{6}{s(s+5)}$ b. $\dfrac{4}{(s+3)(s+8)}$

    c. $\dfrac{8s+5}{2s^2 + 20s + 48}$ d. $\dfrac{4s+13}{s^2 + 8s + 116}$

3.22 Invert the following transforms:

    a. $\dfrac{3s+2}{s^2(s+10)}$ b. $\dfrac{5}{(s+4)^2(s+1)}$

    c. $\dfrac{s^2 + 3s + 5}{s^3(s+2)}$ d. $\dfrac{s^3 + s + 6}{s^4(s+2)}$

3.23 Solve the following problems for $x(t)$:

    a. $5\dot{x} + 3x = 10 + t^2$ $x(0) = 2$

    b. $4\dot{x} + 7x = 6te^{-5t} + e^{-3t}$ $x(0) = 5$

    c. $4\ddot{x} + 3x = te^{-3t}\sin 5t$ $x(0) = 10$ $\dot{x}(0) = -2$

3.24 Compare the LCD method with equation (3.5.4) for obtaining the inverse
Laplace transform of

$$X(s) = \frac{7s + 4}{2s^2 + 16s + 30}$$

Section 3.6 The Impulse and Numerator Dynamics

3.25 Solve the following problems for $x(t)$. Compare the values of $x(0+)$ and $x(0)$.
For parts (b) through (d), also compare the values of $\dot{x}(0+)$ and $\dot{x}(0)$.
   a. $7\dot{x} + 5x = 4\delta(t)$ $x(0) = 3$
   b. $3\ddot{x} + 30\dot{x} + 63x = 5\delta(t)$ $x(0) = \dot{x}(0) = 0$
   c. $\ddot{x} + 14\dot{x} + 49x = 3\delta(t)$ $x(0) = 2$ $\dot{x}(0) = 3$
   d. $\ddot{x} + 14\dot{x} + 58x = 4\delta(t)$ $x(0) = 4$ $\dot{x}(0) = 7$

3.26 Solve the following problems for $x(t)$. The input $g(t)$ is a unit-step function,
$g(t) = u_s(t)$. Compare the values of $x(0+)$ and $x(0)$. For parts (c) and (d), also
compare the values of $\dot{x}(0+)$ and $\dot{x}(0)$.
   a. $7\dot{x} + 5x = 4\dot{g}(t)$ $x(0) = 3$
   b. $7\dot{x} + 5x = 4\dot{g}(t) + 6g(t)$ $x(0) = 3$
   c. $3\ddot{x} + 30\dot{x} + 63x = 4\dot{g}(t)$ $x(0) = 2$ $\dot{x}(0) = 3$
   d. $3\ddot{x} + 30\dot{x} + 63x = 4\dot{g}(t) + 6g(t)$ $x(0) = 4$ $\dot{x}(0) = 7$

Section 3.7 Additional Examples

3.27 Solve the following problem for $x(t)$ and $y(t)$:

$$3\dot{x} = y \qquad x(0) = 5$$
$$\dot{y} = 4u_s(t) - 3y - 15x \qquad y(0) = 10$$

3.28 Solve the following problem for $x(t)$ and $y(t)$:

$$\dot{x} = -2x + 5y \qquad x(0) = 5$$
$$\dot{y} = -6y - 4x + 10u_s(t) \qquad y(0) = 2$$

3.29 Determine the general form of the solution of the following equation, where
the initial conditions $y(0)$ and $\dot{y}(0)$ have arbitrary values:

$$\ddot{y} + y = e^{-t}$$

3.30 a. Use the Laplace transform to obtain the form of the solution of the
following equation:

$$\ddot{x} + 4x = 3t$$

   b. Obtain the solution to the equation in part (a) for the following conditions:
   $x(0) = 10$, $x(5) = 30$.

3.31 Obtain the inverse Laplace transform of

$$X(s) = \frac{30}{(s^2 + 6s + 34)(s^2 + 36)}$$

3.32 Solve the following problem for $x(t)$:

$$\ddot{x} + 12\dot{x} + 40x = 3 \sin 5t \qquad x(0) = \dot{x}(0) = 0$$

3.33   Obtain the inverse transform in the form $x(t) = A \sin(\omega t + \phi)$, where $A > 0$.

$$X(s) = \frac{4s + 9}{s^2 + 25}$$

3.34   Use the Laplace transform to solve the following problem:

$$\ddot{x} + 6\dot{x} + 34x = 5 \sin 6t \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

3.35   Express the oscillatory part of the solution of the following problem in the form of a sine function with a phase angle:

$$\ddot{x} + 12\dot{x} + 40x = 10 \qquad x(0) = \dot{x}(0) = 0$$

3.36   Find the steady-state difference between the input $f(t)$ and the response $x(t)$, if $f(t) = 6t$.

$$\ddot{x} + 8\dot{x} + x = f(t) \qquad x(0) = \dot{x}(0) = 0$$

3.37   Invert the following transform:

$$X(s) = \frac{1 - e^{-3s}}{s^2 + 6s + 8}$$

3.38   Obtain the Laplace transform of the function plotted in Figure P3.38.
3.39   Obtain the Laplace transform of the function plotted in Figure P3.39.

Figure P3.38
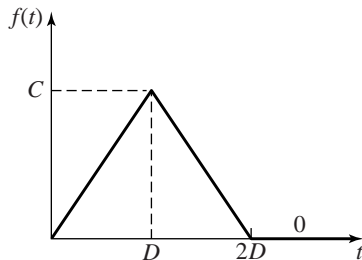
Figure P3.39
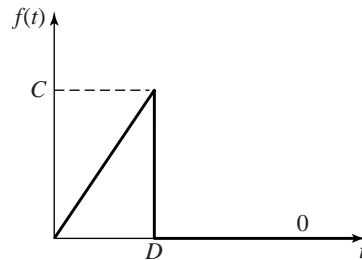
3.40   Obtain the Laplace transform of the function plotted in Figure P3.40.

Figure P3.40
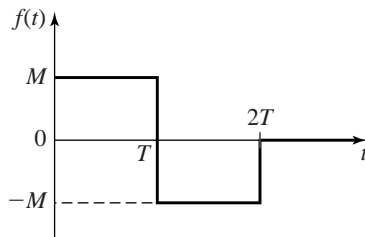
3.41   Obtain the response $x(t)$ of the following model, where the input $P(t)$ is a rectangular pulse of height 3 and duration 5:

$$4\dot{x} + x = P(t) \qquad x(0) = 0$$

3.42 The Taylor series expansion for $\tan t$ is

$$\tan t = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \cdots \qquad |t| < \frac{\pi}{2}$$

Use the first three terms in the series to obtain an approximate closed-form solution of the following problem over the interval $0 \le t \le 0.5$:

$$\dot{x} + x = \tan t \qquad x(0) = 0$$

Compare your answer at $t = 0.5$ with that obtained in Example 3.7.6, which was obtained by using two terms in the series.

3.43 Derive the initial value theorem:

$$\lim_{s \to \infty} s X(s) = x(0+)$$

3.44 Derive the final value theorem:

$$\lim_{s \to 0} s X(s) = \lim_{t \to \infty} x(t)$$

3.45 Derive the integral property of the Laplace transform:

$$\mathcal{L}\left[\int_0^t x(t)\, dt \right] = \frac{X(s)}{s} + \frac{1}{s} \int x(t)\, dt \bigg|_{t=0}$$

Section 3.8 Partial-Fraction Expansion with MATLAB

3.46 Use MATLAB to obtain the inverse transform of the following. If the denominator of the transform has complex roots, express $x(t)$ in terms of a sine and a cosine.

a. $X(s) = \dfrac{8s + 5}{2s^2 + 20s + 48}$

b. $X(s) = \dfrac{4s + 13}{s^2 + 8s + 116}$

c. $X(s) = \dfrac{3s + 2}{s^2(s + 10)}$

d. $X(s) = \dfrac{s^3 + s + 6}{s^4(s + 2)}$

e. $X(s) = \dfrac{4s + 3}{s(s^2 + 6s + 34)}$

f. $X(s) = \dfrac{5s^2 + 3s + 7}{s^3 + 12s^2 + 44s + 48}$

3.47 Use MATLAB to obtain the inverse transform of the following. If the denominator of the transform has complex roots, express $x(t)$ in terms of a sine and a cosine. Hint: you will find it convenient to use the `conv` function to multiply two polynomials.

a. $X(s) = \dfrac{5}{(s + 4)^2(s + 1)}$

b. $X(s) = \dfrac{4s + 9}{(s^2 + 6s + 34)(s^2 + 4s + 20)}$

Section 3.9 Transfer-Function Analysis with MATLAB

3.48 Use MATLAB to solve for and plot the unit-step response of the following models:

  a. $3\ddot{x} + 21\dot{x} + 30x = f(t)$
  b. $5\ddot{x} + 20\dot{x} + 65x = f(t)$
  c. $4\ddot{x} + 32\dot{x} + 60x = 3\dot{f}(t) + 2f(t)$

3.49 Use MATLAB to solve for and plot the unit-impulse response of the following models:

  a. $3\ddot{x} + 21\dot{x} + 30x = f(t)$
  b. $5\ddot{x} + 20\dot{x} + 65x = f(t)$

3.50 Use MATLAB to solve for and plot the impulse response of the following model, where the strength of the impulse is 5:

$$3\ddot{x} + 21\dot{x} + 30x = f(t)$$

3.51 Use MATLAB to solve for and plot the step response of the following model, where the magnitude of the step input is 5:

$$3\ddot{x} + 21\dot{x} + 30x = f(t)$$

3.52 Use MATLAB to solve for and plot the response of the following models for $0 \le t \le 1.5$, where the input is $f(t) = 5t$ and the initial conditions are zero:

  a. $3\ddot{x} + 21\dot{x} + 30x = f(t)$
  b. $5\ddot{x} + 20\dot{x} + 65x = f(t)$
  c. $4\ddot{x} + 32\dot{x} + 60x = 3\dot{f}(t) + 2f(t)$

3.53 Use MATLAB to solve for and plot the response of the following models for $0 \le t \le 6$, where the input is $f(t) = 6\cos 3t$ and the initial conditions are zero:

  a. $3\ddot{x} + 21\dot{x} + 30x = f(t)$
  b. $5\ddot{x} + 20\dot{x} + 65x = f(t)$
  c. $4\ddot{x} + 32\dot{x} + 60x = 3\dot{f}(t) + 2f(t)$

4

Spring and Damper Elements in Mechanical Systems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Model elements containing elasticity as ideal (massless) spring elements.

2. Model elements containing damping as ideal (massless) damper elements.

3. Obtain equations of motion for systems having spring and damper elements.

4. Apply energy methods to obtain equations of motion.

5. Obtain the free and forced response of mass-spring-damper systems.

6. Utilize MATLAB to assist in the response analysis.

I n Chapter 2 we applied Newton's laws of motion to situations in which the masses in question are assumed to be rigid and where the object's motion is relatively uncomplicated, namely, simple translations and simple rotation about a single axis. However, in many applications the mass either deforms somewhat under the action of forces or is connected to another object by an element that deforms. Such deformable elements exert a resisting force that is a function of displacement and are called *spring* elements or *elastic* elements. We treat the modeling of spring elements in Section 4.1 of this chapter, and in Section 4.2 we show how to obtain equations of motion for systems consisting of one or more masses and one or more spring elements.

In Chapter 2 we introduced energy-based methods and the concepts of equivalent mass and equivalent inertia, which simplify the modeling of systems having both translating and rotating components. In Section 4.3 we extend these methods and concepts to spring elements.

Velocity-dependent forces such as fluid drag are the subject of Section 4.4. Elements exerting a resisting force that is a function of velocity are called *damping* or *damper* elements. This section and Section 4.5 provide additional examples of how to model systems containing mass, spring, and damping elements.

MATLAB can be used to perform some of the algebra required to obtain transfer functions of multimass systems, and can be used to find the forced and free response. Section 4.7 shows how this is accomplished. ∎

4.1 SPRING ELEMENTS

All physical objects deform somewhat under the action of externally applied forces. When the deformation is negligible for the purpose of the analysis, or when the corresponding forces are negligible, we can treat the object as a rigid body. Sometimes, however, an elastic element is intentionally included in the system, as with a spring in a vehicle suspension. Sometimes the element is not intended to be elastic, but deforms anyway because it is subjected to large forces or torques. This can be the case with the boom or cables of a large crane that lifts a heavy load. In such cases, we must include the deformation and corresponding forces in our analysis.

The most familiar spring is probably the helical coil spring, such as those used in vehicle suspensions and those found in retractable pens. The purpose of the spring in both applications is to provide a restoring force. However, considerably more engineering analysis is required for the vehicle spring application because the spring can cause undesirable motion of the wheel and chassis, such as vibration. Because the pen motion is constrained and cannot vibrate, we do not need as sophisticated an analysis to see if the spring will work.

Many engineering applications involving elastic elements, however, do not contain coil springs but rather involve the deformation of beams, cables, rods, and other mechanical members. In this section we develop the basic elastic properties of many of these common elements.

FORCE-DEFLECTION RELATIONS

Figure 4.1.1 A spring element.

A coil spring has a *free length,* denoted by L in Figure 4.1.1. The free length is the length of the spring when no tensile or compressive forces are applied to it. When a spring is compressed or stretched, it exerts a restoring force that opposes the compression or extension. The general term for the spring's compression or extension is *deflection*. The greater the deflection (compression or extension), the greater the restoring force. The simplest model of this behavior is the so-called *linear force-deflection model,*

$$f = kx \tag{4.1.1}$$

where f is the restoring force, x is the compression or extension distance (the change in length from the free length), and k is the *spring constant,* or *stiffness,* which is defined to be always positive. Typical units for k are lb/ft and N/m. Some references, particularly in the automotive industry, use the term *spring rate* instead of spring constant. Equation (4.1.1) is commonly known as *Hooke's Law,* named after Englishman Robert Hooke (1635–1703).

When $x = 0$, the spring assumes its free length. We must decide whether extension is represented by positive or negative values of x. This choice depends on the particular application. If $x > 0$ corresponds to extension, then a positive value of f represents the force of the spring pulling against whatever is causing the extension. Conversely,

because of Newton's law of action-reaction, the force causing the extension has the same magnitude as f but is in the opposite direction.

The formula for a coil spring is derived in references on machine design. For convenience, we state it here without derivation, for a spring made from round wire.

$$k = \frac{Gd^4}{64nR^3}$$

where d is the wire diameter, R is the radius of the coil, and n is the number of coils. The *shear modulus of elasticity G* is a property of the wire material.

As we will see, other mechanical elements that have elasticity, such as beams, rods, and rubber mounts, can be modeled as springs, and are usually represented pictorially as a coil spring.

TENSILE TEST OF A ROD

A plot of the data for a tension test on a rod is given in Figure 4.1.2. The elongation is the change in the rod's length due to the tension force applied by the testing machine. As the tension force was increased the elongation followed the curve labeled "Increasing." The behavior of the elongation under decreasing tension is shown by the curve labeled "Decreasing." The rod was stretched beyond its *elastic limit,* so that a permanent elongation remained after the tension force was removed.

For the smaller elongations the "Increasing" curve is close to a straight line with a slope of 3500 pounds per one thousandth of an inch, or 3.5×10^6 lb/in. This line is labeled "Linear" in the plot. If we let x represent the elongation in inches and f the tension force in pounds, then the model $f = 3.5 \times 10^6 x$ represents the elastic behavior of the rod. We thus see that the rod's spring constant is 3.5×10^6 lb/in.

This experiment could have been repeated using compressive instead of tensile force. For small compressive deformations x, we would find that the deformations are related to the compressive force f by $f = kx$, where k would have the same value as before. This example indicates that mechanical elements can be described by the linear law $f = kx$, for both compression and extension, as long as the deformations are not
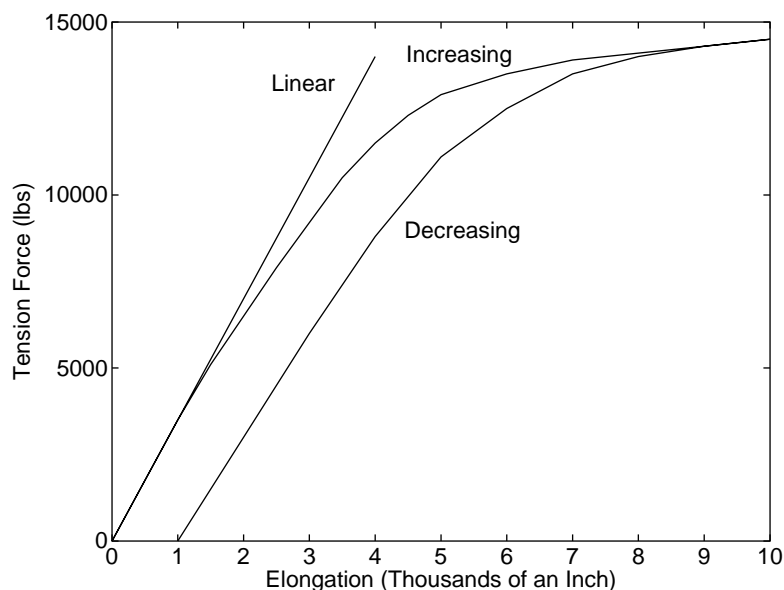
Figure 4.1.2 Plot of tension test data.

too large, that is, deformations not beyond the elastic limit. Note that the larger the deformation, the greater the error that results from using the linear model.

ANALYTICAL DETERMINATION OF THE SPRING CONSTANT

In much engineering design work, we do not have the elements available for testing, and thus we must be able to calculate their spring constant from the geometry and material properties. To do this we can use results from the study of mechanics of materials. Examples 4.1.1 and 4.1.2 show how this is accomplished.

EXAMPLE 4.1.1   Rod with Axial Loading

■ Problem
Derive the spring constant expression for a cylindrical rod subjected to an axial force (either tensile or compressive). The rod length is L and its area is A.

■ Solution
From mechanics of materials references, for example [Roark, 2001], we obtain the force-deflection relation of a cylindrical rod:

$$x = \frac{L}{EA} f = \frac{4L}{\pi E D^2} f$$

where x is the axial deformation of the rod, f is the applied axial force, A is the cross-sectional area, D is the diameter, and E is the *modulus of elasticity* of the rod material. Rewrite this equation as

$$f = \frac{EA}{L} x = \frac{\pi E D^2}{4L} x$$

Thus we see that the spring constant is $k = EA/L = \pi E D^2/4L$.

The modulus of elasticity of steel is approximately 3×10^7 lb/in.^{2}. Thus a steel rod 20 in. long and 1.73 in. in diameter would have a spring constant of 3.5×10^6 lb/in., the same as the rod whose curve is plotted in Figure 4.1.2.

Beams used to support objects can act like springs when subjected to large forces. Beams can have a variety of shapes and can be supported in a number of ways. The beam geometry, beam material, and the method of support determine its spring constant.

EXAMPLE 4.1.2   Spring Constant of a Fixed-End Beam

■ Problem
Derive the spring constant expression of a fixed-end beam of length L, thickness h, and width w, assuming that the force f and deflection x are at the center of the beam (Figure 4.1.3).

Figure 4.1.3 A fixed-end beam.
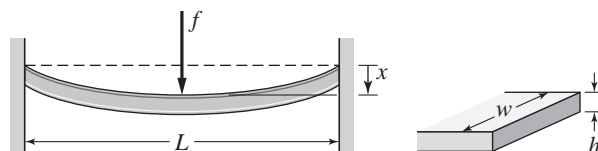
■ Solution

The force-deflection relation of a fixed-end beam is found in mechanics of materials references. It is

$$x = \frac{L^3}{192EI_A} f$$

where x is the deflection in the middle of the beam, f is the force applied at the middle of the beam, and I_A is the area moment of inertia about the beam's longitudinal axis [Roark, 2001]. The area moment I_A is computed with an integral of an area element dA.

$$I_A = \int r^2 \, dA$$

Formulas for the area moments are available in most engineering mechanics texts. For a beam having a rectangular cross section with a width w and thickness h, the area moment is

$$I_A = \frac{wh^3}{12}$$

Thus the force-deflection relation reduces to

$$x = \frac{12L^3}{192Ewh^3} f = \frac{L^3}{16Ewh^3} f$$

The spring constant is the ratio of the applied force f to the resulting deflection x, or

$$k = \frac{f}{x} = \frac{16Ewh^3}{L^3}$$

Table 4.1.1 lists the expressions for the spring constants of several common elements. Note that two beams of identical shape and material, one a cantilever and one fixed-end, have spring constants that differ by a factor of 64. The fixed-end beam is thus 64 times "stiffer" than the cantilever beam! This illustrates the effect of the support arrangement on the spring constant.

A single leaf spring is shown in Table 4.1.1. Springs for vehicle suspensions are often constructed by strapping together several layers of such springs, as shown in Figure 4.1.4. The value of the total spring constant depends not only on the spring constants of the individual layers, but also on the how they are strapped together, the method of attachment to the axle and chassis, and whether any material to reduce friction has been placed between the layers. There is no simple formula for k that accounts for all these variables.

Figure 4.1.4 A leaf spring.

TORSIONAL SPRING ELEMENTS

Table 4.1.2 shows a hollow cylinder subjected to a twisting torque. The resulting twist in the cylinder is called *torsion*. This cylinder is an example of a *torsional* spring, which resists with an opposing torque when twisted. For a torsional spring element we will use the "curly" symbol shown in Figure 4.1.5. The spring relation for a torsional spring is usually written as

$$T = k_T \theta \tag{4.1.2}$$

where θ is the net angular twist in the element, T is the torque causing the twist, and k_T is the *torsional spring constant*. We assign $\theta = 0$ at the spring position where there is no torque in the spring. This is analogous to the free length position of a translational spring. Note that the units of the torsional and translational spring constants are not the same. FPS units for k_T are lb-ft/rad; in SI the units are N · m/rad.

Figure 4.1.5 Symbol for a torsional spring element.
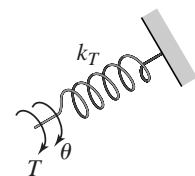
Table 4.1.1 Spring constants of common elements.
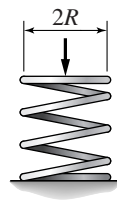
Coil spring

$$k = \frac{Gd^4}{64nR^3}$$

d = wire diameter
n = number of coils

Solid rod
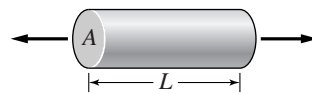
$$k = \frac{EA}{L}$$

Simply supported beam
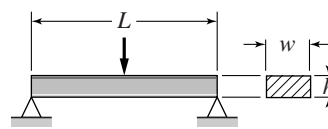
$$k = \frac{4Ewh^3}{L^3}$$

Cantilever beam
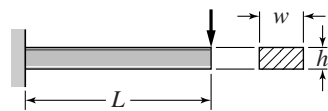
$$k = \frac{Ewh^3}{4L^3}$$

w = beam width
h = beam thickness

Fixed-end beam
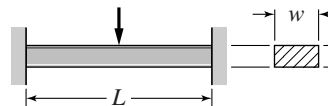
$$k = \frac{16Ewh^3}{L^3}$$

Parabolic leaf spring
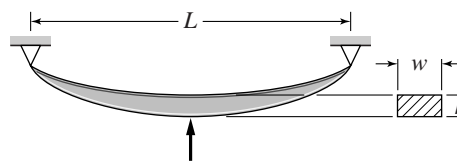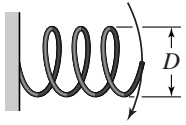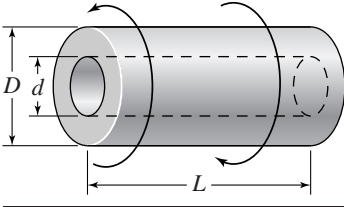
$$k = \frac{2Ewh^3}{L^3}$$

Torsional spring constants of two elements are given in Table 4.1.2. They depend on the geometry of the element and its material properties, namely, E and G, the *shear* modulus of elasticity.

If the cylinder is solid, the formula for k_T given in Table 4.1.2 becomes

$$k_T = \frac{\pi GD^4}{32L}$$

Table 4.1.2 Torsional spring constants of common elements.

Coil spring

$$k_T = \frac{Ed^4}{64nD}$$
$d =$ wire diameter
$n =$ number of coils

Hollow shaft

$$k_T = \frac{\pi G(D^4 - d^4)}{32L}$$

Figure 4.1.6 A torsion-bar suspension.
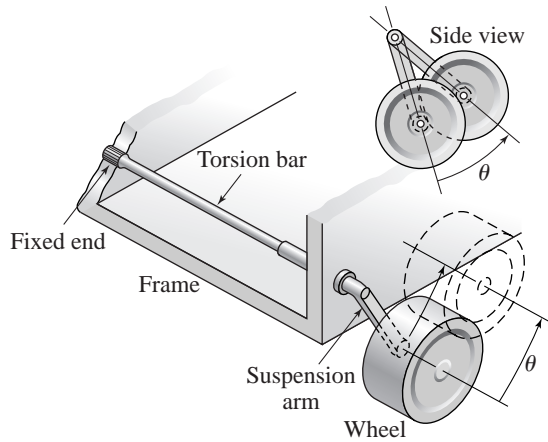
Note that a rod can be designed for axial or torsional loading, such as with a torsion-bar vehicle suspension. Thus, there are two spring constants for rods, a translational constant $k = \pi ED^2/4L$, given previously, and a torsional constant k_T. Figure 4.1.6 shows an example of a torsion-bar suspension, which was invented by Dr. Ferdinand Porsche in the 1930s. As the ground motion pushes the wheel up, the torsion bar twists and resists the motion.

A coil spring can also be designed for axial or torsional loading. Thus there are two spring constants for coil springs, a translational constant k, given in Table 4.1.1, and the torsional constant k_T, given in Table 4.1.2.

SERIES AND PARALLEL SPRING ELEMENTS

In many applications, multiple spring elements are used, and in such cases we must obtain the equivalent spring constant of the combined elements. When two springs are connected side-by-side, as in Figure 4.1.7a, we can determine their equivalent spring constant as follows. Assuming that the force f is applied so that both springs have the same deflection x but different forces f_1 and f_2, then

$$x = \frac{f_1}{k_1} = \frac{f_2}{k_2}$$

Figure 4.1.7 Parallel spring elements.

(a)   (b)
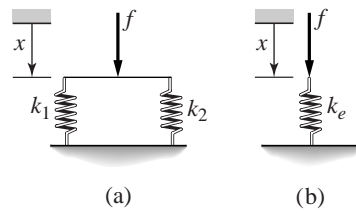
Figure 4.1.8 Series spring elements.

(a)   (b)
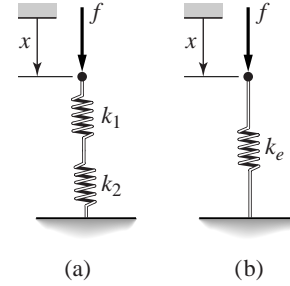
If the system is in static equilibrium, then

$$f = f_1 + f_2 = k_1 x + k_2 x = (k_1 + k_2)x$$

For the equivalent system shown in part (b) of the figure, $f = k_e x$, and thus we see that its equivalent spring constant is given by $k_e = k_1 + k_2$. This formula can be extended to the case of n springs connected side-by-side as follows:

$$k_e = \sum_{i=1}^{n} k_i \tag{4.1.3}$$

When two springs are connected end-to-end, as in Figure 4.1.8a, we can determine their equivalent spring constant as follows. Assuming both springs are in static equilibrium, then both springs are subjected to the same force f, but their deflections f/k_1 and f/k_2 will not be the same unless their spring constants are equal. The total deflection x of the system is obtained from

$$x = \frac{f}{k_1} + \frac{f}{k_2} = \left(\frac{1}{k_1} + \frac{1}{k_2} \right) f$$

For the equivalent system shown in part (b) of the figure, $f = k_e x$, and thus we see that its equivalent spring constant is given by

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

This formula can be extended to the case of n springs connected end-to-end as follows:

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i} \tag{4.1.4}$$

The derivations of (4.1.3) and (4.1.4) assumed that the product of the spring mass times its acceleration is zero, which means that either the system is in static equilibrium or the spring masses are very small compared to the other masses in the system.

The symbols for springs connected end-to-end look like the symbols for electrical resistors similarly connected. Such resistors are said to be in *series* and therefore, springs connected end-to-end are sometimes said to be in series. However, the equivalent electrical resistance is the sum of the individual resistances, whereas series springs obey the reciprocal rule (4.1.4). This similarity in appearance of the symbols often leads people to mistakenly add the spring constants of springs connected end-to-end, just as series resistances are added. Springs connected side-by-side are sometimes called *parallel* springs, and their spring constants should be added.

According to this usage then, parallel springs have the same deflection; series springs experience the same force or torque.

Spring Constant of a Stepped Shaft | **EXAMPLE 4.1.3**

■ Problem
Determine the expression for the equivalent torsional spring constant for the stepped shaft shown in Figure 4.1.9.

■ Solution
Each shaft sustains the same torque but has a different twist angle θ. Therefore, they are in series so that $T = k_{T_1}\theta_1 = k_{T_2}\theta_2$, and the equivalent spring constant is given by

$$\frac{1}{k_{T_e}} = \frac{1}{k_{T_1}} + \frac{1}{k_{T_2}}$$

where k_{T_1} and k_{T_2} are given in Table 4.1.2 as

$$k_{T_i} = \frac{G\pi D_i^4}{32L_i} \qquad i = 1, 2$$

Thus

$$k_{T_e} = \frac{k_{T_1} k_{T_2}}{k_{T_1} + k_{T_2}}$$
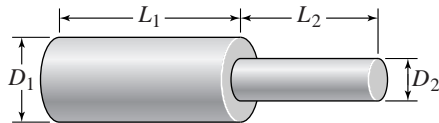
Figure 4.1.9 A stepped shaft.

Spring Constant of a Lever System | **EXAMPLE 4.1.4**

■ Problem
Figure 4.1.10 shows a horizontal force f acting on a lever that is attached to two springs. Assume that the resulting motion is small enough to be only horizontal and determine the expression for the equivalent spring constant that relates the applied force f to the resulting displacement x.

■ Solution
From the triangle shown in part (b) of the figure, for small angles θ, the upper spring deflection is x and the deflection of the lower spring is $x/2$. Thus the free body diagram is as shown in part (c) of the figure. For static equilibrium, the net moment about point O must be zero. This gives

$$fL - kxL - k\frac{x}{2}\frac{L}{2} = 0$$
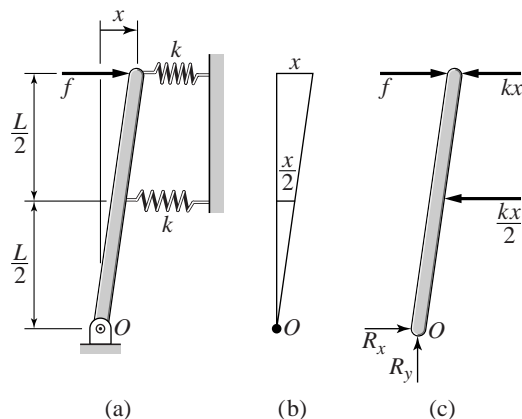
Figure 4.1.10 A lever-spring system.

(a)   (b)   (c)

Therefore,

$$f = k\left(x + \frac{x}{4}\right) = \frac{5}{4}kx$$

and the equivalent spring constant is $k_e = 5k/4$.

Note that although these two springs appear to be connected side-by-side, they are not in parallel, because they do not have the same deflection. Thus their equivalent spring constant is not given by the sum, $2k$.

NONLINEAR SPRING ELEMENTS

Up to now we have used the linear spring model $f = kx$. Even though this model is sometimes only an approximation, nevertheless it leads to differential equation models that are linear and therefore relatively easy to solve. Sometimes, however, the use of a nonlinear model is unavoidable. This is the case when a system is designed to utilize two or more spring elements to achieve a spring constant that varies with the applied load. Even if each spring element is linear, the combined system will be nonlinear.

An example of such a system is shown in Figure 4.1.11a. This is a representation of systems used for packaging and in vehicle suspensions, for example. The two side springs provide additional stiffness when the weight W is too heavy for the center spring.

EXAMPLE 4.1.5

Deflection of a Nonlinear System

■ Problem
Obtain the deflection of the system model shown in Figure 4.1.11a, as a function of the weight W. Assume that each spring exerts a force that is proportional to its compression.

■ Solution
When the weight W is gently placed, it moves through a distance x before coming to rest. From statics, we know that the weight force must balance the spring forces at this new position. Thus,

$$W = k_1 x \qquad \text{if } x < d$$

$$W = k_1 x + 2k_2(x - d) \qquad \text{if } x \geq d$$

We can solve these relations for x as follows:

$$x = \frac{W}{k_1} \qquad \text{if } x < d$$

$$x = \frac{W + 2k_2 d}{k_1 + 2k_2} \qquad \text{if } x \geq d$$

These relations can be used to generate the plot of x versus W, shown in part (b) of the figure.

Figure 4.1.11 A nonlinear spring arrangement.
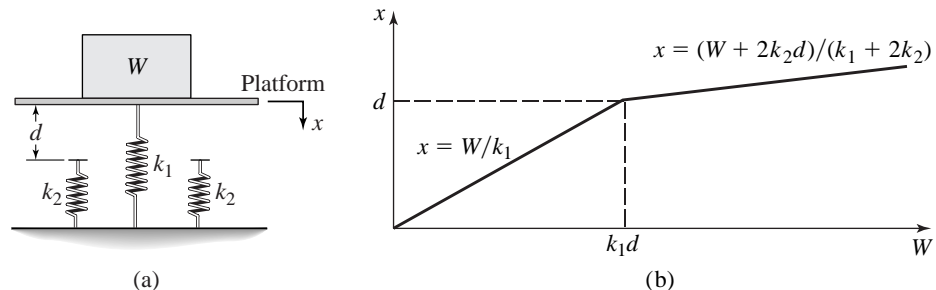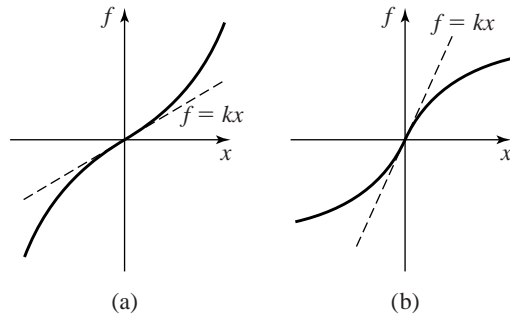
The system in Example 4.1.5 consists of linear spring elements but it has a nonlinear force-deflection relation because of the way the springs are arranged. However, most spring elements display nonlinear behavior if the deflection is large enough. Figure 4.1.12 is a plot of the force-deflection relations for three types of spring elements: the linear spring element, a *hardening* spring element, and a *softening* spring element. The stiffness k is the slope of the force-deflection curve and is constant for the linear spring element. A nonlinear spring element does not have a single stiffness value because its slope is variable. For the hardening element, sometimes called a *hard* spring, its slope and thus its stiffness increases with deflection. The stiffness of a softening element, also called a *soft* spring, decreases with deflection.

4.2 MODELING MASS-SPRING SYSTEMS

If we assume that an object is a rigid body and if we neglect the force distribution within the object, we can treat the object as if its mass were concentrated at its mass center. This is the *point-mass* assumption, which makes it easier to obtain the translational equations of motion, because the object's dimensions can be ignored and all external forces can be treated as if they acted through the mass center. If the object can rotate, then the translational equations must be supplemented with the rotational equations of motion, which were treated in Sections 2.2 and 2.4.

If the system cannot be modeled as a rigid body then we must develop a distributed-parameter model that consists of a *partial* differential equation, which is more difficult to solve.

REAL VERSUS IDEAL SPRING ELEMENTS

By their very nature, all real spring elements have mass and are not rigid bodies. Thus, because it is much easier to derive an equation of motion for a rigid body than for a distributed-mass, flexible system, the basic challenge in modeling mass-spring systems is to first decide whether and how the system can be modeled as one or more rigid bodies.

If the system consists of an object attached to a spring, the simplest way to do this is to neglect the spring mass relative to the mass of the object and take the mass center of the system to be located at the mass center of the object. This assumption is accurate in many practical applications, but to be comfortable with this assumption you should know the numerical values of the masses of the object and the spring element. In some of the homework problems and some of the examples to follow, the numerical values are not given. In such cases, unless otherwise explicitly stated, you should assume that the spring mass can be neglected.

Figure 4.2.1 Models of
mass-spring systems.

(a)   (b)   (c)   (d)
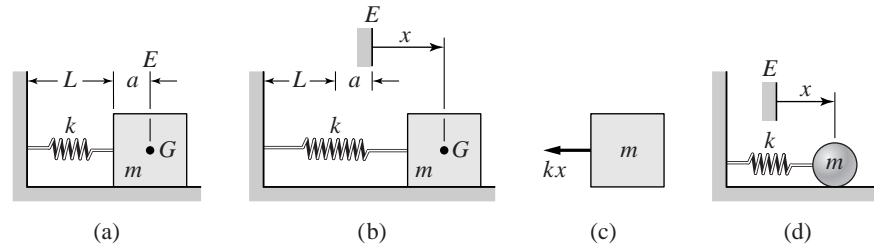
In Section 4.3 we will develop a method to account for the spring mass without
the need to develop a partial differential equation model. This method will be useful
for applications where the spring mass is neither negligible nor the dominant mass in
the system.

An *ideal* spring element is massless. A real spring element can be represented by
an ideal element either by neglecting its mass or by including it in another mass in the
system.

EFFECT OF SPRING FREE LENGTH AND OBJECT GEOMETRY

Suppose we attach a cube of mass *m* and side length $2a$ to a linear spring of negligible
mass, and we fix the other end of the spring to a rigid support, as shown in Figure 4.2.1a.
We assume that the horizontal surface is frictionless. If the mass is homogeneous its
center of mass is at the geometric center *G* of the cube. The free length of the spring is
L and the mass *m* is in equilibrium when the spring is at its free length. The equilibrium
location of *G* is the point marked *E*. Part (b) of the figure shows the mass displaced a
distance *x* from the equilibrium position. In this position, the spring has been stretched a
distance *x* from its free length, and thus its force is kx. The free body diagram, displaying
only the horizontal force, is shown in part (c) of the figure. From this diagram we can
obtain the following equation of motion:

$$m\ddot{x} = -kx \qquad (4.2.1)$$

Note that neither the free length *L* nor the cube dimension *a* appears in the equation
of motion. These two parameters need to be known only to locate the equilibrium
position *E* of the mass center. Therefore we can represent the object as a point mass,
as shown in Figure 4.2.1d.

Unless otherwise specified you should assume that the objects in our diagrams can
be treated as point masses and therefore their geometric dimensions need not be known
to obtain the equation of motion. You should also assume that the location of the equi-
librium position is known. The shaded-line symbol shown in Figure 4.2.1a is used to
indicate a rigid support, such as the horizontal surface and the vertical wall, and also
to indicate the location of a fixed coordinate origin, such as the origin of *x*.

EFFECT OF GRAVITY

Now suppose the object slides on an inclined frictionless surface. In Figure 4.2.2a
the spring is at its free length. If we let the object slide until it reaches equilibrium
(Figure 4.2.2b), the spring stretches a distance δ_{st}, which is called the *static spring
deflection*. Because the mass is in equilibrium, the sum of the forces acting on it must

Figure 4.2.2 Effect of inclination
on a mass-spring model.

(a)   (b)   (c)   (d)
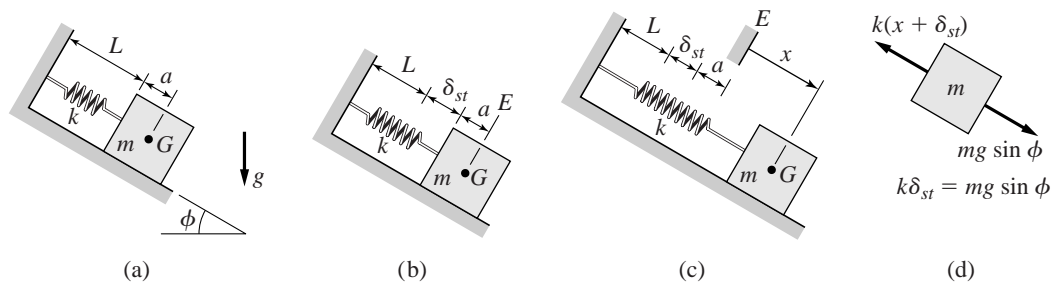
be zero. Thus, for the forces parallel to the inclined surface,

$$mg \sin \phi - k\delta_{st} = 0$$

Figure 4.2.2c shows the object displaced a distance x from the equilibrium position. In this position the spring has been stretched a distance $x + \delta_{st}$ from its free length, and thus its force is $k(x + \delta_{st})$. The free-body diagram displaying only the forces parallel to the plane is shown in part (d) of the figure. From this diagram we can obtain the following equation of motion:

$$m\ddot{x} = -k(x + \delta_{st}) + mg \sin \phi = -kx + (mg \sin \phi - k\delta_{st})$$

Because $mg \sin \phi = k\delta_{st}$ the term within parentheses is zero and the equation of motion reduces to $m\ddot{x} = -kx$, the same as for the system shown in Figure 4.2.1.

CHOOSING THE EQUILIBRIUM POSITION AS COORDINATE REFERENCE

The example in Figure 4.2.2 shows that for a mass connected to a *linear* spring element, the force due to gravity is canceled out of the equation of motion by the force in the spring due to its static deflection, *as long the displacement of the mass is measured from the equilibrium position*. We will refer to the spring force caused by its static deflection as the *static* spring force and the spring force caused by the variable displacement x as the *dynamic* spring force.

We need not choose the equilibrium location as the coordinate reference. If we choose another coordinate, however, the corresponding equation of motion will contain additional terms that correspond to the static forces in the system. For example, in Figure 4.2.3a if we choose the coordinate y, the corresponding free body diagram is shown in part (b) of the figure. The resulting equation of motion is

$$m\ddot{y} = -k(y - L) + mg \sin \phi = -ky + kL + mg \sin \phi$$

Note that $kL + mg \sin \phi \neq 0$ so the static force terms do not cancel out of the equation.

The advantages of choosing the equilibrium position as the coordinate origin are (1) that we need not specify the geometric dimensions of the mass and (2) that this choice simplifies the equation of motion by eliminating the static forces.

Figure 4.2.3 Choice of coordinate origin for a mass-spring model.

(a)

(b)
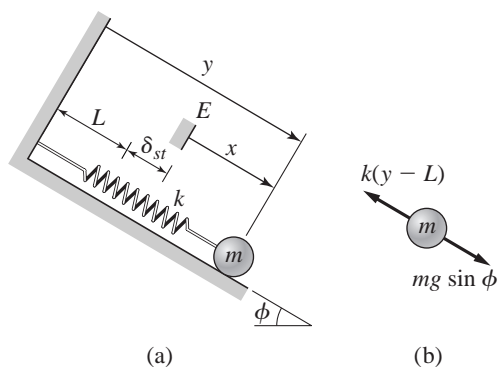
Figure 4.2.4 Static deflection in a mass-spring system.

(a)   (b)   (c)   (d)
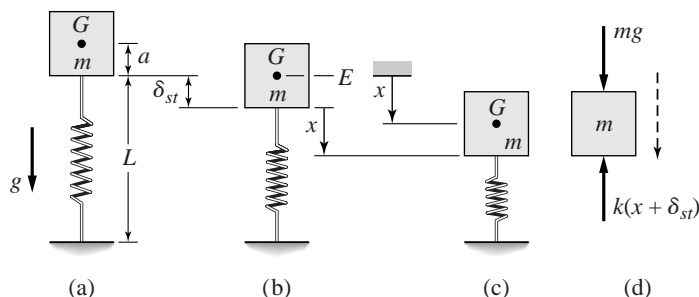
Now suppose we place the mass m on a spring as shown in Figure 4.2.4a. Assume that the mass is constrained to move in only the vertical direction. If we let the mass settle down to its equilibrium position at E, the spring compresses an amount δ_{st} and thus $mg = k\delta_{st}$ (Figure 4.2.4b). If the mass is displaced a distance x down from equilibrium (Figure 4.2.4c), the resulting spring force is $k(x + \delta_{st})$ and the resulting free body diagram is shown in part (d) of the figure. Thus the equation of motion is

$$m\ddot{x} = -k(x + \delta_{st}) + mg = -kx + (mg - k\delta_{st}) = -kx$$

So the equation of motion reduces to $m\ddot{x} = -kx$.

In Figure 4.2.4c, we imagined the mass to be displaced downward from equilibrium and thus we chose the coordinate x to be positive downward. However, we are free to imagine the mass being displaced upward; in this case, we would choose x to be positive upward and would obtain the same equation of motion: $m\ddot{x} = -kx$. You should draw the free body diagram for this case to make sure that you understand the principles.

Figure 4.2.5 shows three situations, and the corresponding free body diagrams, that have the same equation of motion, $m\ddot{x} = -kx$.

It is important to understand that any forces acting on the mass, other than gravity and the spring force, are not to be included when determining the location of the equilibrium position. For example, in Figure 4.2.6a a force f acts on the mass. The equilibrium position E is the location of the mass at which $k\delta_{st} = mg \sin \phi$ when $f = 0$. From the free body diagram shown in part (b), the equation of motion is $m\ddot{x} = f - kx$.

The previous analysis is based on a system model that contains a linear spring and a constant gravity force. For nonlinear spring elements, the gravity forces may or may not appear in the equation of motion. The gravity force acts like a spring in some applications and thus it might appear in the equation of motion. For example, the equation of motion for a pendulum, derived in Chapter 2 for small angles, is $mL^2\ddot{\theta} = -mgL\theta$. The gravity term is not canceled out in this equation because the effect of gravity here is not a constant torque but rather is a torque $mgL\theta$ that is a function of the coordinate θ.

Figure 4.2.5 Choice of coordinate direction for a mass-spring model.
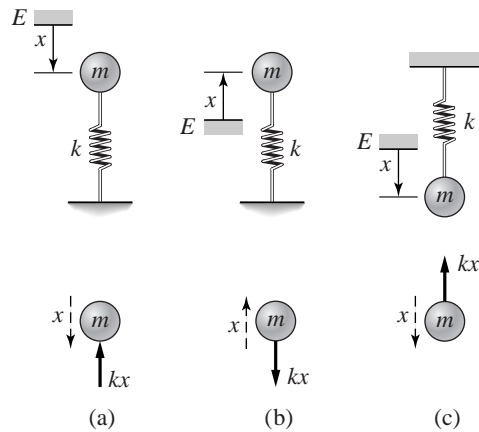
Figure 4.2.6 Modeling an external force on a mass-spring system.
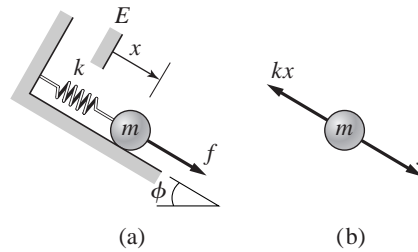
Equation of Motion of a Two-Mass System | **EXAMPLE 4.2.1**

■ Problem

Derive the equations of motion of the two-mass system shown in Figure 4.2.7a.

■ Solution

Choose the coordinates x_1 and x_2 as the displacements of the masses from their equilibrium positions. In equilibrium the static forces in the springs cancel the weights of the masses. Thus the free body diagrams showing the dynamic forces, and not the static forces, are as shown in Figure 4.2.7b. These diagrams have been drawn with the assumption that the displacement x_1 of mass m_1 from its equilibrium position is greater than the displacement of m_2. From these diagrams we obtain the equations of motion:

$$m_1\ddot{x}_1 = f - k_1(x_1 - x_2)$$
$$m_2\ddot{x}_2 = k_1(x_1 - x_2) - k_2 x_2$$

If we move all terms to the left side of the equal sign except for the external force f, we obtain

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = f \tag{1}$$

$$m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2 x_2 = 0 \tag{2}$$

In drawing the free body diagrams of multimass systems, you must make an assumption about the relative motions of each mass. For example, we could have assumed that the displacement x_1 of mass m_1 from its equilibrium position is *less* than the displacement of m_2. Figure 4.2.7c shows the free body diagrams drawn for this assumption. If your assumptions are correct, the forces shown on the diagram must be *positive*. Note that the directions of the forces associated with spring k_1 are the opposite of those in part (b) of the figure. You should confirm that the diagram in part (c) results in equations of motion that are identical to equations (1) and (2).

You must be consistent in your assumptions made about the relative motion when drawing the free-body diagrams. A common mistake is to use one assumption to obtain the free-body diagram for mass m_1 but another assumption for mass m_2.

Figure 4.2.7 A system with
two masses.

(a)   (b)   (c)
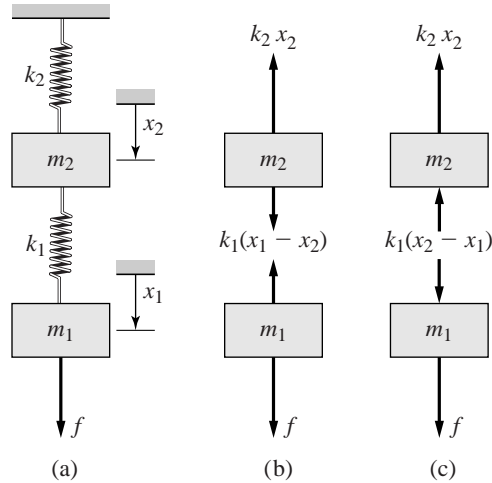
SOLVING THE EQUATION OF MOTION

We have seen that the equation of motion for many mass-spring systems has the form
$m\ddot{x} + kx = f$, where f is an applied force other than gravity and the spring force.
Suppose that the force f is zero and that we set the mass in motion at time $t = 0$ by pulling
it to a position $x(0)$ and releasing it with an initial velocity $\dot{x}(0)$. The solution form of
the equation can be obtained from Table 3.1.3 and is $x(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t$,
where we have defined

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.2.2}$$

Using the initial conditions we find that the constants are $C_1 = \dot{x}(0)/\omega_n$ and $C_2 = x(0)$.
Thus the solution is

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t \tag{4.2.3}$$

This solution shows that the mass oscillates about the rest position $x = 0$ with a
frequency of $\omega_n = \sqrt{k/m}$ radians per unit time. The period of the oscillation is $2\pi/\omega_n$.
The frequency of oscillation ω_n is called the *natural frequency*. The natural frequency
is greater for stiffer springs (larger k values). The amplitude of the oscillation depends
on the initial conditions $x(0)$ and $\dot{x}(0)$.

As shown in Example 3.7.3, the solution (4.2.3) can be put into the following form:

$$x(t) = A \sin(\omega_n t + \phi) \tag{4.2.4}$$

where

$$\sin \phi = \frac{x(0)}{A} \qquad \cos \phi = \frac{\dot{x}(0)}{A\omega_n} \tag{4.2.5}$$

$$A = \sqrt{[x(0)]^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \tag{4.2.6}$$

If f is a unit-step function, and if the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$ are zero, then you should be able to show that the unit-step response is given by

$$x(t) = \frac{1}{k}(1 - \cos \omega_n t) = \frac{1}{k}\left[1 + \sin\left(\omega_n t - \frac{\pi}{2}\right)\right] \qquad (4.2.7)$$

The displacement oscillates about $x = 1/k$ with an amplitude of $1/k$ and a radian frequency ω_n.

| Beam Vibration | **EXAMPLE 4.2.2** |

■ **Problem**

The vertical motion of the mass m attached to the beam in Figure 4.2.8a can be modeled as a mass supported by a spring, as shown in part (b) of the figure. Assume that the beam mass is negligible compared to m so that the beam can be modeled as an ideal spring. Determine the system's natural frequency of oscillation.

■ **Solution**

The spring constant k is that of the fixed-end beam, and is found from Table 4.1.1 to be $k = 16Ewh^3/L^3$. The mass m has the same equation of motion as (4.2.1), where x is measured from the equilibrium position of the mass. Thus, if the mass m on the beam is initially displaced vertically, it will oscillate about its rest position with a frequency of

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16Ewh^3}{mL^3}}$$

The source of disturbing forces that initiate such motion will be examined in later chapters. If the beam mass is appreciable, then we must modify the equation of motion. We will see how to do this in Section 4.3.

(a)   (b)
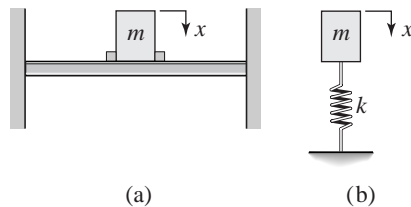
Figure 4.2.8 Model of a mass supported by a fixed-end beam.

| A Torsional Spring System | **EXAMPLE 4.2.3** |

■ **Problem**

Consider a torsional system like that shown in Figure 4.2.9a. A cylinder having inertia I is attached to a rod, whose torsional spring constant is k_T. The angle of twist is θ. Assume that the inertia of the rod is negligible compared to the inertia I so that the rod can be modeled as an ideal torsional spring. Obtain the equation of motion in terms of θ and determine the natural frequency.

■ **Solution**

Because the rod is modeled as an ideal torsional spring, this system is conceptually identical to that shown in Figure 4.2.9b. The free body diagram is shown in part (c) of the figure. From this diagram we obtain the following equation of motion.

$$I\ddot{\theta} = -k_T\theta$$

Figure 4.2.9 A torsional
spring system.

(a)   (b)   (c)
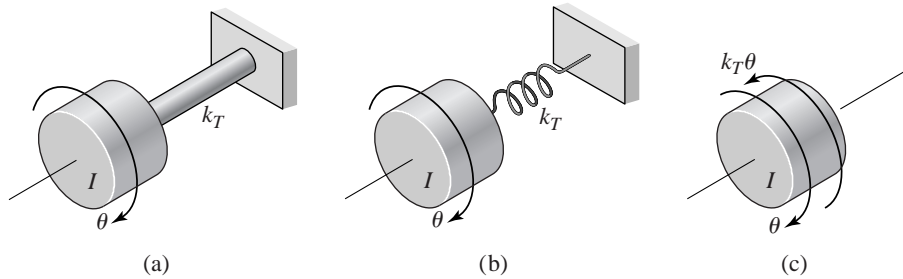
This has the same form as (4.2.1), and thus we can see immediately that the natural frequency is

$$\omega_n = \sqrt{\frac{k_T}{I}}$$

If the cylinder is twisted and then released, it will oscillate about the equilibrium $\theta = 0$ with a frequency of $\sqrt{k_T/I}$ radians per unit time. This result assumes that the inertia of the rod is very small compared to the inertia I of the attached cylinder. If the rod inertia is appreciable, then we must modify the equation of motion, as will be discussed in Section 4.3.

DISPLACEMENT INPUTS AND SPRING ELEMENTS

Consider the mass-spring system and its free body diagram shown in Figure 4.2.10a. This gives the equation of motion $m\ddot{x} + kx = f$. To solve this equation for $x(t)$, we must know the force $f(t)$.

Now consider the system shown in Figure 4.2.10b, where we are given the displacement $y(t)$ of the left-hand end of the spring. This represents a practical application in which a rotating cam causes the follower to move the left-hand end of the spring, as in part (c) of the figure. If we know the cam profile and its rotational speed then we can determine $y(t)$. Suppose that when $x = y = 0$, both springs are at their free

Figure 4.2.10 Force and
displacement inputs.

(a)

(b)

(c)
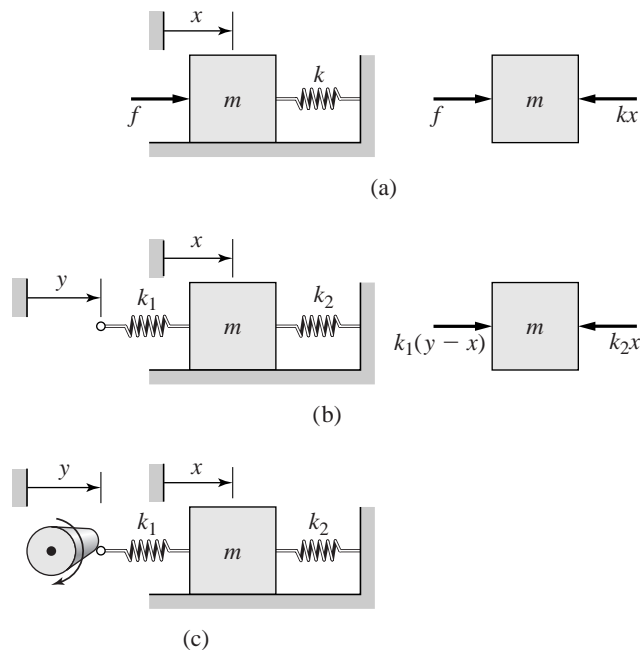
lengths. To draw the free body diagram we must make an assumption about the relative displacements of the endpoints of the spring element. The free body diagram has been drawn with the assumption that $y > x$. Here we are not given an applied force as an input, but nevertheless we must draw the free body diagram showing the forces acting on the mass. The force produced by the given displacement $y(t)$ is the resulting spring force $k(y - x)$. The equation of motion is $m\ddot{x} = k_1(y - x) - k_2 x$. Note that we must be given $y(t)$ to solve this equation for $x(t)$. If we need to obtain the force acting on the follower as a result of the motion, we must first solve for $x(t)$ and then compute the follower force from $k_1(y - x)$. This force is of interest to designers because it indicates how much wear will occur on the follower surface.

When displacement inputs are given, it is important to realize that ultimately the displacement is generated by a force (or torque) and that this force must be great enough to generate the specified displacement in the presence of any resisting forces or system inertia. For example, the motor driving the cam must be able to supply enough torque to generate the motion $y(t)$.

SIMPLE HARMONIC MOTION

From the equation of motion $m\ddot{x} = -kx$ we can see that the acceleration is $\ddot{x} = -kx/m = -\omega_n^2 x$. This type of motion, where the acceleration is proportional to the displacement but opposite in sign, is called *simple harmonic motion*. It occurs when the restoring force—here, the spring force—is proportional to the displacement. It is helpful to understand the relation between the displacement, velocity, and acceleration in simple harmonic motion. Expressions for the velocity and acceleration are obtained by differentiating $x(t)$, whose expression is given by (4.2.4):

$$\dot{x}(t) = A\omega_n \cos(\omega_n t + \phi) = A\omega_n \sin\left(\omega_n t + \phi + \frac{\pi}{2}\right)$$

$$\ddot{x}(t) = -A\omega_n^2 \sin(\omega_n t + \phi) = A\omega_n^2 \sin(\omega_n t + \phi + \pi)$$

The displacement, velocity, and acceleration all oscillate with the same frequency ω_n but they have different amplitudes and are shifted in time relative to one another.
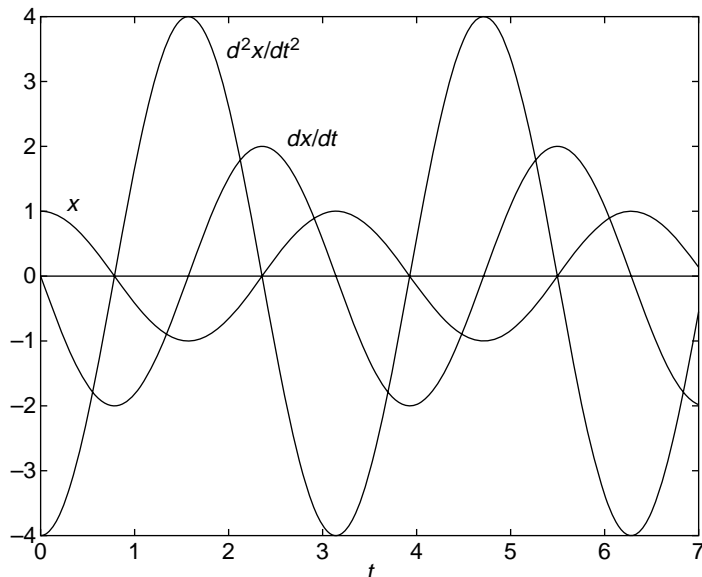
Figure 4.2.11 Plots of displacement, velocity, and acceleration for simple harmonic motion.

The velocity is zero when the displacement and acceleration reach their extreme values. The sign of the acceleration is the opposite of that of the displacement, and the magnitude of the acceleration is ω_n^2 times the magnitude of the displacement. These functions are plotted in Figure 4.2.11 for the case where $x(0) = 1$, $\dot{x}(0) = 0$, and $\omega_n = 2$.

4.3 ENERGY METHODS

The force exerted by a spring is a conservative force. If the spring is linear, then its resisting force is given by $f = -kx$ and thus the potential energy of a linear spring is given by

$$V(x) = \frac{1}{2}kx^2 \tag{4.3.1}$$

where x is the deflection from the *free length* of the spring.

A *torsional* spring exerts a moment M if it is twisted. If the spring is linear the moment is given by $M = k_T\theta$, where θ is the twist angle. The work done by this moment and stored as potential energy in the spring is

$$V(\theta) = \int_0^\theta M\, d\theta = \int_0^\theta k_T\theta\, d\theta = \frac{1}{2}k_T\theta^2 \tag{4.3.2}$$

So the potential energy stored in a torsional spring is $V(\theta) = k_T\theta^2/2$.

The conservation of energy principle states that $T + V = T_0 + V_0 =$ constant, where T and V are the system's kinetic and potential energies. For the system shown in Figure 4.3.1a, for a frictionless surface, the principle gives

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}kx_0^2 = \text{constant}$$

This relation can be rearranged as follows:

$$\frac{m}{2}\left(\dot{x}^2 - \dot{x}_0^2\right) + \frac{k}{2}\left(x^2 - x_0^2\right) = 0$$

which states that $\Delta T + \Delta V = 0$.

For the system shown in Figure 4.3.1b, we must include the effect of gravity, and thus the potential energy is the sum of the spring's potential energy V_s and the gravitational potential energy V_g, which we may choose to be zero at $y = 0$. The conservation of energy principle gives

$$T + V_g + V_s = \text{constant}$$

or

$$\Delta T + \Delta V = \Delta T + \Delta V_g + \Delta V_s = 0$$

The spring is at its free length when $y = 0$, so we can write

$$\frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}ky^2 = \text{constant}$$

Figure 4.3.1 (a) A system having kinetic and elastic potential energy. (b) A system having kinetic, elastic potential, and gravitational potential energy.

(a)   (b)
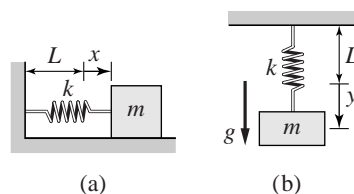
Note that the gravitational potential energy has a negative sign because we have selected y to be positive downward.

The numerical value of the gravitational potential energy depends on the location of the datum and it may be negative. We are free to select the location because only the *change* in gravitational potential energy is significant. Note, however, that a spring's potential energy is always nonnegative and that the potential energy is positive whenever the deflection from the free length is nonzero.

A Force Isolation System

EXAMPLE 4.3.1

■ Problem

Figure 4.3.2 shows a representation of a spring system to isolate the foundation from the force of a falling object. Suppose the weight W is dropped from a height h above the platform attached to the center spring. Determine the maximum spring compression and the maximum force transmitted to the foundation. The given values are $k_1 = 10^4$ N/m, $k_2 = 1.5 \times 10^4$ N/m, $d = 0.1$ m, and $h = 0.5$ m. Consider two cases: (a) $W = 64$ N and (b) $W = 256$ N.

Figure 4.3.2 A force-isolation system.

■ Solution

The velocity of the weight is zero initially and also when the maximum compression is attained. Therefore $\Delta T = 0$ and we have

$$\Delta T + \Delta V = \Delta T + \Delta V_s + \Delta V_g = 0$$

or

$$\Delta V_s + \Delta V_g = 0$$

That is, if the weight is dropped from a height h above the platform and if we choose the gravitational potential energy to be zero at that height, then the maximum spring compression x can be found by adding the change in the weight's gravitational potential energy $0 - W(h + x) = -W(h + x)$ to the change in potential energy stored in the springs. Thus

$$\frac{1}{2}k_1(x^2 - 0) + [0 - W(h + x)] = 0 \qquad \text{if } x < d$$

which gives the following quadratic equation to solve for x:

$$\frac{1}{2}k_1 x^2 - Wx - Wh = 0 \qquad \text{if } x < d \tag{1}$$

If $x \geq d$, $\Delta V_s + \Delta V_g = 0$ gives

$$\frac{1}{2}k_1(x^2 - 0) + \frac{1}{2}(2k_2)[(x - d)^2 - 0] + [0 - W(h + x)] = 0 \qquad \text{if } x \geq d$$

which gives the following quadratic equation to solve for x:

$$(k_1 + 2k_2)x^2 - (2W + 4k_2 d)x + 2k_2 d^2 - 2Wh = 0 \qquad \text{if } x \geq d \tag{2}$$

For the given values, equation (1) becomes

$$10^4 x^2 - 2Wx - W = 0 \qquad \text{if } x < 0.1 \tag{3}$$

and from equation (2),

$$4 \times 10^4 x^2 - (2W + 6000)x + 300 - W = 0 \qquad \text{if } x \geq 0.1 \tag{4}$$

For case (a), the positive root of equation (3) gives $x = 0.0867$, which is less than 0.1. So only the middle spring is compressed, and it is compressed 0.0867 m. The resulting maximum force on the foundation is the spring force $k_1 x = 10^4 (0.0867) = 867$ N.

For case (b), the positive root of equation (3) gives $x = 0.188$, which is greater than 0.1. So all three springs will be compressed. From equation (4),

$$4 \times 10^4 x^2 - (512 + 6000)x + 300 - 256 = 0$$

which has the solutions $x = 0.156$ and $x = 0.007$. We discard the second solution because it is less than 0.1 and thus corresponds to compression in the middle spring only. So the outer springs will be compressed $0.156 - 0.1 = 0.056$ m and the middle spring will be compressed 0.156 m. The resulting maximum force on the foundation is the total spring force $k_1 x + 2k_2(x - 0.1) = 10^4 (0.156) + 2(1.5 \times 10^4)(0.156 - 0.1) = 3240$ N.

OBTAINING THE EQUATION OF MOTION

In mass-spring systems with negligible friction and damping, we can often use the principle of conservation of energy to obtain the equation of motion and, for simple harmonic motion, to determine the frequency of vibration without obtaining the equation of motion.

EXAMPLE 4.3.2

Equation of Motion of a Mass-Spring System

Figure 4.3.3 A mass-spring system.
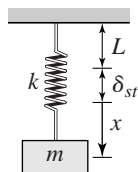
■ **Problem**

Use the energy method to derive the equation of motion of the mass m attached to a spring and moving in the vertical direction, as shown in Figure 4.3.3.

■ **Solution**

With the displacement x measured from the equilibrium position, and taking the gravitational potential energy to be zero at $x = 0$, the total potential energy of the system is

$$V = V_s + V_g = \frac{1}{2}k(x + \delta_{st})^2 - mgx = \frac{1}{2}kx^2 + k\delta_{st}x + \frac{1}{2}k\delta_{st}^2 - mgx$$

Because $k\delta_{st} = mg$ the expression for V becomes

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\delta_{st}^2$$

The total energy of the system is

$$T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{2}k\delta_{st}^2$$

From conservation of mechanical energy, $T + V$ is constant and thus its time derivative is zero. Therefore,

$$\frac{d}{dt}(T + V) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) + \frac{d}{dt}\left(\frac{1}{2}kx^2\right) + \frac{d}{dt}\left(\frac{1}{2}k\delta_{st}^2\right) = 0$$

Evaluating the derivatives gives

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

Canceling $\dot{x}$ gives the equation of motion $m\ddot{x} + kx = 0$.

Example 4.3.2 shows that if we can obtain the expression for the sum of the kinetic and potential energies, $T + V$, the equation of motion can be found by differentiating $T + V$ with respect to time. Although this was a simple example, it illustrates that with this method we need not draw the free body diagrams of every member of a multibody system whose motion can be described by a single coordinate.

RAYLEIGH'S METHOD

We can use the principle of conservation of energy to obtain the natural frequency of a mass-spring system if the spring is linear. This approach is sometimes useful because it does not require that we first obtain the equation of motion.

The method was developed by Lord Rayleigh (John William Strutt) and was presented in his *Theory of Sound* in 1847. A modern reprint is [Rayleigh, 1945]. Rayleigh is considered one of the founders of the study of acoustics and vibration. We illustrate Rayleigh's method here for a second-order system, but it is especially useful for estimating the lowest natural frequency of higher-order systems with several degrees of freedom and distributed parameter systems with an infinite number of natural frequencies.

In simple harmonic motion, the kinetic energy is maximum and the potential energy is minimum at the equilibrium position $x = 0$. When the displacement is maximum, the potential energy is maximum but the kinetic energy is zero. From conservation of energy,

$$T_{max} + V_{min} = T_{min} + V_{max}$$

Thus

$$T_{max} + V_{min} = 0 + V_{max}$$

or

$$T_{max} = V_{max} - V_{min} \qquad (4.3.3)$$

For example, for the mass-spring system oscillating vertically as shown in Figure 4.3.3, $T = m\dot{x}^2/2$ and $V = k(x + \delta_{st})^2/2 - mgx$, and from (4.3.3) we have,

$$T_{max} = \frac{1}{2}m(\dot{x}_{max})^2 = V_{max} - V_{min} = \frac{1}{2}k(x_{max} + \delta_{st})^2 - mgx_{max} - \frac{1}{2}k\delta_{st}^2$$

or

$$\frac{1}{2}m(\dot{x}_{max})^2 = \frac{1}{2}k(x_{max})^2$$

where we have used the fact that $k\delta_{st} = mg$. In simple harmonic motion $|\dot{x}_{max}| = \omega_n |x_{max}|$, and thus,

$$\frac{1}{2}m(\omega_n|x_{max}|)^2 = \frac{1}{2}k|x_{max}|^2$$

Cancel $|x_{max}|^2$ and solve for ω_n to obtain $\omega_n = \sqrt{k/m}$.

In this simple example, we merely obtained the expression for ω_n that we already knew. However, in other applications the expressions for T and V may be different, but if the motion is simple harmonic, we can directly determine the natural frequency by using the fact that $|\dot{x}_{max}| = \omega_n|x_{max}|$ to express T_{max} as a function of $|x_{max}|$ and then equating T_{max} to $V_{max} - V_{min}$. This approach is called *Rayleigh's method*.

EXAMPLE 4.3.3

Natural Frequency of a Suspension System

■ Problem

Figure 4.3.4 shows the suspension of one front wheel of a car in which $L_1 = 0.4$ m and $L_2 = 0.6$ m. The coil spring has a spring constant of $k = 3.6 \times 10^4$ N/m and the car weight associated with that wheel is 3500 N. Determine the suspension's natural frequency for vertical motion.

■ Solution

Imagine that the frame moves down by a distance A_f, while the wheel remains stationary. Then from similar triangles the amplitude A_s of the spring deflection is related to the amplitude A_f of the frame motion by $A_s = L_1 A_f / L_2 = 0.4 A_f / 0.6 = 2 A_f / 3$.

Using the fact that $k\delta_{st} = mg$, the change in potential energy can be written as

$$V_{\max} - V_{\min} = \frac{1}{2}k(A_s + \delta_{st})^2 - mg A_s - \frac{1}{2}k\delta_{st}^2 = \frac{1}{2}kA_s^2 = \frac{1}{2}k\left(\frac{2}{3}A_f\right)^2$$

The amplitude of the velocity of the mass in simple harmonic motion is $\omega_n A_f$, and thus the maximum kinetic energy is

$$T_{\max} = \frac{1}{2}m(\omega_n A_f)^2$$

From Rayleigh's method, $T_{\max} = V_{\max} - V_{\min}$, we obtain

$$\frac{1}{2}m(\omega_n A_f)^2 = \frac{1}{2}k\left(\frac{2}{3}A_f\right)^2$$

Solving this for ω_n we obtain

$$\omega_n = \frac{2}{3}\sqrt{\frac{k}{m}} = \frac{2}{3}\sqrt{\frac{3.6 \times 10^4}{3500/9.8}} = 6.69 \text{ rad/s}$$

Figure 4.3.4 A vehicle suspension.

Upper control arm
Frame
Wheel
L_1
L_2
Lower control arm
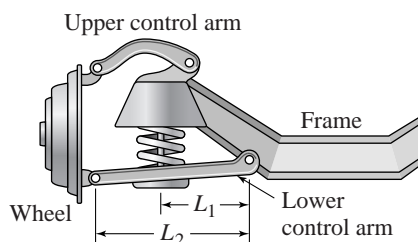
Figure 4.3.5 An example of a spring element with distributed mass.

(a)   (b)
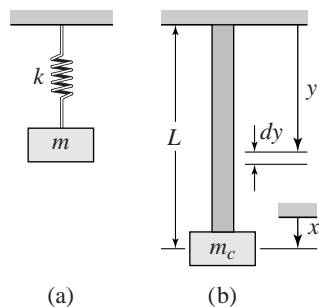
EQUIVALENT MASS OF ELASTIC ELEMENTS

If an elastic element is represented as in Figure 4.3.5a, we assume that the mass of the element either is negligible compared to the rest of the system's mass or has been included in the mass attached to the element. This included mass is called the *equivalent* mass of the element. We do this so that we can obtain a lumped-parameter model of the system. As we did with rigid-body systems in Chapter 2, we compute the equivalent mass by using kinetic energy equivalence, because mass is associated with kinetic energy.

| Equivalent Mass of a Rod | **EXAMPLE 4.3.4** |

■ Problem

The rod shown in Figure 4.3.5b acts like a spring when an axially applied force stretches or compresses the rod. Determine the equivalent mass of the rod.

■ Solution

In Figure 4.3.5b, the mass of an infinitesimal element of thickness dy is $dm_r = \rho \, dy$, where ρ is the mass density per unit length of the material. Thus the kinetic energy of the element is $(dm_r)\dot{y}^2/2$, and the kinetic energy of the entire rod is

$$\text{KE} = \frac{1}{2} \int_0^L \dot{y}^2 \, dm_r = \frac{1}{2} \int_0^L \dot{y}^2 \rho \, dy$$

If we assume that the velocity $\dot{y}$ of the element is linearly proportional to its distance from the support, then

$$\dot{y} = \dot{x} \frac{y}{L}$$

where $\dot{x}$ is the velocity of the end of the rod. Thus

$$\text{KE} = \frac{1}{2} \int_0^L \left(\dot{x} \frac{y}{L} \right)^2 \rho \, dy = \frac{1}{2} \frac{\rho \dot{x}^2}{L^2} \int_0^L y^2 \, dy = \frac{1}{2} \frac{\rho \dot{x}^2}{L^2} \frac{y^3}{3} \bigg|_0^L$$

or

$$\text{KE} = \frac{1}{2} \frac{\rho \dot{x}^2}{L^2} \frac{L^3}{3} = \frac{1}{2} \left(\frac{\rho L}{3} \right) \dot{x}^2 = \frac{1}{2} \frac{m_r}{3} \dot{x}^2$$
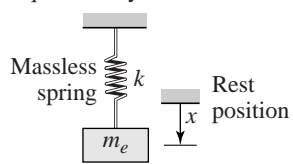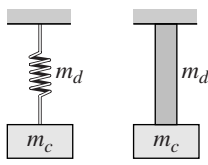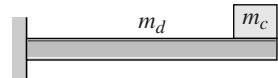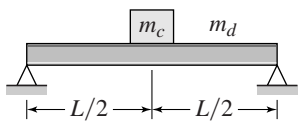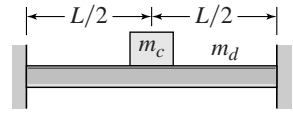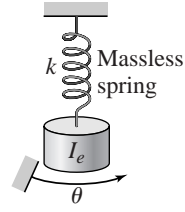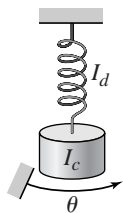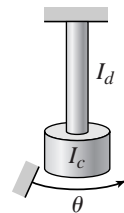
because $\rho L = m_r$, the mass of the rod. For an equivalent mass m_e concentrated at the end of the rod, its kinetic energy is $m_e \dot{x}^2/2$. Thus the equivalent mass of the rod is $m_e = m_r/3$. So the mass m in Figure 4.3.5a is $m = m_c + m_e = m_c + m_r/3$.

The approach of this example can be applied to a coil spring of mass m_s; its equivalent mass is $m_s/3$. A similar approach using the expression for the kinetic energy of rotation, $I_r \dot{\theta}^2/2$, will show that the equivalent inertia of a rod in torsion is $I_r/3$, where I_r is the rod inertia.

This type of analysis can also be used to find the equivalent mass of a beam by using the appropriate formula for the beam's static load-deflection curve to obtain an expression for the velocity of a beam element as a function of its distance from a support. The derivations of such formulas are given in basic references on the mechanics of materials and are beyond the scope of this text. The expressions for the equivalent beam masses given in Table 4.3.1 were derived in this manner. Because the static load-deflection curve describes the *static* deflection, it does not account for inertia effects, and therefore the expressions given in Table 4.3.1 are approximations. However, they are accurate enough for many applications.

Table 4.3.1 Equivalent masses and inertias of common elements.

<div align="center">

Translational systems

</div>

Nomenclature:

m_c = concentrated mass
m_d = distributed mass
m_e = equivalent lumped mass

System model:

$m_e \ddot{x} + kx = 0$

Equivalent system

Helical spring, or rod in tension/compression

$m_e = m_c + m_d/3$

Cantilever beam

$m_e = m_c + 0.23 m_d$

Simply supported beam

$m_e = m_c + 0.50 m_d$

Fixed-end beam

$m_e = m_c + 0.38 m_d$

<div align="center">

Rotational systems

</div>

Nomenclature:

I_c = concentrated inertia
I_d = distributed inertia
I_e = equivalent lumped inertia

System model:

$I_e \ddot{\theta} + k\theta = 0$

Equivalent system

Helical spring

$I_e = I_c + I_d/3$

Rod in torsion

$I_e = I_c + I_d/3$

	EXAMPLE 4.3.5
Equivalent Mass of a Fixed-End Beam	

■ Problem

Figure 4.3.6a shows a motor mounted on a beam with two fixed-end supports. An imbalance in the motor's rotating mass will produce a vertical force f that oscillates at the same frequency as the motor's rotational speed. The resulting beam motion can be excessive if the frequency is near the natural frequency of the beam, as we will see in a later chapter, and excessive beam motion can eventually cause beam failure. Determine the natural frequency of the beam-motor system.

■ Solution

Treating this system as if it were a single mass located at the middle of the beam results in the equivalent system is shown in Figure 4.3.6b, where x is the displacement of the motor from its equilibrium position. The equivalent mass of the system is the motor mass (treated as a concentrated mass m_c) plus the equivalent mass of the beam. From Table 4.3.1, the beam's equivalent mass is $0.38m_d$. Thus the system's equivalent mass is $m_e = m_c + 0.38m_d$.

The equivalent spring constant of the beam is found from Table 4.1.1. It is

$$k = \frac{16Ewh^3}{L^3}$$

where h is the beam's thickness (height) and w is its width (into the page). Thus the system model is $m_e\ddot{x} + kx = f$ where x is the vertical displacement of the beam end from its equilibrium position. The natural frequency is $\omega_n = \sqrt{k/m_e}$, which gives

$$\omega_n = \sqrt{\frac{k}{m_e}} = \sqrt{\frac{16Ewh^3/L^3}{m_c + 0.38m_d}}$$
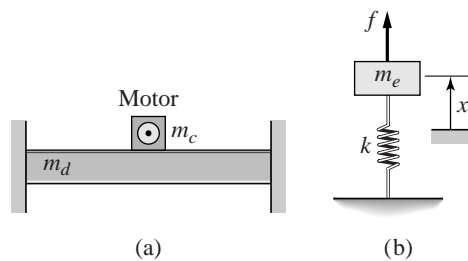
Figure 4.3.6 A motor supported by a fixed-end beam.

(a) (b)

	EXAMPLE 4.3.6
Torsional Vibration with Fixed Ends	

■ Problem

Figure 4.3.7a shows an inertia I_1 rigidly connected to two shafts, each with inertia I_2. The other ends of the shafts are rigidly attached to the supports. The applied torque is T_1. (a) Derive the
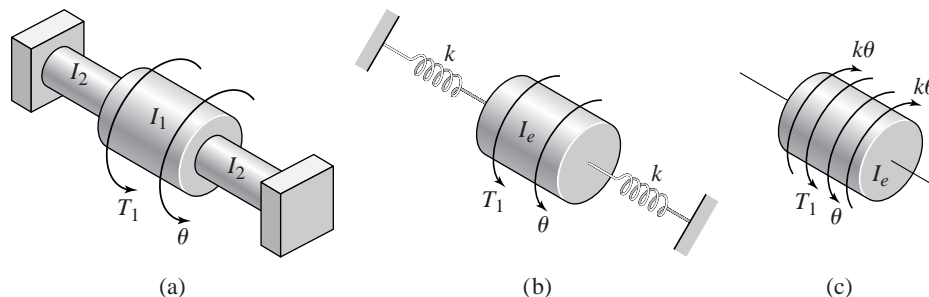
Figure 4.3.7 An inertia fixed to two torsional spring elements.

(a) (b) (c)

equation of motion. (b) Calculate the system's natural frequency if I_1 is a cylinder 5 in. in diameter and 3 in. long; the shafts are cylinders 2 in. in diameter and 6 in. long. The three cylinders are made of steel with a shear modulus $G = 1.73 \times 10^9$ lb/ft^2 and a density $\rho = 15.2$ slug/ft^3.

■ **Solution**

a. As in Table 4.3.1, we add one-third of each the shaft's inertia to the inertia of the cylinder in the middle. Thus the equivalent inertia of the system is

$$I_e = I_1 + 2\left(\frac{1}{3}I_2\right)$$

The equivalent representation is shown in part (b) of the figure, and the free body diagram is shown in part (c) of the figure. From this we obtain the equation of motion:

$$I_e\ddot{\theta} = T - k\theta - k\theta = T - 2k\theta$$

where from Table 4.1.2,

$$k = \frac{G\pi D^4}{32L}$$

b. The value of k is

$$k = \frac{1.73 \times 10^9 \pi \, (2/12)^4}{32(6/12)} = 2.62 \times 10^5 \text{ lb-ft/rad}$$

The moment of inertia of a cylinder of diameter D, length L, and mass density ρ is

$$I = \frac{1}{2}m\left(\frac{D}{2}\right)^2 = \frac{1}{2}\pi\rho L\left(\frac{D}{2}\right)^4$$

The moments of inertia are

$$I_1 = \frac{\pi(15.2)}{2}\frac{3}{12}\left(\frac{5}{24}\right)^4 = 1.12 \times 10^{-2} \text{ slug-ft}^2$$

$$I_2 = \frac{\pi(15.2)}{2}\frac{6}{12}\left(\frac{2}{24}\right)^4 = 5.76 \times 10^{-4} \text{ slug-ft}^2$$

Thus

$$I_e = 1.12 \times 10^{-2} + 2\left(\frac{1}{3}5.76 \times 10^{-4}\right) = 1.16 \times 10^{-2} \text{ slug/ft}^2$$

This system's natural frequency is $\sqrt{2k/I_e} = 6720$ rad/sec, or $6720/2\pi = 1070$ cycles per second. This gives a period of 9.35×10^{-4} sec.

4.4 DAMPING ELEMENTS

A spring element exerts a reaction force in response to a *displacement,* either compression or extension, of the element. On the other hand, a *damping* element is an element that resists relative *velocity* across it. A common example of a damping element, or *damper,* is a cylinder containing a fluid and a piston with one or more holes (Figure 4.4.1a). If we hold the piston rod in one hand and the cylinder in the other hand, and move the piston and the cylinder at the same velocity, we will feel no reaction force. However, if we move the piston and the cylinder at different velocities, we will feel a
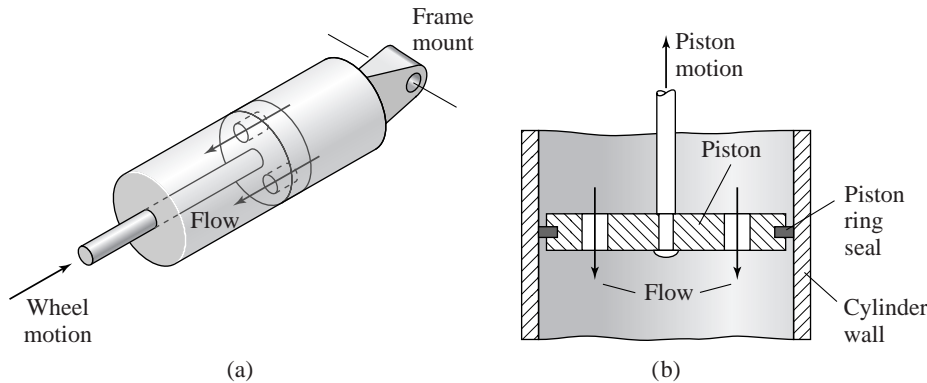
Figure 4.4.1 A piston damper.

(a)　　　　　　　　　　(b)

Figure 4.4.2 A pneumatic door closer.

(a)　　　　　　　　　　(b)
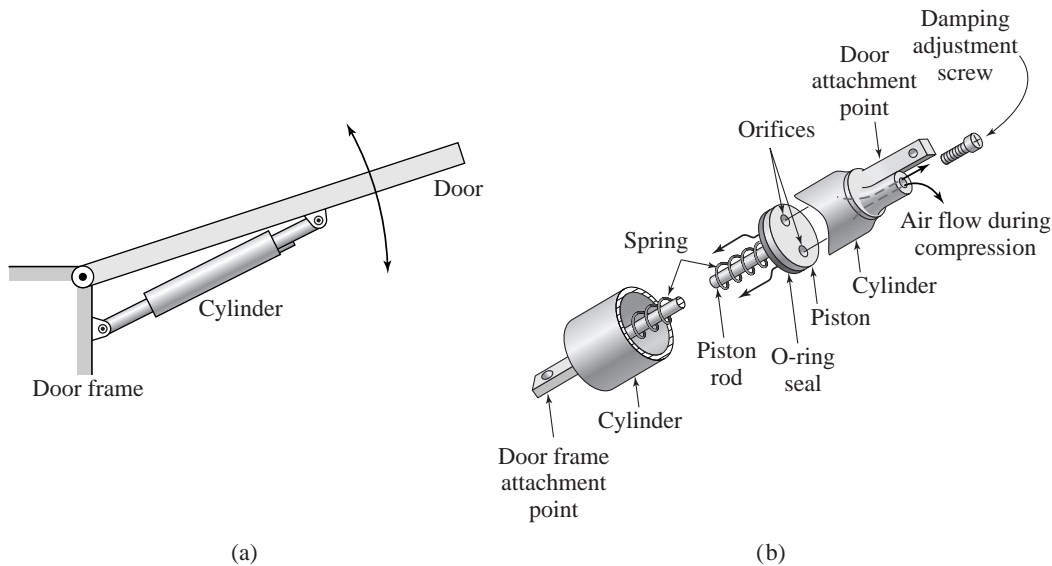
resisting force that is caused by the fluid moving through the holes from one side of the piston to the other (Figure 4.4.1b). From this example, we can see that the resisting force in a damper is caused by fluid friction (in this case, friction between the fluid and the walls of the piston holes) and that the force depends on the relative velocity of the piston and the cylinder. The faster we move the piston relative to the cylinder, the greater is the resisting force.

A DOOR CLOSER

An example from everyday life of a device that contains a damping element as well as a spring element is the door closer (Figure 4.4.2). In some models, the working fluid is air, while others use a hydraulic fluid. The cylinder is attached to the door and the piston rod is fixed to the door frame. As the door is closed, the air is forced both through the piston holes and out past the adjustment screw, which can be used to adjust the amount of damping resistance (a smaller passageway provides more resistance to the flow and thus more damping force). The purpose of the spring is to close the door; if there were no spring, the door would remain stationary because the damper does not exert any

Figure 4.4.3 A rotary damper.

(a)   (b)
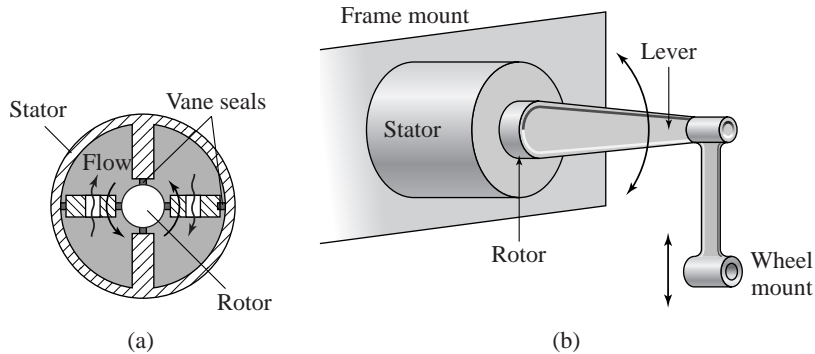
force unless its endpoints are moving relative to each other. The purpose of the damper is to exert a force that prevents the door from being opened or closed too quickly (such as may happen due to a gust of wind). If you have such a door closer, try closing the door at different speeds and notice the change in resisting force.

A *rotary* or torsional damper exerts a resisting *torque* in response to an *angular* velocity difference across it. A common example is the vane-type damper shown in Figure 4.4.3a. The rotating part (the rotor) has vanes with holes through which the fluid can flow. The stator is the stationary housing. This device is the basis of some door closers in larger buildings. It is also used to provide damping of wheel motion in some vehicle suspensions [part (b) of the figure].

SHOCK ABSORBERS

Figure 4.4.4 A damper piston with spring-loaded valves.

The telescopic shock absorber is used in many vehicles. A cutaway view of a typical shock absorber is very complex but the basic principle of its operation is the damper concept illustrated in Figure 4.4.1. The damping resistance can be designed to be dependent on the sign of the relative velocity. For example, Figure 4.4.4 shows a piston containing spring-loaded valves that partially block the piston passageways. If the two spring constants are different or if the two valves have different shapes, then the flow resistance will be dependent on the direction of motion. This design results in a force versus velocity curve like that shown in Figure 4.4.5. During compression (as when the wheel hits a road bump) the resisting force is different than during rebound (when the wheel is forced back to its neutral position by the suspension spring). The resisting force during compression should be small to prevent a large force from being transmitted to the passenger compartment, whereas during rebound the resisting force should be greater to prevent wheel oscillation. An aircraft application of a shock absorber is the *oleo strut,* shown in Figure 4.4.6.

Damping can exist whenever there is a fluid resistance force produced by a fluid layer moving relative to a solid surface. The fluid's viscosity produces a shear stress that exerts a resisting force on the solid surface. Viscosity is an indication of the "stickiness" of the fluid; molasses and oil have greater viscosities than water, for example. Other examples of damping include aerodynamic drag as, discussed in part (b) of Example 1.3.4 in Chapter 1, and hydrodynamic drag. Damping can also be caused by nonfluid effects, such as the energy loss that occurs due to internal friction in solid but flexible materials.

Engineering systems can exhibit damping in bearings and other surfaces lubricated to prevent wear. Damping elements can be deliberately included as part of the design. Such is the case with shock absorbers, fluid couplings, and torque converters.

Figure 4.4.5 Force-velocity curves for
a damper during rebound and during
compression.
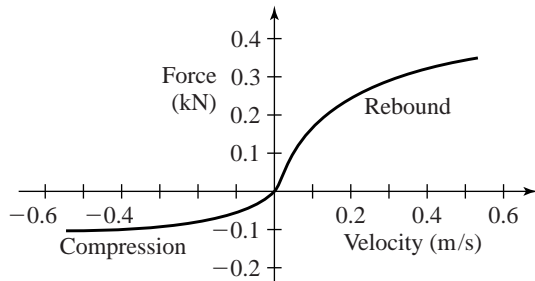
Figure 4.4.6 An oleo strut.
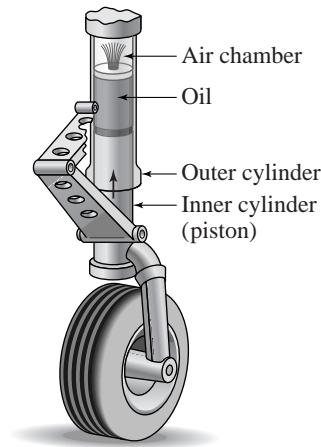
IDEAL DAMPERS

As with spring elements, real damping elements have mass, such as the masses of the
piston and the cylinder in a shock absorber. If the system consists of an object attached to
a damper (such as a vehicle chassis), a simplifying assumption is to neglect the damper
mass relative to the mass of the object and take the mass center of the system to be
located at the mass center of the object. This assumption is accurate in many practical
applications, but to be comfortable with it you should know the numerical values of
the masses of the object and the damper element. In some of the homework problems
and some of the examples to follow, the numerical values are not given. In such cases,
unless otherwise explicitly stated, you should assume that the damper mass can be
neglected. In other cases, where the piston mass and cylinder mass are substantial, for
example, the damper must be modeled as two masses, one for the piston and one for
the cylinder. An *ideal* damping element is one that is massless.

DAMPER REPRESENTATIONS

The dependence of the damping force on the relative velocity can be quite complicated,
and detailed analysis requires application of fluid mechanics principles. Sometimes, to
obtain a linear system model, we model the damping as a linear function of the relative
velocity. This approach enables us to obtain equations of motion that are easier to solve,
without ignoring altogether the effect of the velocity-dependent damping force. The
linear model for the damping force f as a function of the relative velocity v is

$$f = cv \tag{4.4.1}$$

where c is the *damping coefficient*. The units of c are force/velocity; for example,
$N \cdot s/m$ or lb-sec/ft. In applying this equation to obtain free body diagrams, you must
remember that the damping force always opposes the relative velocity.

Using the methods of Chapter 7, Section 7.4, we can derive the following expression
for the damping coefficient of a piston-type damper with a single hole.

$$c = 8\pi \mu L \left[\left(\frac{D}{d} \right)^2 - 1 \right]^2 \tag{4.4.2}$$

Figure 4.4.7 Symbols for translational and torsional damper elements.
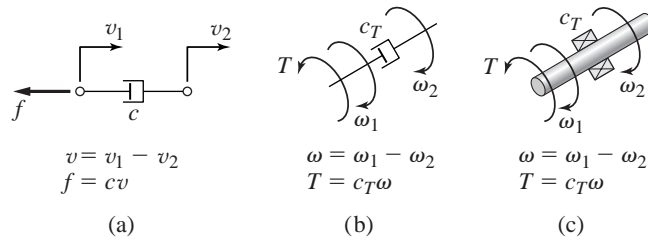
where μ is the viscosity of the fluid, L is the length of the hole through the piston, d is the diameter of the hole, and D is the diameter of the piston. For two holes, as shown in Figure 4.4.1, multiply the result by 2 (this is an example of two damping elements in *series;* their damping coefficients add).

The symbol shown in Figure 4.4.7a is used as the general symbol for a damping element, because it resembles the piston-cylinder device. The symbol is used even when the damping is produced by something other than a piston and cylinder.

The linear model of a torsional damper is

$$T = c_T \omega \tag{4.4.3}$$

where c_T is the *torsional damping coefficient, ω* is the angular velocity, and T is the torque. Torsional dampers are represented by a slightly different symbol, shown in Figure 4.4.7b. When rotational resistance is due to viscous friction in bearings, the bearing symbol shown in Figure 4.4.7c is often used with the symbol c_T to represent the damping. For torsional damping, the units of c_T are torque/angular velocity; for example, $N \cdot m \cdot s/rad$ or lb-ft-sec/rad.

MODELING MASS-DAMPER SYSTEMS

The equations of motion for systems containing damping elements are derived as with spring elements, except that consistent assumptions must also be made about the relative velocities, as well as about the relative displacements, if there is a spring present.

EXAMPLE 4.4.1

Damped Motion on an Inclined Surface

■ **Problem**
Derive and solve the equation of motion of the block sliding on an inclined, lubricated surface (Figure 4.4.8a). Assume that the damping force is linear. For this application the damping coefficient c depends on the contact area of the block, the viscosity of the lubricating fluid, and the thickness of the fluid layer.

Figure 4.4.8 A mass sliding on a lubricated, inclined surface.
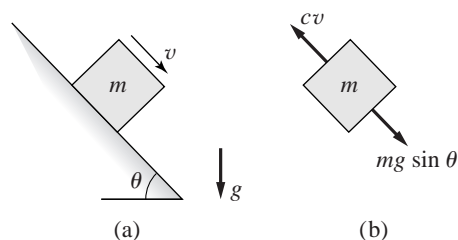
■ Solution

We define v to be the velocity of the block parallel to the surface. The free body diagram, displaying only the forces parallel to the surface, is shown in Figure 4.4.8b. Note that the direction of the damping force must be opposite that of the velocity. The equation of motion is

$$m\dot{v} = mg\sin\theta - cv$$

Because the gravity force $mg\sin\theta$ is constant, the solution is

$$v(t) = \left[v(0) - \frac{mg\sin\theta}{c}\right]e^{-ct/m} + \frac{mg\sin\theta}{c}$$

Eventually the block will reach a constant velocity of $mg\sin\theta/c$ regardless of its initial velocity. At this velocity the damping force is great enough to equal the gravity force component. Because $e^{-4} = 0.02$, when $t = 4m/c$ the velocity will be approximately 98% of its final velocity. Thus a system with a larger mass or less resistance (a smaller damping constant c) will require more time to reach its constant velocity. This apparently unrealistic result is explained by noting that the constant velocity attained is $mg\sin\theta/c$. Thus a system with a larger mass or a smaller c value will attain a higher velocity, and thus should take longer to reach it.

A Wheel-Axle System with Bearing Damping | **EXAMPLE 4.4.2**

■ Problem

Figure 4.4.9a illustrates a wheel-axle system in which the axle is supported by two sets of bearings that produce damping. Each bearing set has a torsional damping coefficient c_T. The torque T is supplied by a motor. The force F is the friction force due to the road surface. Derive the equation of motion.

■ Solution

Part (b) of the figure shows the free body diagram. Note that the damping torque $c_T\omega$ from each bearing set opposes the angular velocity. The inertia I is the combined inertia of the wheel and the two shafts of the axle: $I = I_w + 2I_s$. The equation of motion is

$$I\dot{\omega} = T - RF - 2c_T\omega$$
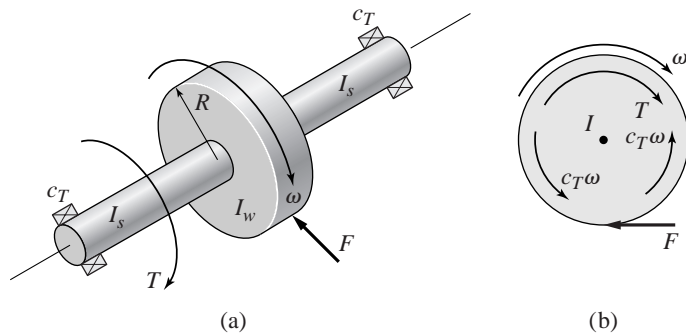
Figure 4.4.9 A wheel-axle system.

(a)   (b)

In a simple bearing, called a journal bearing, the axle passes through an opening in a support. A commonly used formula for the damping coefficient of such a bearing is *Petrov's* law:

$$c_T = \frac{\pi D^3 L \mu}{4\epsilon} \tag{4.4.4}$$

where L is the length of the opening, D is the axle diameter, and ϵ is the thickness of the lubricating layer (the radial clearance between the axle and the support).

EXAMPLE 4.4.3

A Generic Mass-Spring-Damper System

■ Problem
Figure 4.4.10a represents a generic mass-spring-damper system with an external force f. Derive its equation of motion and determine its characteristic equation.

■ Solution
The free body diagram is shown in Figure 4.4.10b. Note that because we have defined x, the displacement from equilibrium, to be positive downward, we must take the velocity $\dot{x}$ also to be positive downward. The damper force opposes the velocity. Thus the equation of motion is

$$m\ddot{x} = -c\dot{x} - k(x + \delta_{st}) + mg + f = -c\dot{x} - kx + f$$

because $k\delta_{st} = mg$. The equation can be rearranged as

$$m\ddot{x} + c\dot{x} + kx = f$$

From this we can recognize the characteristic equation to be $ms^2 + cs + k = 0$.

Figure 4.4.10 A mass-spring-damper system.

(a)   (b)
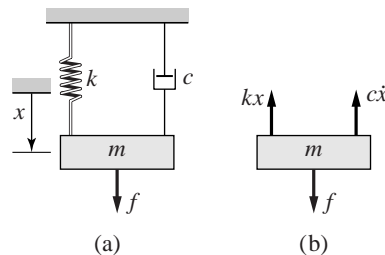
In Section 4.2 we solved the equation of motion for the case where there is no damping ($c = 0$). Now let us investigate the effects of damping.

EXAMPLE 4.4.4

Effects of Damping

■ Problem
Suppose that for the system shown in Figure 4.4.10a the mass is $m = 1$ and the spring constant is $k = 16$. Investigate the free response as we increase the damping for the four cases: $c = 0, 4, 8$, and 10. Use the initial conditions: $x(0) = 1$ and $\dot{x}(0) = 0$.

■ Solution
The characteristic equation is $s^2 + cs + 16 = 0$. The roots are

$$s = \frac{-c \pm \sqrt{c^2 - 64}}{2}$$

The free response can be obtained with the trial-solution method explained in Section 3.1 (see Examples 3.2.1, 3.2.2, and 3.2.3) or with the Laplace transform method covered in Section 3.2.
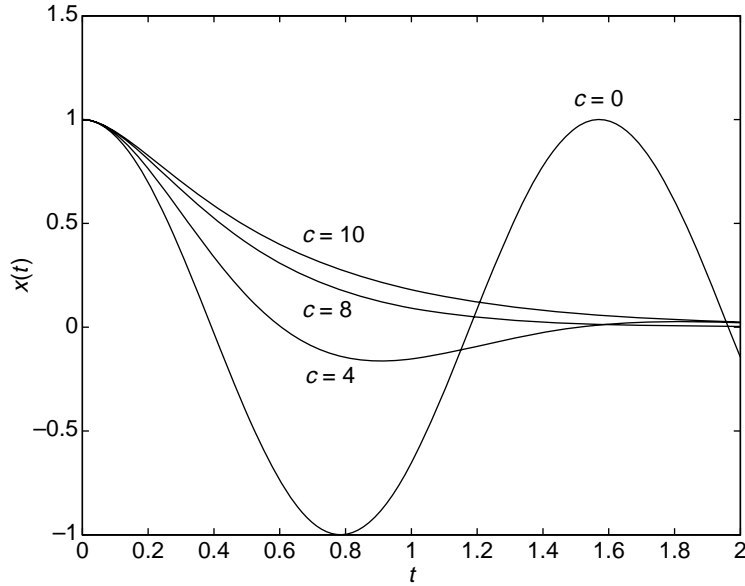
The free responses for the given initial conditions, for each value of c, are as follows:

$$\text{For } c = 0 \quad x(t) = \cos 4t$$

$$\text{For } c = 4 \quad x(t) = 1.155e^{-2t} \sin\left(\sqrt{12}t + 1.047\right)$$

$$\text{For } c = 8 \quad x(t) = (1 + 4t)e^{-4t}$$

$$\text{For } c = 10 \quad x(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$$

The solutions are plotted in Figure 4.4.11. For no damping, the system is neutrally stable and the mass oscillates with a constant amplitude and a radian frequency of $\sqrt{k/m} = 4$, which is the natural frequency ω_n. As the damping is increased slightly to $c = 4$, the system becomes stable and the mass still oscillates but with a smaller radian frequency $\left(\sqrt{12} = 3.464\right)$. The oscillations die out eventually as the mass returns to rest at the equilibrium position $x = 0$. As the damping is increased further to $c = 8$, the mass no longer oscillates because the damping force is large enough to limit the velocity and thus the momentum of the mass to a value that prevents the mass from overshooting the equilibrium position. For a larger value of c ($c = 10$), the mass takes longer to return to equilibrium because the damping force greatly slows down the mass.

A Two-Mass System | **EXAMPLE 4.4.5**

■ Problem
Derive the equations of motion of the two-mass system shown in Figure 4.4.12a.

■ Solution
Choose the coordinates x_1 and x_2 as the displacements of the masses from their equilibrium positions. In equilibrium, the static forces in the springs cancel the weights of the masses. Note that the dampers have no effect in equilibrium and thus do not determine the location of the equilibrium position. Therefore the free body diagrams showing the dynamic forces, and not the static forces, are as shown in Figure 4.4.12b. These diagrams have been drawn with the assumption that the displacement x_2 of mass m_2 from its equilibrium position is greater than the

Figure 4.4.12 A two-mass system.

(a)   (b)   (c)
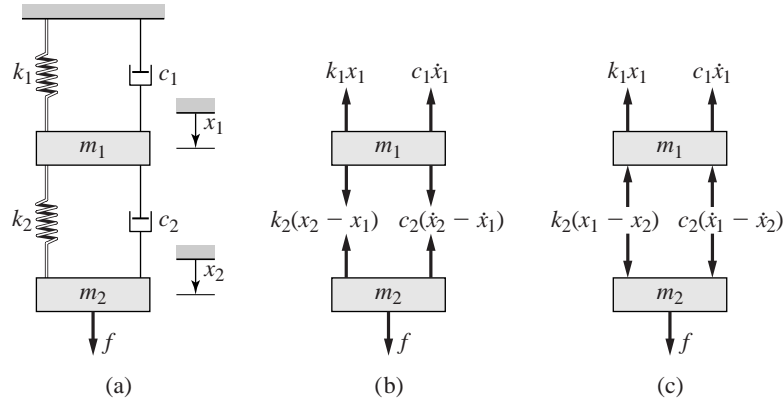
displacement of m_1. Because of the dampers, an additional assumption is required concerning the relative velocities of the masses. The diagrams in part (b) of the figure are based on the assumption that the velocity $\dot{x}_2$ of mass m_2 is greater than the velocity $\dot{x}_1$ of m_1. If your assumptions are correct, the force values shown on the diagram must be *positive*. From these diagrams we obtain the equations of motion:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1)$$

$$m_2\ddot{x}_2 = f - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1)$$

If we move all terms to the left side of the equal sign except for the external force f, and collect terms, we obtain

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0 \tag{1}$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 - c_2\dot{x}_1 + c_2\dot{x}_2 = f \tag{2}$$

In drawing the free body diagrams of multimass systems having springs and dampers between the masses, you must make assumptions about the relative displacements and relative velocities of each mass. For example, we could have assumed that the displacement x_2 of mass m_2 is *less* than the displacement of m_1, and assumed that the velocity $\dot{x}_2$ of mass m_2 is *less* than the velocity $\dot{x}_1$ of m_1. Figure 4.4.12c shows the free body diagrams drawn for this assumption. Note that the directions of the forces associated with spring k_2 and damper c_2 are the opposite of those in part (b) of the figure. You should confirm that the diagram in part (c) results in equations of motion that are identical to equations (1) and (2).

You must be consistent in your assumptions made to draw the free body diagrams. A common mistake is to use one assumption to obtain the free body diagram for mass m_1 but another assumption for mass m_2.

MOTION INPUTS WITH DAMPING ELEMENTS

Sometimes we are given the motion, either the displacement or the velocity, of the endpoint of a damping element. For example, consider the system shown in Figure 4.4.13, where we are given the displacement $y(t)$ of the left-hand end of the damper. Suppose that when $x = 0$ the spring is at its free length. Here we are not given an applied force as an input, but nevertheless we must draw the free body diagram showing the forces acting on the mass. To draw the free body diagram we must make an assumption about the relative velocities of the endpoints of the damping element. The free body diagram has been drawn with the assumption that $\dot{y} > \dot{x}$. The force produced by the
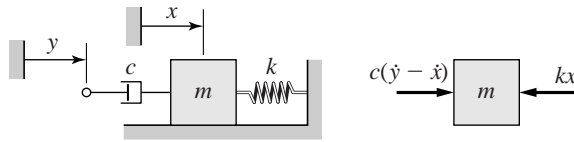
Figure 4.4.13 A system with velocity input.

given displacement $y(t)$ is the resulting damper force $c(\dot{y} - \dot{x})$. The equation of motion is $m\ddot{x} = c(\dot{y} - \dot{x}) - kx$, or $m\ddot{x} + c\dot{x} + kx = c\dot{y}$. Note that we must be given the velocity $\dot{y}$, or be able to compute $\dot{y}$ from $y(t)$, to solve this equation for $x(t)$.

When motion inputs are given, it is important to realize that ultimately such motion is generated by a force (or torque) and that this force must be great enough to generate the specified motion in the presence of any resisting forces or system inertia. For example, from the principle of action and reaction, we see that the mechanism supplying the velocity $\dot{y}(t)$ must generate a force equal to the damper force $c(\dot{y} - \dot{x})$.

4.5 ADDITIONAL MODELING EXAMPLES

This section contains examples for additional practice in deriving equations of motion for systems containing spring and damper elements.

A Translational System with Displacement Input | **EXAMPLE 4.5.1**

■ Problem
Derive the equation of motion for the system shown in Figure 4.5.1a. The input is the displacement y of the right-end of the spring. The output is the displacement x of the mass. The spring is at its free length when $x = y$.

■ Solution
The free body diagram in Figure 4.5.1b displays only the horizontal forces, and it has been drawn assuming that $y > x$. From this diagram we can obtain the equation of motion.

$$m\ddot{x} = k(y - x) - c\dot{x} \qquad \text{or} \qquad m\ddot{x} + c\dot{x} + kx = ky$$

(a)   (b)
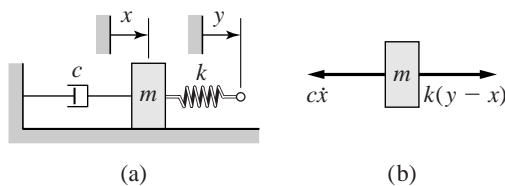
Figure 4.5.1 A translational system with displacement input.

A Rotational System with Displacement Input | **EXAMPLE 4.5.2**

■ Problem
Derive the equation of motion for the system shown in Figure 4.5.2a. The input is the angular displacement ϕ of the left-end of the rod, which is modeled as a torsional spring. The output is the angular displacement θ of the inertia I. Neglect the inertia of the rod. There is no torque in the rod when $\phi = \theta$.

■ Solution
The free body diagram in Figure 4.5.2b has been drawn assuming that $\phi > \theta$. From this diagram we can obtain the equation of motion.

$$I\ddot{\theta} = k_T(\phi - \theta) - c_T\dot{\theta} \qquad \text{or} \qquad I\ddot{\theta} + c_T\dot{\theta} + k_T\theta = k_T\phi$$

Figure 4.5.2 A rotational system with displacement input.

(a)   (b)
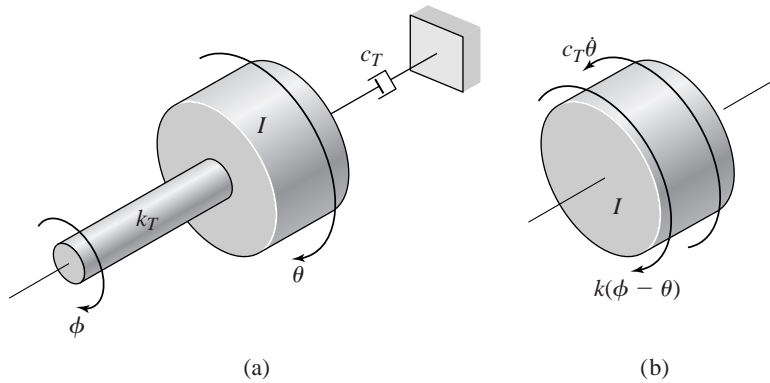
Sometimes we need to determine the motion of a point in the system where there is no mass. In such problems, it is helpful to place a "fictitious" mass at the point in question, draw the free body diagram of the fictitious mass, and set the mass value to zero in the resulting equation of motion. Although the same result can be obtained by applying the principles of statics to the point in question, this method helps to organize the process.

EXAMPLE 4.5.3

Displacement Input and Negligible System Mass

■ Problem
Obtain the equation of motion of point A for the system shown in Figure 4.5.3a. We are given the displacement $y(t)$. The spring is at its free length when $x = y$.

■ Solution
We place a fictitious mass m_A at point A, and draw its free body diagram, shown in part (b) of the figure. The diagram has been drawn with the assumption that $y > x$. The corresponding equation of motion is

$$m_A\ddot{x} = k(y - x) - c\dot{x}$$

Let $m_A = 0$ to obtain the answer: $0 = k(y - x) - c\dot{x}$ or $c\dot{x} + kx = ky$. This can be solved for $x(t)$ if we know $y(t)$.

Figure 4.5.3 A system with negligible mass.

(a)   (b)
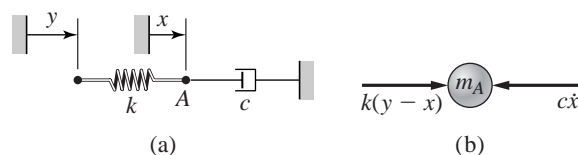
EXAMPLE 4.5.4

A Two-Mass System with Displacement Input

■ **Problem**

Figure 4.5.4a shows a two-mass system where the displacement $y(t)$ of the right-hand end of the spring is a given function. The masses slide on a frictionless surface. When $x_1 = x_2 = y = 0$, the springs are at their free lengths. Derive the equations of motion.

■ **Solution**

The free body diagrams shown in part (b) of the figure display only the horizontal forces, and they were drawn with the assumptions that $y > x_2 > x_1$. These diagrams give the equations of motion:

$$m_1 \ddot{x}_1 = k_1(x_2 - x_1)$$
$$m_2 \ddot{x}_2 = -k_1(x_2 - x_1) + k_2(y - x_2)$$
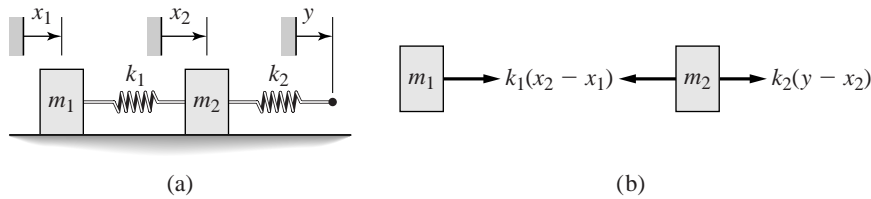
Figure 4.5.4 A two-mass system with displacement input.

A Two-Inertia System with Angular Displacement Input

EXAMPLE 4.5.5

■ **Problem**

Figure 4.5.5a shows a system with two inertia elements and two torsional dampers. The left-hand end of the shaft is twisted by the angular displacement ϕ, which is a specified function of time. The shaft has a torsional spring constant k_T and negligible inertia. The equilibrium position corresponds to $\phi = \theta_1 = \theta_2 = 0$. Derive the equations of motion.

■ **Solution**

From the free body diagrams in part (b) of the figure, which are drawn for $\phi > \theta_1 > \theta_2$ and $\dot{\theta}_1 > \dot{\theta}_2$, we obtain

$$I_1 \ddot{\theta}_1 = k_T(\phi - \theta_1) - c_{T_1}(\dot{\theta}_1 - \dot{\theta}_2)$$
$$I_2 \ddot{\theta}_2 = c_{T_1}(\dot{\theta}_1 - \dot{\theta}_2) - c_{T_2}\dot{\theta}_2$$
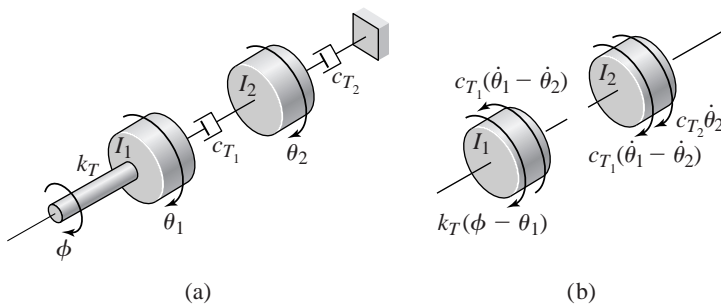
Figure 4.5.5 A system with two inertias.

| EXAMPLE 4.5.6 | A Single-Inertia Fluid-Clutch Model |

■ Problem

Figure 4.5.6a shows a driving disk rotating at a specified speed ω_d. There is a viscous fluid layer between this disk and the driven disk whose inertia plus that of the shaft is I_1. Through the action of the viscous damping, the rotation of the driving disk causes the driven disk to rotate, and this rotation is opposed by the torque T_1, which is due to whatever load is being driven. This model represents the situation in a fluid clutch, which avoids the wear and shock that occurs in friction clutches. (a) Derive a model for the speed ω_1. (b) Find the speed $\omega_1(t)$ for the case where the load torque $T_1 = 0$ and the speed ω_d is a step function of magnitude Ω_d.

■ Solution

a. Lacking a more detailed model of the fluid forces in the viscous fluid layer, we will assume that the effect can be modeled as a massless rotational damper obeying the linear damping law. Thus we will model the system as shown in Figure 4.5.6b. The given input variable is the velocity ω_d.

   To draw the free body diagram of I_1, note that the viscous fluid layer opposes any velocity difference across it. The torque that it exerts as a result has a magnitude $c_T |\omega_d - \omega_1|$ and acts in the direction that will reduce the speed difference. Figure 4.5.6c shows the free body diagram, which is drawn assuming that $\omega_d > \omega_1$. The equation of motion is

$$I_1 \dot{\omega}_1 = c_T (\omega_d - \omega_1) - T_1$$

b. If ω_d is a step function of magnitude Ω_d, and if $T_1 = 0$, the solution for $\omega(t)$ can be found from Table 3.1.3. It is

$$\omega_1(t) = \Omega_d - [\omega_1(0) - \Omega_d]e^{-c_T t/I_1} \tag{1}$$

   This shows that the speed ω_1 will eventually equal Ω_d as $t \to \infty$. If the inertia I_1 is initially rotating in the direction opposite to that of ω_d, it eventually reverses direction. When $t = 4I_1/c_T$, the magnitude of the difference, $|\omega_1(t) - \Omega_d|$, is only 2% of the initial magnitude $|\omega_1(0) - \Omega_d|$.

   The model derived in this example is a first-order model, and it is based on the assumption that the torque on the driving shaft is high enough to drive the disk on that side at the specified speed ω_d, regardless of the effects of the damping and the inertia of the driven side. If this is not true, then the model derived in Example 4.5.7 is a better model.

Figure 4.5.6 A single-inertia model of a fluid clutch.

(a)   (b)   (c)
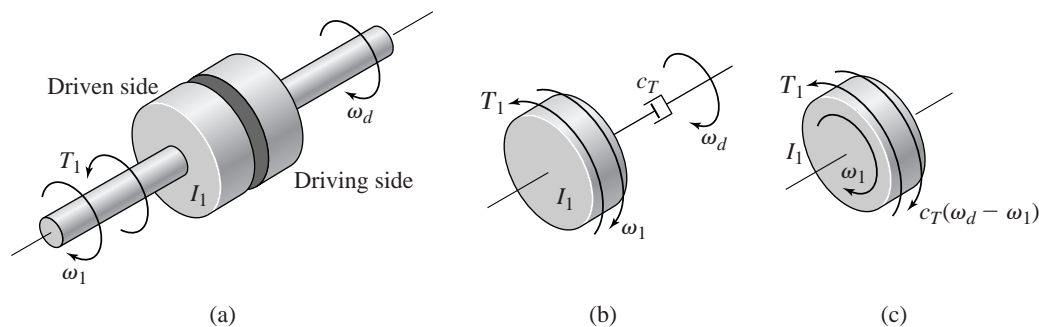
| | A Two-Inertia Fluid-Clutch Model | **EXAMPLE 4.5.7** |

■ Problem

Figure 4.5.7a is a fluid clutch model that can be used when the torque on the driving shaft is not sufficient to drive the disk on that side at the specified speed ω_d. To account for this situation, we must include the inertia of the driving side in the model. Assume that the torques T_d and T_1 are specified functions of time. (a) Derive the equations of motion for the speeds ω_d and ω_1. (b) Obtain the transfer functions $\Omega_1(s)/T_1(s)$ and $\Omega_1(s)/T_d(s)$. (c) Obtain the expression for the response $\omega_1(t)$ if the initial conditions are zero, $T_1 = 0$, and T_d is a step function of magnitude M.

■ Solution

a.   We can model this system with two inertias, as shown in Figure 4.5.7b. The damper torque acts to reduce the difference between the speeds ω_1 and ω_d. The free body diagrams in Figure 4.5.7c are drawn for the case where $\omega_d > \omega_1$, and we can obtain the following equations of motion from them:

$$I_d\dot{\omega}_d = T_d - c_T(\omega_d - \omega_1)$$

$$I_1\dot{\omega}_1 = -T_1 + c_T(\omega_d - \omega_1)$$

b.   Applying the Laplace transform to the equations of motion with zero initial conditions, we obtain

$$(I_d s + c_T)\Omega_d(s) - c_T\Omega_1(s) = T_d(s)$$

$$-c_T\Omega_d(s) + (I_1 s + c_T)\Omega_1(s) = -T_1(s)$$

Eliminating $\Omega_d(s)$ from these equations and solving for $\Omega_1(s)$, we obtain

$$\Omega_1(s) = \frac{c_T}{s(I_1 I_d s + c_T I_1 + c_T I_d)}T_d(s) - \frac{I_d s + c_T}{s(I_1 I_d s + c_T I_1 + c_T I_d)}T_1(s)$$

Thus, the two transfer functions are

$$\frac{\Omega_1(s)}{T_d(s)} = \frac{c_T}{s(I_1 I_d s + c_T I_1 + c_T I_d)} \qquad \frac{\Omega_1(s)}{T_1(s)} = -\frac{I_d s + c_T}{s(I_1 I_d s + c_T I_1 + c_T I_d)}$$

c.   Setting $T_1(s) = 0$ and $T_d(s) = M/s$ gives

$$\Omega_1(s) = \frac{c_T}{s(I_1 I_d s + c_T I_1 + c_T I_d)}\frac{M}{s} = \frac{b}{s(s+a)}\frac{M}{s}$$

Figure 4.5.7 A two-inertia model of a fluid clutch.

(a)   (b)   (c)
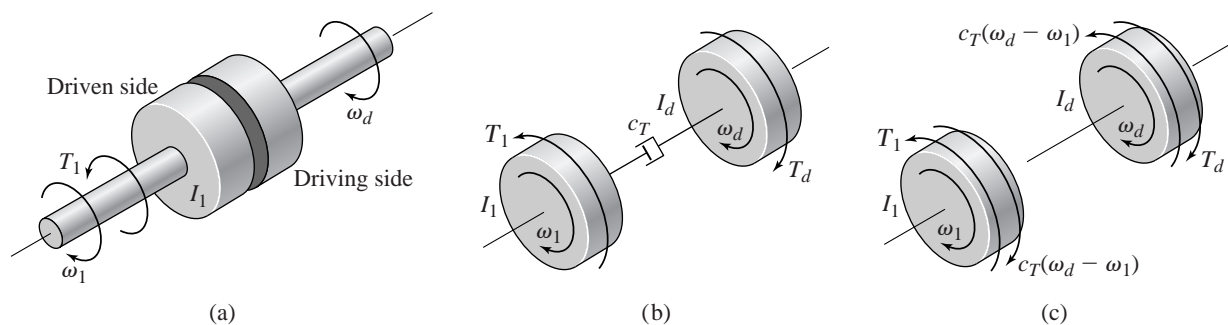
where

$$a = \frac{c_T I_1 + c_T I_d}{I_1 I_d} \qquad b = \frac{c_T}{I_1 I_d}$$

A partial fraction expansion gives

$$\Omega_1(s) = \frac{bM}{a} \left[\frac{1}{s^2} - \frac{1}{as} + \frac{1}{a(s+a)} \right]$$

The corresponding response is

$$\omega_1(t) = \frac{bM}{a} \left(t - \frac{1}{a} + \frac{1}{a} e^{-at} \right)$$

The time constant is $\tau = 1/a$. For $t > 4/a$ approximately, the speed increases linearly with time.

EXAMPLE 4.5.8

The Quarter-Car Model

■ Problem

The *quarter-car model* of a vehicle suspension is shown in Figure 4.5.8a. In this simplified model, the masses of the wheel, tire, and axle are neglected, and the mass m represents one-fourth of the vehicle mass. The spring constant k models the elasticity of both the tire and the suspension spring. The damping constant c models the shock absorber. The equilibrium position of m when $y = 0$ is $x = 0$. The road surface displacement $y(t)$ can be derived from the road surface profile and the car's speed. Derive the equation of motion of m with $y(t)$ as the input, and obtain the transfer function.

■ Solution

Figure 4.5.8b shows the free body diagram, which is drawn assuming that $\dot{y} > \dot{x}$ and that $y > x$. Only the dynamic spring force is shown because the static spring force is canceled by the gravity force. From this free body diagram we obtain the equation of motion:

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \qquad \text{or} \qquad m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

The transfer function is found by applying the Laplace transform to the equation, with the initial conditions set to zero:

$$ms^2 X(s) + cs X(s) + k X(s) = cs Y(s) + k Y(s)$$

from which we obtain

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k}$$

Figure 4.5.8 A quarter-car model with a single mass.

(a)   (b)
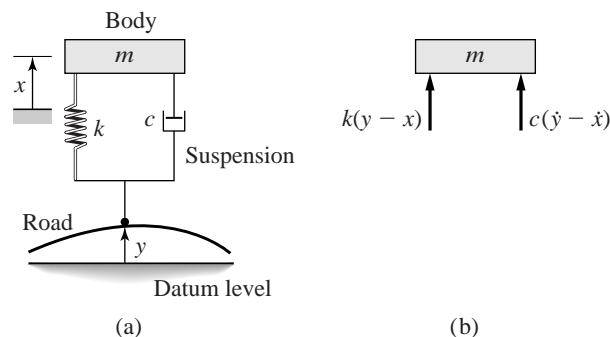
A Quarter-Car Model with Two-Masses | **EXAMPLE 4.5.9**

■ Problem

The suspension model shown in Figure 4.5.9 includes the mass of the wheel-tire-axle assembly. The mass m_1 is one-fourth the mass of the car body, and m_2 is the mass of the wheel-tire-axle assembly. The spring constant k_1 represents the suspension's elasticity, and k_2 represents the tire's elasticity. Derive the equations of motion for m_1 and m_2 in terms of the displacements from equilibrium, x_1 and x_2.

■ Solution

Assuming that $x_2 > x_1, \dot{x}_2 > \dot{x}_1$, and $y > x_2$, we obtain the free body diagram shown. The equation of motion for mass m_1 is

$$m_1 \ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1)$$

For mass m_2,

$$m_2 \ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) + k_2(y - x_2)$$
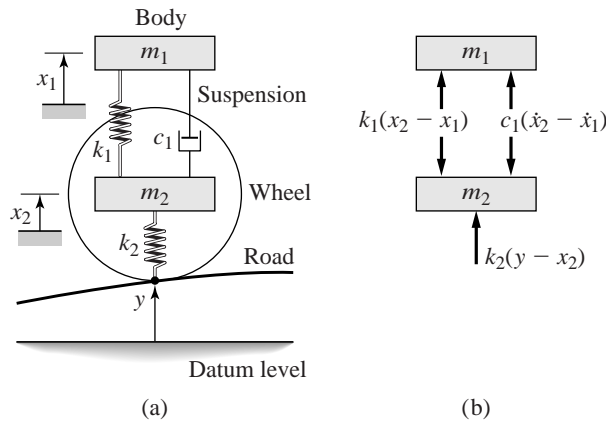
Figure 4.5.9 A quarter-car model with two masses.

In Chapter 2 we derived the following equation of motion of a pendulum whose mass is concentrated a distance L from the pivot point (see Figure 4.5.10).

$$L\ddot{\theta} = -g \sin \theta$$

We also obtained the following linear model that is approximately correct when the pendulum is hanging nearly vertical at $\theta = 0$.

$$L\ddot{\theta} = -g\theta$$

To obtain this model we used the small-angle approximation $\sin \theta \approx \theta$. Because the characteristic roots of the model are imaginary $(s = \pm j\sqrt{g/L})$, the equilibrium position at $\theta = 0$ is neutrally stable. Because this conclusion is based on the approximate model, we cannot use this model to predict what will happen if the mass is displaced far from $\theta = 0$. On the other hand, if we move the mass exactly to $\theta = 180°$ common sense tells us that it will stay there, but if it is slightly disturbed the mass will fall. Thus we can see that the equilibrium position at $\theta = 180°$ is unstable.

Figure 4.5.10 A pendulum with a concentrated mass.
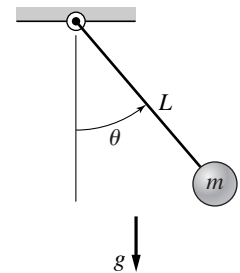
Let us now see how the stability properties are affected if we introduce stiffness and damping into the system.

EXAMPLE 4.5.10

Stability of an Inverted Pendulum

■ Problem

Determine the effects of stiffness and damping on the stability properties of an inverted pendulum (Figure 4.5.11a). Assume that the angle ϕ is small.

■ Solution

Note that for small values of ϕ the motion of the attachment point of the spring and damper is approximately horizontal; its displacement is $L_1\phi$ and its velocity is $L_1\dot{\phi}$. The free body diagram is shown in Figure 4.5.11b. Note that the moment arm of the spring and damper forces is L_1. From the diagram we can write the equation of motion.

$$I_O\ddot{\phi} = M_O \qquad \text{or} \qquad mL^2\ddot{\phi} = mgL\phi - L_1(cL_1\dot{\phi}) - L_1(kL_1\phi)$$

or

$$mL^2\ddot{\phi} + cL_1^2\dot{\phi} + \left(kL_1^2 - mgL\right)\phi = 0$$

which has the form

$$\ddot{\phi} + a\dot{\phi} + b\phi = 0 \qquad a = \frac{cL_1^2}{mL^2} \qquad b = \frac{kL_1^2 - mgL}{mL^2}$$

From the results of Problem 3.6 in Chapter 3, the system will be stable if both a and b are positive.

Note that a cannot be negative for physically realistic values of the other parameters, but it can be zero if there is no damping ($c = 0$). Thus we conclude that some damping is necessary for the system to be stable.

However, b can be positive, negative, or zero depending on the relative values of kL_1^2 and mgL^2. If $b < 0$ the system is unstable ($b < 0$ if $kL_1^2 < mgL$). This indicates that the torque from the spring is not great enough to overcome the torque due to gravity, and thus the mass will fall. We cannot tell how far it will fall because eventually ϕ will become so large that the approximation $\sin\phi \approx \phi$ will no longer be accurate and thus the linear model will be useless.

If $c = 0$ and $b = 0$, the two roots are both zero, which indicates neutral stability. If slightly displaced with zero initial velocity, the mass will remain in that position. If $c = 0$ but $b > 0$, the two roots are imaginary ($s = \pm j\sqrt{b}$) and thus the system is neutrally stable. If slightly displaced, the mass will oscillate about $\phi = 0$ with a constant amplitude.

Figure 4.5.11 A pendulum with a damper and spring element.

(a)   (b)
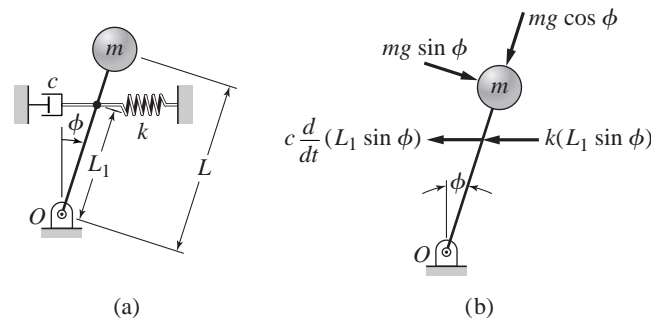
STEP RESPONSE WITH AN INPUT DERIVATIVE

The following first-order model (4.5.1) contains an input derivative, and so we must be careful in modeling the input and in solving for the response.

$$m\dot{v} + cv = b\dot{f}(t) + f(t) \tag{4.5.1}$$

The transfer function is

$$\frac{V(s)}{F(s)} = \frac{bs + 1}{ms + c} \tag{4.5.2}$$

Thus we see that the presence of an input derivative is indicated by an s term in the numerator. This presence is called "numerator dynamics." In mechanical systems, numerator dynamics occurs when a displacement input acts directly on a damper.

An example of a device having numerator dynamics is shown in Figure 4.5.12. From the free body diagram,

$$c(\dot{y} - \dot{x}) + k_1(y - x) - k_2 x = 0$$

or

$$c\dot{x} + (k_1 + k_2)x = c\dot{y} + k_1 y$$

Its transfer function is

$$\frac{X(s)}{Y(s)} = \frac{cs + k_1}{cs + k_1 + k_2}$$

Constant Inputs versus Step Inputs   Note that a step function changes value at $t = 0$. Thus it is not a constant. If, however, the input f in (4.5.1) is a constant F for $-\infty \le t \le \infty$, then $\dot{f} = 0$ for $-\infty \le t \le \infty$, and the existence of the input derivative in the model does not affect the response, because the model reduces to $m\dot{v} + cv = f(t)$. Thus, with $v_{ss} = F/c$ and $\tau = m/c$, we have

$$v(t) = v(0)e^{-t/\tau} + v_{ss}(1 - e^{-t/\tau}) \tag{4.5.3}$$

If, however, the input is a step function, then we cannot use (4.5.3). In this case, we can use the Laplace transform to derive the response. Assuming that $f(t)$ is a step function of magnitude F, we obtain from (4.5.1)

$$m[sV(s) - v(0)] + cV(s) = b[sF(s) - f(0)] + F(s) = b\left[s\frac{F}{s} - f(0)\right] + \frac{F}{s}$$

With $f(0) = 0$ we obtain

$$V(s) = \frac{mv(0)}{ms + c} + \frac{bF}{ms + c} + \frac{F}{s(ms + c)}$$

$$k_1(y - x) + c(\dot{y} - \dot{x}) \longrightarrow k_2 x$$

(a)   (b)
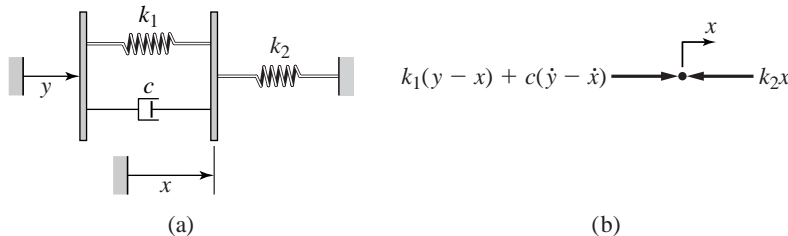
Figure 4.5.12 A device having numerator dynamics.

The inverse transform gives

$$v(t) = v(0)e^{-ct/m} + \frac{bF}{m}e^{-ct/m} + \frac{F}{c}(1 - e^{-ct/m}) \tag{4.5.4}$$

Comparing this with (4.5.3) we see that the effect of the term $b\dot{f}(t)$ is to increase the initial value of $v(t)$ by the amount bF/m.

Of course, no physical variable can be discontinuous, and therefore the step function is only an approximate description of an input that changes quickly. For example, the displacement $y(t)$ in Figure 4.5.12a must be continuous. With some models, a step input may produce a physically unrealistic discontinuity in the response. An example is given by (4.5.4), where $v(t)$ is discontinuous at $t = 0$. Taking the limit of $v(t)$ as $t \to 0+$, we find that $v(0+) = v(0) + bF/m$.

The reason for using a step function is to reduce the complexity of the mathematics required to find the response. The following example illustrates this.

EXAMPLE 4.5.11

An Approximation to the Step Function

■ Problem
Consider the following model:

$$\dot{v} + 10v = \dot{f} + f \tag{1}$$

where $f(t)$ is the input and $v(0) = 0$.

a. Obtain the expression for the unit-step response.
b. Obtain the response to the input $f(t) = 1 - e^{-100t}$ and compare with the results of part (a).

■ Solution
a. Comparing equation (1) with (4.5.1), we see that $m = 1$, $c = 10$, and $b = 1$. Thus, from (4.5.3),

$$v(t) = 0.1 + 0.9e^{-10t} \tag{2}$$

Note that this equation gives $v(0+) = 1$. Thus, for this model, the effect of the step input acting on the input derivative $\dot{f}$ creates an instantaneous jump in the value of v at $t = 0$ from $v(0) = 0$ to $v(0+) = 1$.

b. The function $f(t) = 1 - e^{-100t}$ is plotted in the top graph of Figure 4.5.13. It resembles a step function except that it is continuous. The corresponding response is obtained as follows:

$$V(s) = \frac{s+1}{s+10}\frac{100}{s(s+100)} = \frac{0.1}{s} - \frac{1.1}{s+100} + \frac{1}{s+10}$$

This gives

$$v(t) = 0.1 - 1.1e^{-100t} + e^{-10t} \tag{3}$$

This response gives $v(0+) = 0$ and is plotted along with the step response in the bottom graph of Figure 4.5.13. This response resembles the step response except for the latter's unrealistic discontinuity at $t = 0$. The response curves are very close for $t > 0.1$.

Although the response for the approximate step function was easily obtained in this example, this might not be true for higher-order models. In such cases, the approximation introduced by using the step function might be justified to obtain an expression for the response.
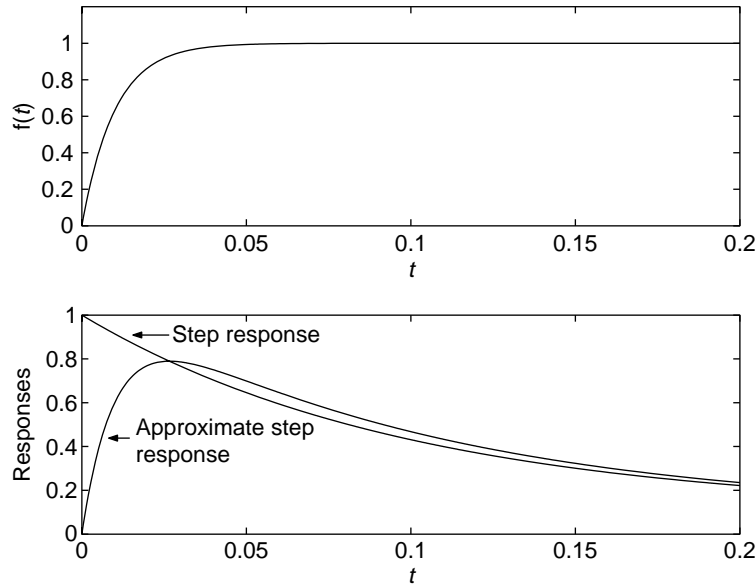
There will be cases, such as with nonlinear models containing input derivatives, where the Laplace transform cannot be used to find the response. In such cases we must use an approximate method or a numerical method. Input derivatives with step inputs, however, are difficult to handle numerically. In such cases the approximate step function $1 - e^{-t/\tau}$ and its derivative $e^{-t/\tau}/\tau$ can be used to obtain the response. For this approximate method to work, the value of τ must be chosen to be much smaller than the estimated response time of the system. An example of such an application is the system shown in Figure 4.5.12a if one or both spring elements are nonlinear. For example, if the force-displacement relation of the leftmost spring is a cubic function, then the equation of motion is

$$c(\dot{y} - \dot{x}) + k_{11}(y - x) + k_{12}(y - x)^3 - k_2 x = 0$$

which is nonlinear, and therefore not solvable with the Laplace transform. This equation must be solved numerically, and so using a pure step function for $y(t)$ gives no advantage and may cause numerical difficulties. The use of MATLAB to solve such an equation is discussed in Section 5.3.

Damper Location and Numerator Dynamics | **EXAMPLE 4.5.12**

■ **Problem**

By obtaining the equations of motion and the transfer functions of the two systems shown in Figure 4.5.14, investigate the effect of the location of the damper on the step response of the system. The displacement $y(t)$ is a given function. Obtain the unit-step response for each system for the specific case $m = 1$, $c = 6$, and $k = 8$, with zero initial conditions.

■ **Solution**

For the system in part (a) of the figure, $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$ and thus,

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k} \tag{1}$$

Therefore this system has numerator dynamics.

(a)   (b)
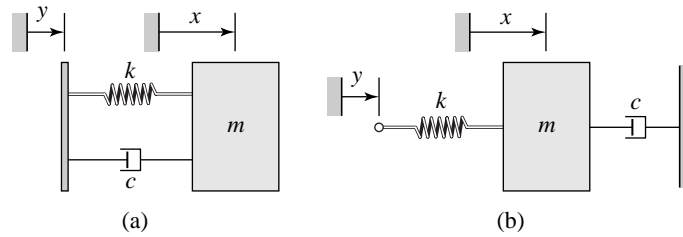
For the system in part (b), $m\ddot{x} + c\dot{x} + kx = ky$ and thus,

$$\frac{X(s)}{Y(s)} = \frac{k}{ms^2 + cs + k} \tag{2}$$

Therefore this system does not have numerator dynamics. Both systems have the same characteristic equation.

Substituting the given values into equation (1), using $Y(s) = 1/s$ and performing a partial-fraction expansion, we obtain

$$X(s) = \frac{6s + 8}{s(s^2 + 6s + 8)} = \frac{1}{s} + \frac{1}{s + 2} - \frac{2}{s + 4}$$

The response is

$$x(t) = 1 + e^{-2t} - 2e^{-4t}$$

For equation (2),

$$X(s) = \frac{8}{s(s^2 + 6s + 8)} = \frac{1}{s} - \frac{2}{s + 2} + \frac{1}{s + 4}$$

The response is

$$x(t) = 1 - 2e^{-2t} + e^{-4t}$$

The responses are shown in Figure 4.5.15. Curve (a) corresponds to equation (1), which has numerator dynamics. Curve (b) corresponds to the system without numerator dynamics, equation (2). The numerator dynamics causes an overshoot in the response but does not affect
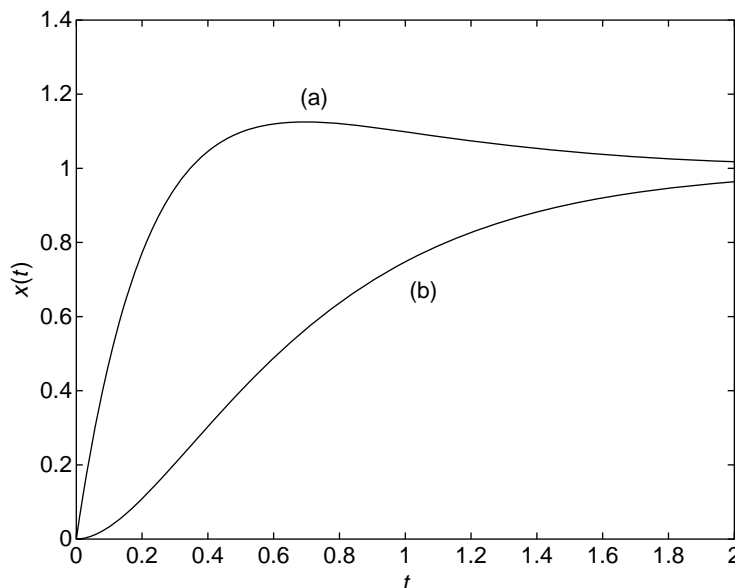
the steady-state response. Thus, the damper location can affect the response. But the damper location does not affect the time constants, because both systems have the same characteristic roots. Thus, they take about the same length of time to approach steady state.

4.6 COLLISIONS AND IMPULSE RESPONSE

An input that changes at a constant rate is modeled by the ramp function. The step function models an input that rapidly reaches a constant value, while the rectangular pulse function models a constant input that is suddenly removed. The impulsive function—called an impulse—is similar to the pulse function, but it models an input that is suddenly applied and removed after a very short (infinitesimal) time.

The *strength* of an impulsive input is the area under its curve. The *Dirac delta function* $\delta(t)$ is an impulsive function with a strength equal to unity. Thus,

$$\int_0^{0+} \delta(t)\, dt = 1$$

The Dirac function is an analytically convenient approximation of an input applied for only a very short time, such as the interaction force between two colliding objects. It is also useful for estimating the system's parameters experimentally and for analyzing the effect of differentiating a step or any other discontinuous input function.

The response to an impulsive input is called the *impulse response*. In particular, the response to $\delta(t)$ is called the *unit* impulse response. Note that the system transfer function $T(s)$ is the Laplace transform of the unit impulse response, because $\mathcal{L}[\delta(t)] = 1$. That is, if $X(s) = T(s)F(s)$ and if $f(t) = \delta(t)$, then $F(s) = 1$ and $X(s) = T(s)$. Therefore, if we can obtain the transform of the response $x(t)$ to an impulse of strength A, then we can determine the transfer function from $T(s) = X(s)/A$. This relation has some applications in determining the transfer function from the measured response.

INITIAL CONDITIONS AND IMPULSE RESPONSE

The given initial conditions $x(0)$, $\dot{x}(0)$, ..., for the dependent variable $x(t)$, represent the situation at the start of the process and are the result of any inputs applied prior to $t = 0$. The effects of any inputs starting at $t = 0$ are not felt by the system until an infinitesimal time later, at $t = 0+$. For some models, $x(t)$ and its derivatives do not change between $t = 0$ and $t = 0+$, and thus the solution $x(t)$ obtained from the differential equation will match the given initial conditions when the solution $x(t)$ and its derivatives are evaluated at $t = 0$. The results obtained from the initial value theorem will also match the given initial conditions.

However, for some other models, $x(0) \neq x(0+)$, or $\dot{x}(0) \neq \dot{x}(0+)$, and so forth for the higher derivatives of $x(t)$, depending on the type of input. The initial value theorem gives the value at $t = 0+$, which for these responses is not necessarily equal to the value at $t = 0$. In these cases the solution of the differential equation is correct only for $t > 0$. This phenomenon occurs in models having impulsive inputs and in models containing derivatives of a discontinuous input.

For example, in Chapter 3 we found that an impulsive input applied to a first-order system will instantaneously change the value of x so that $x(0) \neq x(0+)$. For a second-order system, however, we found that such an input changes the values only of the derivative, so that $x(0) = x(0+)$ but $\dot{x}(0) \neq \dot{x}(0+)$. In keeping with our interpretation

of the initial conditions, we consider the function $\delta(t)$ to start at time $t = 0$ and finish at $t = 0+$, with its effects first felt at $t = 0+$.

Of course, a pure impulsive input does not exist in nature, and, for analytical convenience, we often model as impulsive an input that has a duration that is short compared to the response time of the system.

An example of a force often modeled as impulsive is the force generated when two objects collide. Recall that when Newton's law of motion $m\dot{v} = f(t)$ is integrated over time, we obtain the *impulse-momentum principle* for a system having constant mass:

$$mv(t) - mv(0) = \int_0^t f(u)\, du \qquad (4.6.1)$$

This states that the change in momentum mv equals the time integral of the applied force $f(t)$. In the terminology of mechanics, the force integral—the area under the force-time curve—is called the *linear impulse*. The linear impulse is the strength of an impulsive force, but a force need not be impulsive to produce a linear impulse.

If $f(t)$ is an impulsive input of strength A, that is, if $f(t) = A\delta(t)$, then

$$mv(0+) - mv(0) = \int_0^{0+} A\delta(u)\, du = A \int_0^{0+} \delta(u)\, du = A \qquad (4.6.2)$$

since the area under the $\delta(t)$ curve is 1. So the change in momentum equals the strength of the impulsive force.

In practice, the strength is often all we can determine about an input modeled as impulsive. Sometimes we need not determine the input characteristics at all, as the next example illustrates.

EXAMPLE 4.6.1

Inelastic Collision

■ Problem

Suppose a mass $m_1 = m$ moving with a speed v_1 becomes embedded in mass m_2 after striking it (Figure 4.6.1). Suppose $m_2 = 10m$. Determine the expression for the displacement $x(t)$ after the collision.

■ Solution

Figure 4.6.1 Inelastic collision.

If we take the entire system to consist of both masses, then the force of collision is *internal* to the system. Because the displacement of m_2 immediately after the collision will be small, we may neglect the spring force initially. Thus, the *external* force $f(t)$ in (4.6.1) is zero, and we have

$$(m + 10m)v(0+) - [mv_1 + 10m(0)] = 0$$

or

$$v(0+) = \frac{mv_1}{11m} = \frac{1}{11}v_1$$

The equation of motion for the combined mass is $11m\ddot{x} + kx = 0$. We can solve it for $t \geq 0+$ by using the initial conditions at $t = 0+$; namely, $x(0+) = 0$ and $\dot{x}(0+) = v(0+)$. The solution is

$$x(t) = \frac{v(0+)}{\omega_n} \sin \omega_n t = \frac{v_1}{11} \sqrt{\frac{11m}{k}} \sin \sqrt{\frac{k}{11m}} t$$

Note that it was unnecessary to determine the impulsive collision force. Note also that this force did not change x initially; it changed only $\dot{x}$.
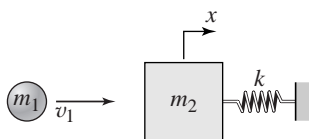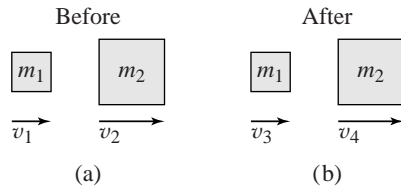
Figure 4.6.2 Two colliding masses.

Consider the two colliding masses shown in Figure 4.6.2. Part (a) shows the situation before collision, and part (b) shows the situation after collision. When the two masses are treated as a single system, no external force is applied to the system, and (4.6.1) shows that the momentum is conserved, so that

$$m_1 v_1 + m_2 v_2 = m_1 v_3 + m_2 v_4$$

or

$$m_1(v_1 - v_3) = -m_2(v_2 - v_4) \tag{4.6.3}$$

If the collision is *perfectly elastic,* kinetic energy is conserved, so that

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_3^2 + \frac{1}{2}m_2 v_4^2$$

or

$$\frac{1}{2}m_1\left(v_1^2 - v_3^2\right) = -\frac{1}{2}m_2\left(v_2^2 - v_4^2\right) \tag{4.6.4}$$

Using the algebraic identities:

$$v_1^2 - v_3^2 = (v_1 - v_3)(v_1 + v_3)$$
$$v_2^2 - v_4^2 = (v_2 - v_4)(v_2 + v_4)$$

we can write (4.6.4) as

$$\frac{1}{2}m_1(v_1 - v_3)(v_1 + v_3) = -\frac{1}{2}m_2(v_2 - v_4)(v_2 + v_4) \tag{4.6.5}$$

Divide (4.6.5) by (4.6.3) to obtain

$$v_1 + v_3 = v_2 + v_4 \qquad \text{or} \qquad v_1 - v_2 = v_4 - v_3 \tag{4.6.6}$$

This relation says that in a perfectly elastic collision the *relative* velocity of the masses changes sign but its magnitude remains the same.

The most common application is where we know v_1 and mass m_2 is initially stationary, so that $v_2 = 0$. In this case, we can solve (4.6.3) and (4.6.6) for the velocities after collision as follows:

$$v_3 = \frac{m_1 - m_2}{m_1 + m_2}v_1 \qquad v_4 = \frac{2m_1}{m_1 + m_2}v_1 \tag{4.6.7}$$

Perfectly Elastic Collision   **EXAMPLE 4.6.2**

■ **Problem**

Consider again the system treated in Example 4.6.1, and shown again in Figure 4.6.3a. Suppose now that the mass $m_1 = m$ moving with a speed v_1 rebounds from the mass $m_2 = 10m$ after striking it. Assume that the collision is perfectly elastic. Determine the expression for the displacement $x(t)$ after the collision.

Figure 4.6.3 Perfectly elastic collision.

(a)   (b)
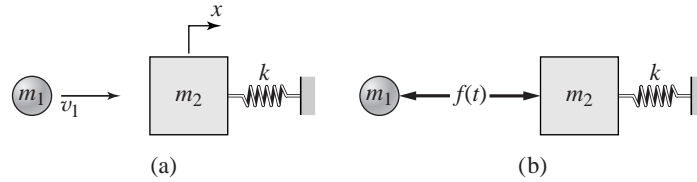
■ Solution

For a perfectly elastic collision, the velocity v_3 of the mass m after the collision is given by (4.6.7).

$$v_3 = \frac{m_1 - m_2}{m_1 + m_2} v_1 = \frac{m - 10m}{m + 10m} v_1 = -\frac{9}{11} v_1$$

Thus the change in the momentum of m is

$$m\left(-\frac{9}{11} v_1\right) - m v_1 = \int_0^{0+} f(t)\, dt$$

Thus the linear impulse applied to the mass m during the collision is

$$\int_0^{0+} f(t)\, dt = -\frac{20}{11} m v_1$$

From Newton's law of action and reaction (Figure 4.6.3b), we see that the linear impulse applied to the $10m$ mass is $+20mv_1/11$, and thus its equation of motion is

$$10m\ddot{x} + kx = \frac{20}{11} m v_1 \delta(t)$$

We can solve this equation for $t > 0$ using the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. The Laplace transform gives

$$\left(10ms^2 + k\right) X(s) = \frac{20}{11} m v_1$$

$$X(s) = \frac{20mv_1/11}{10ms^2 + k} = \frac{2v_1}{11} \frac{1}{s^2 + k/10m}$$

Thus

$$x(t) = \frac{2v_1}{11} \sqrt{\frac{10m}{k}} \, \sin \sqrt{\frac{k}{10m}} t$$

This gives $\dot{x}(0+) = 2v_1/11$, which is identical to the solution for v_4 from (4.6.7), as it should be.

4.7 MATLAB APPLICATIONS

We have seen how the `residue` function can be used to obtain the coefficients of a partial-fraction expansion. In addition, with the `conv` function, which multiplies polynomials, you can use MATLAB to perform some of the algebra to obtain a closed-form expression for the response.

MATLAB has the `step` and `impulse` functions to compute the step and impulse responses from the transfer functions. However, MATLAB does not have a function for computing the free response from a transfer function model. We now show how to use MATLAB to obtain the free response.

EXAMPLE 4.7.1

Obtaining the Free Response with the `step` Function

■ Problem
Use the MATLAB `step` function to obtain a plot of the free response of the following model, where $x(0) = 4$ and $\dot{x}(0) = 2$.

$$5\ddot{x} + 3\dot{x} + 10x = 0$$

■ Solution
Applying the Laplace transform gives

$$(5s^2 + 3s + 10)X(s) = 20s + 22$$

or

$$X(s) = \frac{20s + 22}{5s^2 + 3s + 10}$$

If we multiply the numerator and denominator by s, we obtain

$$X(s) = \frac{20s^2 + 22s}{5s^2 + 3s + 10} \frac{1}{s} = T(s)\frac{1}{s}$$

Thus we may compute the free response by using the `step` function with the transfer function

$$T(s) = \frac{20s^2 + 22s}{5s^2 + 3s + 10}$$

The MATLAB program is

```
sys = tf([20, 22, 0],[5, 3, 10]);
step(sys)
```

EXAMPLE 4.7.2

Determining the Free Response with MATLAB

■ Problem
For the system shown in Figure 4.7.1, suppose that $m_1 = m_2 = 1$, $c_1 = 2$, $c_2 = 3$, $k_1 = 1$, and $k_2 = 4$. a) Obtain the plot of the unit-step response of x_2 for zero initial conditions. b) Use two methods with MATLAB to obtain the free response for $x_1(t)$, for the initial conditions $x_1(0) = 3$, $\dot{x}_1(0) = 2$, $x_2(0) = 1$, and $\dot{x}_2(0) = 4$.

Figure 4.7.1 A two-mass system.

■ Solution
The equations of motion are given by equations (1) and (2) of Example 4.4.5. With the given parameter values, they become

$$\ddot{x}_1 + 5\dot{x}_1 + 5x_1 - 3\dot{x}_2 - 4x_2 = 0$$

$$\ddot{x}_2 + 3\dot{x}_2 + 4x_2 - 3\dot{x}_1 - 4x_1 = f(t)$$

Because we need to find the free response, we must now keep the initial conditions in the analysis. So, transforming the equations with the given initial conditions gives

$$s^2 X_1(s) - sx_1(0) - \dot{x}_1(0) + 5[sX_1(s) - x_1(0)] + 5X_1(s) - 3[sX_2(s) - x_2(0)] - 4X_2(s) = 0$$

$$s^2 X_2(s) - sx_2(0) - \dot{x}_2(0) + 3[sX_2(s) - x_2(0)] + 4X_2(s) - 3[sX_1(s) - x_1(0)] - 4X_1(s) = F(s)$$

Collecting terms results in

$$(s^2 + 5s + 5)X_1(s) - (3s + 4)X_2(s) = x_1(0)s + \dot{x}_1(0) + 5x_1(0) - 3x_2(0) = I_1(s) \qquad (1)$$

$$-(3s + 4)X_1(s) + (s^2 + 3s + 4)X_2(s) = x_2(0)s + \dot{x}_2(0) + 3x_2(0) - 3x_1(0) = I_2(s) + F(s) \quad (2)$$

where the terms due to the initial conditions are denoted by $I_1(s)$ and $I_2(s)$.

$$I_1(s) = x_1(0)s + \dot{x}_1(0) + 5x_1(0) - 3x_2(0)$$

$$I_2(s) = x_2(0)s + \dot{x}_2(0) + 3x_2(0) - 3x_1(0)$$

Equations (1) and (2) are two algebraic equations in two unknowns. Their solution can be obtained in several ways, but the most general method for our purposes is to solve them using determinants (this is known as *Cramer's method*).

The determinant of the left-hand side of the equations (1) and (2) is

$$D(s) = \begin{vmatrix} (s^2 + 5s + 5) & -(3s + 4) \\ -(3s + 4) & (s^2 + 3s + 4) \end{vmatrix}$$

$$= (s^2 + 5s + 5)(s^2 + 3s + 4) - (3s + 4)^2$$

The solutions for $X_1(s)$ and $X_2(s)$ can be expressed as

$$X_1(s) = \frac{D_1(s)}{D(s)} \qquad X_2(s) = \frac{D_2(s)}{D(s)}$$

where

$$D_1(s) = \begin{vmatrix} I_1(s) & -(3s + 4) \\ I_2(s) + F(s) & (s^2 + 3s + 4) \end{vmatrix}$$

$$= (s^2 + 3s + 4)I_1(s) + (3s + 4)I_2(s) + (3s + 4)F(s)$$

$$D_2(s) = \begin{vmatrix} (s^2 + 5s + 5) & I_1(s) \\ -(3s + 4) & I_2(s) + F(s) \end{vmatrix}$$

$$= (s^2 + 5s + 5)I_2(s) + (3s + 4)I_1(s) + (s^2 + 5s + 5)F(s)$$

a. The transfer functions for the input $f(t)$ are found by setting $I_1(s) = I_2(s) = 0$ in $D_1(s)$ and $D_2(s)$:

$$\frac{X_1(s)}{F(s)} = \frac{D_1(s)}{D(s)} = \frac{3s + 4}{D(s)} \qquad \frac{X_2(s)}{F(s)} = \frac{D_2(s)}{D(s)} = \frac{s^2 + 5s + 5}{D(s)}$$

The unit-step response for x_2 is plotted with the following MATLAB program.

```
D = conv([1,5,5], [1,3,4])-conv([0,3,4], [0,3,4]);
x2 = tf([1,5,5],D);
step(x2)
```

b. Using the technique introduced in Example 4.7.1, we multiply the numerator of $X_1(s)$ by s. The following MATLAB program performs the calculations. Note that the multiplication by s is accomplished by appending a 0 to the polynomials used to compute D1 (with $F(s) = 0$).

```
% Specify the initial conditions.
x10 = 3; x1d0 = 2; x20 = 1; x2d0 = 4;
% Form the initial-condition arrays.
I1 = [x10,x1d0+5*x10-3*x20]; I2 = [x20,x2d0+3*x20-3*x10];
```

```
% Form the determinant D1.
D1 = conv([1,3,4,0],I1)+conv([0,3,4,0],I2);
% Form the determinant D.
D = conv([1,5,5],[1,3,4])-conv([0,3,4],[0,3,4]);
sys = tf(D1,D);
step(sys)
```

The first method gives the response plot but not the closed-form solution. The following method gives the closed-form solution for the free response.

The characteristic roots can be found with the following MATLAB file.

```
% Find the roots.
D = conv([1,5,5],[1,3,4])-conv([0,3,4],[0,3,4]);
R = roots(D)
```

The roots are -5.6773, -1.3775, and $-0.4726 \pm 0.5368j$.

Note that if all the characteristic roots are distinct, the partial fraction expansion gives the following solution form, even if the roots are complex.

$$x_1(t) = \sum_{i=1}^{4} r_i e^{p_i t}$$

where the r_i are the residues of the partial-fraction expansion and the p_i are the poles (the roots). So if the roots are all distinct, we can use the following MATLAB file to perform the rest of the algebra.

```
% Specify the initial conditions.
x10 = 3; x1d0 = 2; x20 = 1; x2d0 = 4;
% Form the initial-condition vectors.
I1 = [x10,x1d0+5*x10-3*x20]; I2 = [x20,x2d0+3*x20-3*x10];
% Form the determinant D1.
D1 = conv([1,3,4],I1)+conv([0,3,4],I2);
% Compute the partial fraction expansion.
[r,p,K] = residue(D1,D);
% Use 5 time constants to estimate the simulation time.
tmax = floor(-5/max(real(p)));
t = (0:tmax/500:tmax);
% Evaluate the time functions.
x1 = real(r(1)*exp(p(1)*t)+r(2)*exp(p(2)*t)+...
    r(3)*exp(p(3)*t)+r(4)*exp(p(4)*t));
plot(t,x1),xlabel('t'),ylabel('x_1')
```

Although in the expression for `x1` the imaginary parts should cancel, giving a real result, we use the `real` function because numerical round-off errors can produce a small imaginary part. The plot is shown in Figure 4.7.2. We can use a similar approach to plot $x_2(t)$.

If a closed-form expression for the free response is required, you can use the roots (the poles) in the array `p` and the expansion coefficients (the residues) in the array `r`. If the roots are complex, you can use the Euler identities $e^{jx} = \cos x \pm j \sin x$ with the expansion coefficients to obtain the solution, as shown in Section 3.7 in Chapter 3. Here the residues corresponding in order to the four roots given already are -0.3554, 3.7527, and $-0.1987 \pm 4.6238j$.

Figure 4.7.2 Plot of the free response for Example 4.7.2.
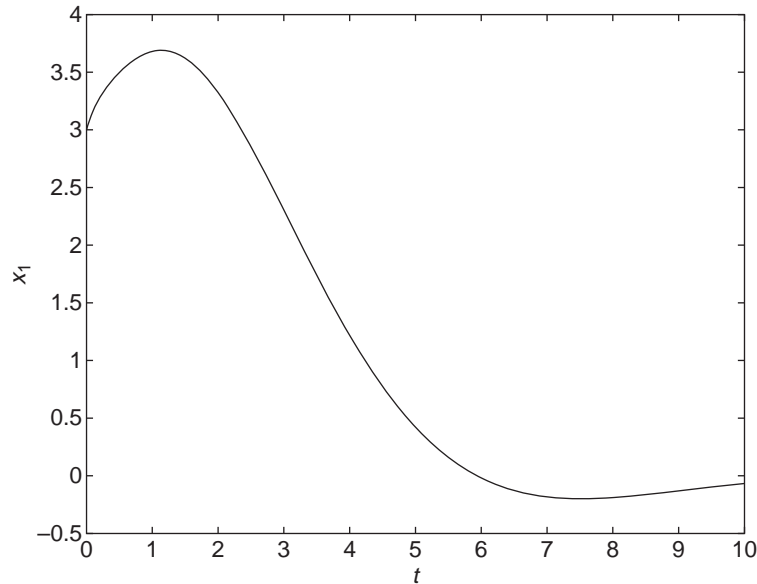
4.8 CHAPTER REVIEW

This chapter showed how to model elements containing elasticity or damping as ideal (massless) spring or ideal damper elements. Spring elements exert a resisting force that is a function of the relative displacement (compression or extension) of the element's endpoints. A damper element exerts a force that depends on the relative velocity of its endpoints. Through many examples we also saw how to obtain the equations of motion of systems containing spring and damper elements. We extended energy-based methods and the concepts of equivalent mass and equivalent inertia to include spring elements.

In general, a spring element exerts a restoring force that causes the system to oscillate, while a damper element acts to prevent oscillations. We analyzed this behavior by solving the equations of motion using the analysis techniques developed in Chapter 3. These techniques can be applied to models of single-mass systems, which can be reduced to a single second-order equation, or to models of multimass systems that may consist of coupled second-order equations.

The algebra and computations required to analyze system response are naturally more difficult for multimass systems, but MATLAB can assist in the analysis. It can be used to perform some of the algebra required to obtain transfer functions and to find the response.

Now that you have finished this chapter, you should be able to

1. Model elements containing elasticity as ideal (massless) spring elements.
2. Model elements containing damping as ideal (massless) damper elements.
3. Obtain equations of motion for systems having spring and damper elements.
4. Apply energy methods to obtain equations of motion.
5. Obtain the free and forced response of mass-spring-damper systems.
6. Utilize MATLAB to assist in the response analysis.

REFERENCES

[Rayleigh, 1945] J. W. S. Rayleigh, *The Theory of Sound*, Vols. 1 and 2, Dover Publications, New York, 1945.

[Roark, 2001] R. J. Roark, R. G. Budynas, and W. C. Young, *Roark's Formulas for Stress and Strain,* 7th ed., McGraw-Hill, New York, 2001.

PROBLEMS

Section 4.1 Spring Elements

4.1 Compute the translational spring constant of a particular steel helical coil spring, of the type used in automotive suspensions. The coil has six turns. The coil diameter is 4 in., and the wire diameter is 0.5 in. For the shear modulus, use $G = 1.7 \times 10^9$ lb/ft^2.

4.2 In the spring arrangement shown in Figure P4.2, the displacement x is caused by the applied force f. Assuming the system is in static equilibrium, sketch the plot of f versus x. Determine the equivalent spring constant k_e for this arrangement, where $f = k_e x$.

4.3 In the arrangement shown in Figure P4.3, a cable is attached to the end of a cantilever beam. We will model the cable as a rod. Denote the translational spring constant of the beam by k_b, and the translational spring constant of the cable by k_c. The displacement x is caused by the applied force f.

   a.   Are the two springs in series or in parallel?

   b.   What is the equivalent spring constant for this arrangement?
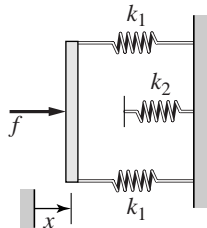
Figure P4.2   **Figure P4.3**   **Figure P4.4**
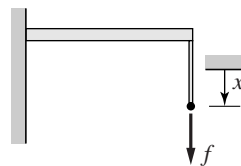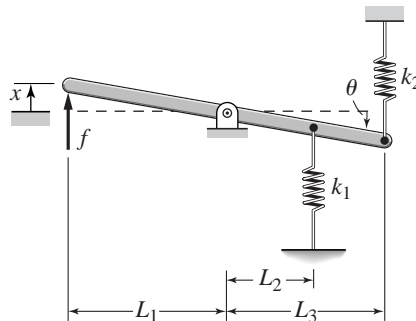
4.4 In the spring arrangement shown in Figure P4.4, the displacement x is caused by the applied force f. Assuming the system is in static equilibrium when $x = 0$ and that the angle θ is small, determine the equivalent spring constant k_e for this arrangement, where $f = k_e x$.

4.5 The two stepped solid cylinders in Figure P4.5 consist of the same material and have an axial force f applied to them. Determine the equivalent translational spring constant for this arrangement. (Hint: Are the two springs in series or in parallel?)

4.6 A table with four identical legs supports a vertical force. The solid cylindrical legs are made of metal with $E = 2 \times 10^{11}$ N/m^2. The legs are 1 m in length and 0.03 m in diameter. Compute the equivalent spring constant due to the legs, assuming the table top is rigid.

Figure P4.5
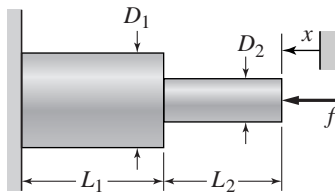
Figure P4.7
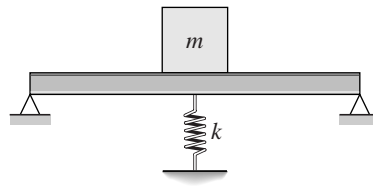
Figure P4.8
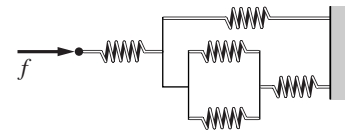
4.7 The beam shown in Figure P4.7 has been stiffened by the addition of a spring support. The steel beam is 3 ft long, 1 in thick, and 1 ft wide, and its mass is 3.8 slugs. The mass m is 40 slugs. Neglecting the mass of the beam,

   a. Compute the spring constant k necessary to reduce the static deflection to one-half its original value before the spring k was added.

   b. Compute the natural frequency ω_n of the combined system.

4.8 Determine the equivalent spring constant of the arrangement shown in Figure P4.8. All the springs have the same spring constant k.

4.9 Compute the equivalent torsional spring constant of the stepped shaft arrangement shown in Figure P4.9. For the shaft material, $G = 8 \times 10^{10}$ N/m^2.

4.10 Plot the spring force felt by the mass shown in Figure P4.10 as a function of the displacement x. When $x = 0$, spring 1 is at its free length. Spring 2 is at its free length in the configuration shown.

Figure P4.9

$D_1 = 0.4$ m
$d_1 = 0.3$ m
$D_2 = 0.35$ m
$d_2 = 0.25$ m
$L_1 = 2$ m
$L_2 = 3$ m
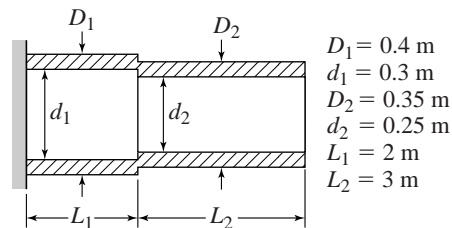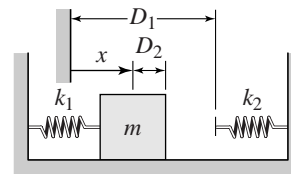
Figure P4.10

Section 4.2 Modeling Mass-Spring Systems

Note: see also the problems for Section 4.5: Additional Modeling Examples.

4.11 For each of the systems shown in Figure P4.11, the input is the force f and the outputs are the displacements x_1 and x_2 of the masses. The equilibrium positions with $f = 0$ correspond to $x_1 = x_2 = 0$. Neglect any friction between the masses and the surface. Derive the equations of motion of the systems.

4.12 The mass m in Figure P4.12 is attached to a rigid lever having negligible mass and negligible friction in the pivot. The input is the displacement x. When x and θ are zero, the springs are at their free length. Assuming that θ is small, derive the equation of motion for θ with x as the input.

4.13 In the pulley system shown in Figure P4.13, the input is the applied force f, and the output is the displacement x. Assume the pulley masses are negligible and derive the equation of motion.

4.14 Figure P4.14 illustrates a cylindrical buoy floating in water with a mass density ρ. Assume that the center of mass of the buoy is deep enough so that the buoy motion is primarily vertical. The buoy mass is m and the diameter

(a)   (b)   (c)
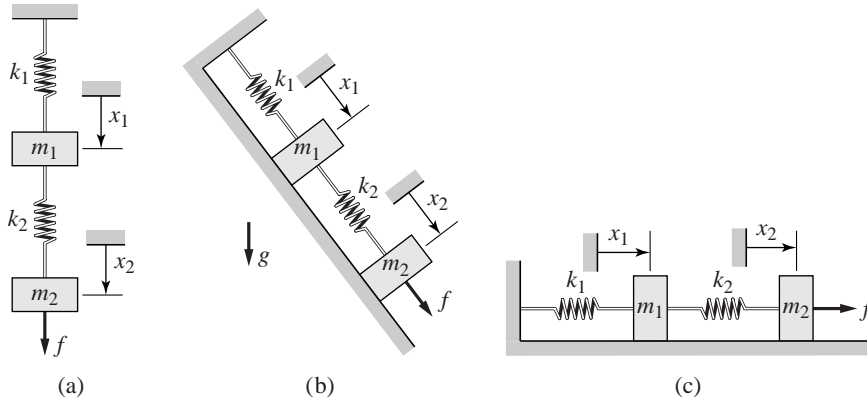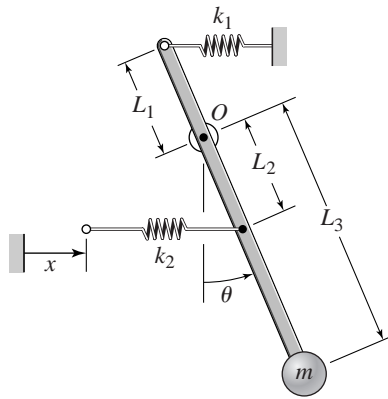
Figure P4.12

Figure P4.13
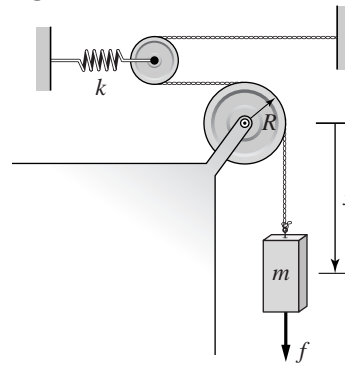
Figure P4.14
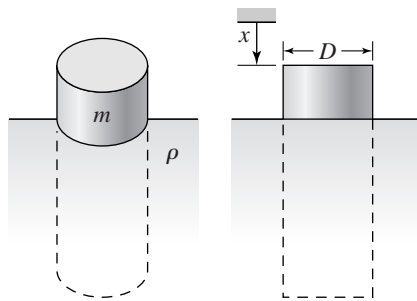
Figure P4.15
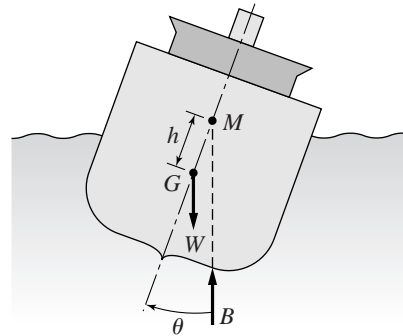
is D. Archimedes' principle states that the buoyancy force acting on a floating object equals the weight of the liquid displaced by the object. (a) Derive the equation of motion in terms of the variable x, which is the displacement from the equilibrium position. (b) Obtain the expression for the buoy's natural frequency. (c) Compute the period of oscillation if the buoy diameter is 2 ft and the buoy weighs 1000 lb. Take the mass density of fresh water to be $\rho = 1.94$ slug/ft^3.

4.15 Figure P4.15 shows the cross-sectional view of a ship undergoing rolling motion. Archimedes' principle states that the buoyancy force B acting on a floating object equals the weight of the liquid displaced by the object. The metacenter

M is the intersection point of the line of action of the buoyancy force and the ship's centerline. The distance *h* of *M* from the mass center *G* is called the metacentric height. (a) Obtain the equation of motion describing the ship's rolling motion in terms of the angle θ. (b) The given parameters are the ship's weight *W*, its metacentric height *h*, and its moment of inertia *I* about the center of gravity. Obtain an expression for the natural frequency of the rolling motion.

4.16 In the system shown in Figure P4.16, the input is the angular displacement ϕ of the end of the shaft, and the output is the angular displacement θ of the inertia *I*. The shafts have torsional stiffnesses k_1 and k_2. The equilibrium position corresponds to $\phi = \theta = 0$. Derive the equation of motion and find the transfer function $\Theta(s)/\Phi(s)$.

4.17 In Figure P4.17, assume that the cylinder rolls without slipping. The spring is at its free length when *x* and *y* are zero. (a) Derive the equation of motion in terms of *x*, with $y(t)$ as the input. (b) Suppose that $m = 10$ kg, $R = 0.3$ m, $k = 1000$ N/m, and that $y(t)$ is a unit-step function. Solve for $x(t)$ if $x(0) = \dot{x}(0) = 0$.

4.18 In Figure P4.18 when $x_1 = x_2 = 0$ the springs are at their free lengths. Derive the equations of motion.

Figure P4.16
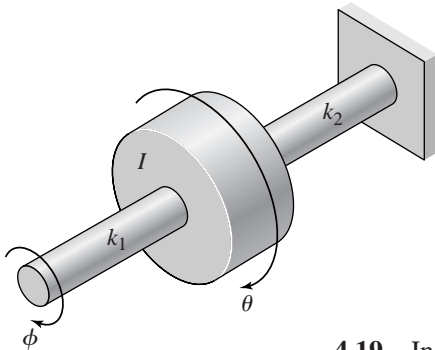
Figure P4.17
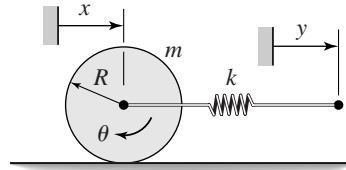
Figure P4.18
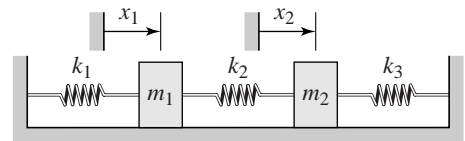
4.19 In Figure P4.19 model the three shafts as massless torsional springs. When $\theta_1 = \theta_2 = 0$ the springs are at their free lengths. Derive the equations of motion with the torque T_2 as the input.

4.20 In Figure P4.20 when $\theta_1 = \theta_2 = 0$ the spring is at its free length. Derive the equations of motion, assuming small angles.

Figure P4.19
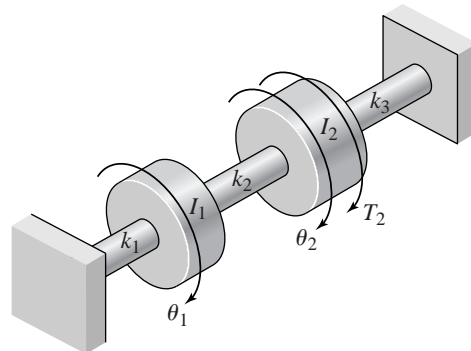
Figure P4.20
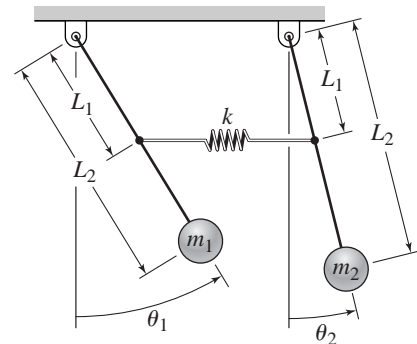
4.21 Consider the torsion-bar suspension shown in Figure 4.1.6. Assume that the torsion bar is a steel rod with a length of 4 ft and diameter 1.5 in. The wheel weighs 40 lb and the suspension arm is 2 ft long. Neglect the masses of the torsion bar and the suspension arm, and calculate the natural frequency of the system. Use $G = 1.7 \times 10^9$ lb/ft^2.

Section 4.3 Energy Methods

4.22 For Figure P4.22, assume that the cylinder rolls without slipping and use conservation of energy to derive the equation of motion in terms of x.

4.23 For Figure P4.23, the equilibrium position corresponds to $x = 0$. Neglect the masses of the pulleys and assume that the cable is inextensible, and use conservation of energy to derive the equation of motion in terms of x.

4.24 For Figure P4.24, the equilibrium position corresponds to $x = 0$. Neglect the masses of the pulleys and assume that the cable is inextensible, and use conservation of energy to derive the equation of motion in terms of x.

4.25 Use the Rayleigh method to obtain an expression for the natural frequency of the system shown in Figure P4.25. The equilibrium position corresponds to $x = 0$.

4.26 For Figure P4.26, assume that the cylinder rolls without slipping and use the Rayleigh method to obtain an expression for the natural frequency of the system. The equilibrium position corresponds to $x = 0$.

4.27 Use the Rayleigh method to obtain an expression for the natural frequency of the system shown in Figure P4.27. The equilibrium position corresponds to $x = 0$.

Figure P4.22
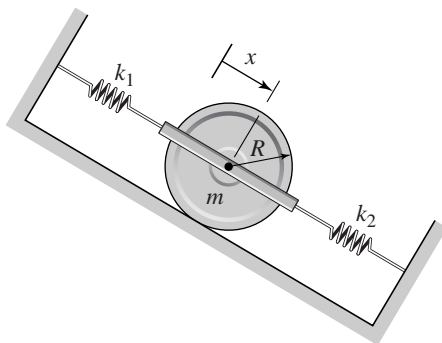
Figure P4.23
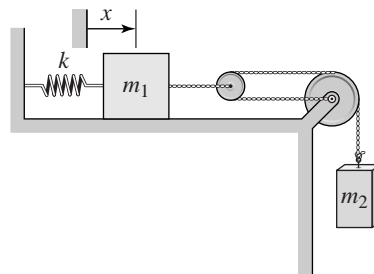
Figure P4.24
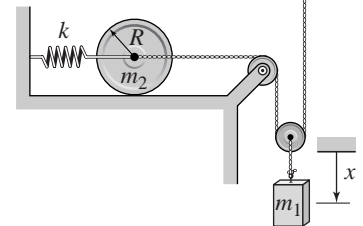
Figure P4.25
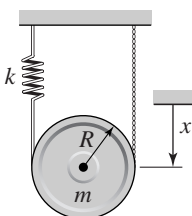
Figure P4.26
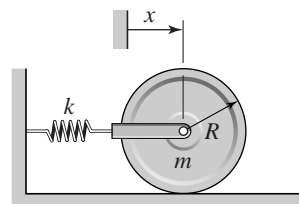
Figure P4.27
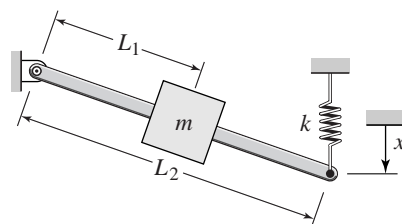
Figure P4.28
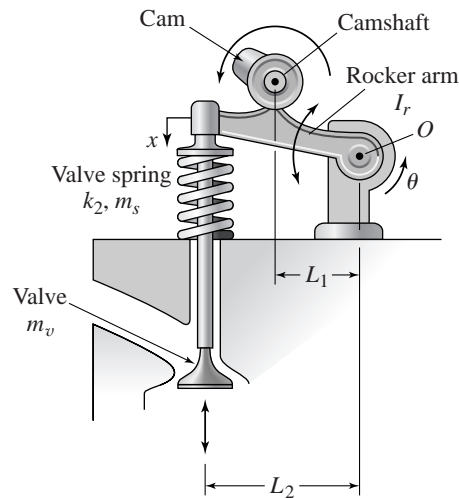
Figure P4.31
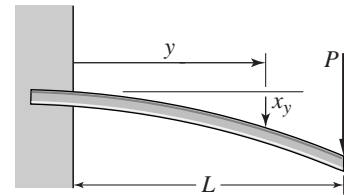
4.28   Figure P4.28 shows an engine valve driven by an overhead camshaft. The rocker arm pivots about the fixed point O and the inertia of the arm about this point is I_r. The valve mass is m_v and the spring mass is m_s; its spring constant is k_s. Let f_c denote the force exerted on the rocker arm by the camshaft. Assuming that $\theta(t)$ and its time derivatives are known (from the cam profile and the cam speed), derive a dynamic model that can be used to solve for the cam force $f_c(t)$. (This information is needed to predict the amount of wear on the cam surface.)

4.29   The vibration of a motor mounted on the end of a cantilever beam can be modeled as a mass-spring system. The motor weighs 30 lb, and the beam weighs 7 lb. When the motor is placed on the beam, it causes an additional static deflection of 0.8 in. Find the equivalent mass m and equivalent spring constant k.

4.30   The vibration of a motor mounted in the middle of a fixed-end beam can be modeled as a mass-spring system. The motor mass is 40 kg, and the beam mass is 13 kg. When the motor is placed on the beam, it causes an additional static deflection of 3 mm. Find the equivalent mass m and equivalent spring constant k.

4.31   The static deflection of a cantilever beam is described by

$$x_y = \frac{P}{6EI_A} y^2 (3L - y)$$

where P is the load applied at the end of the beam, and x_y is the vertical deflection at a point a distance y from the support (Figure P4.31). Obtain an expression for an equivalent mass located at the end of the beam.

4.32   Figure P4.32 shows a winch supported by a cantilever beam at the stern of a ship. The mass of the winch is m_w, the mass of the beam plus winch bracket and motor is m_b. The object hoisted by the winch has a mass m_h; the wire rope mass m_r is assumed to be negligible compared to the other masses. Find the equation of motion for the vertical motion x_1 of the winch: (a) assuming that the rope does not stretch and (b) assuming that the rope stretches and has a spring constant k_r.
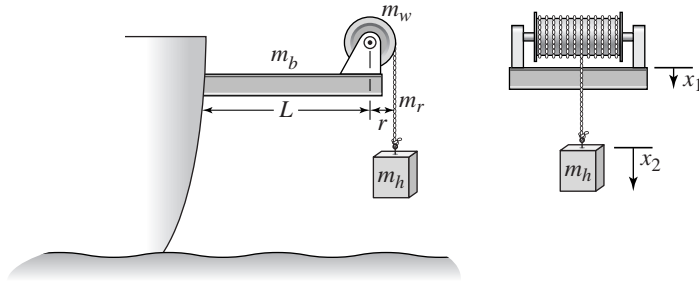
Section 4.4 Damping Elements

4.33 A 50-kg block is placed on an inclined plane whose angle with the horizontal is 25°. The viscous friction coefficient between the block and the plane is $c = 6\,N \cdot s/m$. (a) Derive the equation of motion. (b) Solve the equation of motion for the speed $v(t)$ of the block, assuming that the block is initially given a speed of 5 m/s. (c) Find the steady-state speed of the block and estimate the time required to reach that speed. (d) Discuss the effect of the initial velocity of the steady-state speed.

4.34 A certain mass-spring-damper system has the following equation of motion.

$$40\ddot{x} + c\dot{x} + 1200x = f(t)$$

Suppose that the initial conditions are zero and that the applied force $f(t)$ is a step function of magnitude 5000. Solve for $x(t)$ for the following two cases: (a) $c = 680$ and (b) $c = 400$.

4.35 For each of the systems shown in Figure P4.35, the input is the force f and the outputs are the displacements x_1 and x_2 of the masses. The equilibrium positions with $f = 0$ correspond to $x_1 = x_2 = 0$. Neglect any friction between the masses and the surface. Derive the equations of motion of the systems.

(a)

(b)

(c)

(d)
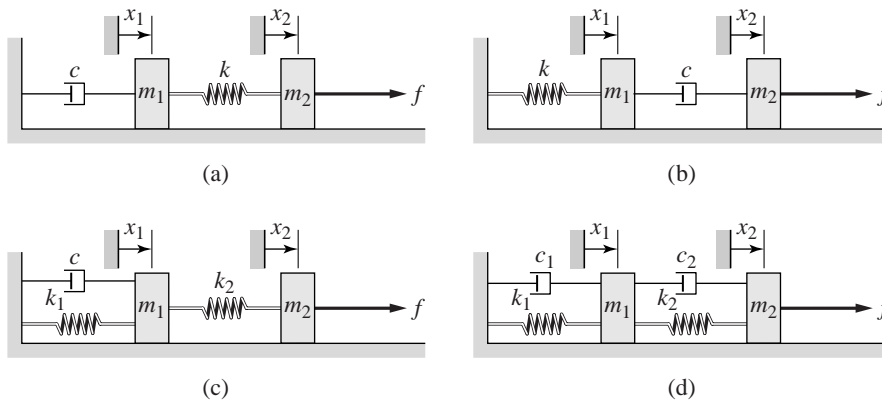
4.36 In Figure P4.36 a motor supplies a torque T to turn a drum of radius R and inertia I about its axis of rotation. The rotating drum lifts a mass m by means of a cable that wraps around the drum. The drum's speed is ω. Viscous torsional damping c_T exists in the drum shaft. Neglect the mass of the cable. (a) Obtain the equation of motion with the torque T as the input and the vertical speed v of the mass as the output. (b) Suppose that $m = 40$ kg, $R = 0.2$ m,

$I = 0.8 \text{ kg} \cdot \text{m}^2$, and $c_T = 0.1 \text{ N} \cdot \text{m} \cdot \text{s}$. Find the speed $v(t)$ if the system is initially at rest and the torque T is a step function of magnitude $300 \text{ N} \cdot \text{m}$.
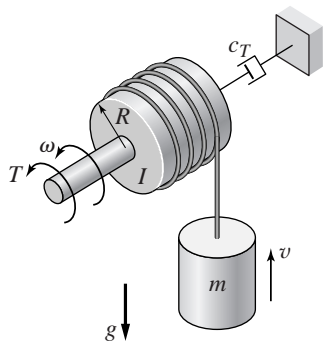
4.37 Derive the equation of motion for the lever system shown in Figure P4.37, with the force f as the input and the angle θ as the output. The position $\theta = 0$ corresponds to the equilibrium position when $f = 0$. The lever has an inertia I about the pivot. Assume small displacements.

4.38 In the system shown in Figure P4.38, the input is the displacement y and the output is the displacement x of the mass m. The equilibrium position corresponds to $x = y = 0$. Neglect any friction between the mass and the surface. Derive the equation of motion and find the transfer function $X(s)/Y(s)$.

Figure P4.37
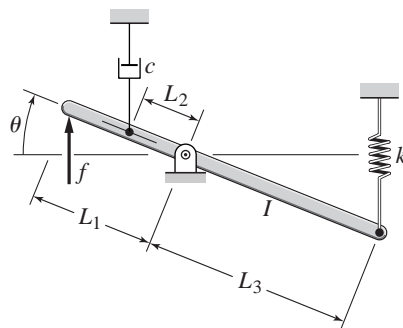
Figure P4.38
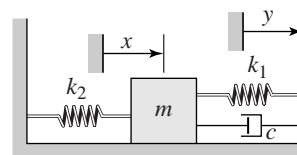
4.39 Figure P4.39a shows a Houdaille damper, which is a device attached to an engine crankshaft to reduce vibrations. The damper has an inertia I_d that is free to rotate within an enclosure filled with viscous fluid. The inertia I_p is the inertia of the fan-belt pulley. Modeling the crankshaft as a torsional spring k_T, the damper system can be modeled as shown in part (b) of the figure. Derive the equation of motion with the angular displacements θ_p and θ_d as the outputs and the crankshaft angular displacement ϕ as the input.

Figure P4.39

(a)

(b)
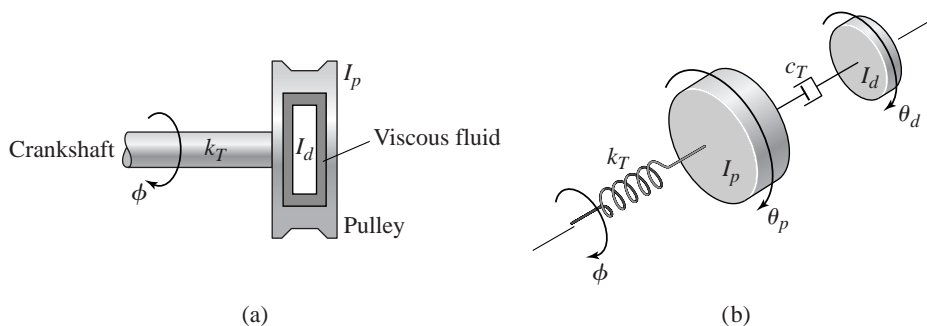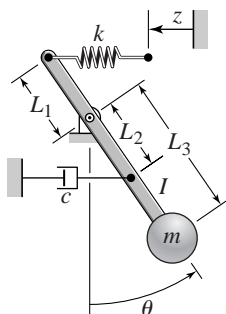
Figure P4.40

Section 4.5 Additional Modeling Examples

4.40 The mass m in Figure P4.40 is attached to a rigid rod having an inertia I about the pivot and negligible pivot friction. The input is the displacement z. When $z = \theta = 0$, the spring is at its free length. Assuming that θ is small, derive the equation of motion for θ with z as the input.

4.41 In the system shown in Figure P4.41, the input is the force f and the output is the displacement x_A of point A. When $x = x_A$ the spring is at its free length. Derive the equation of motion.

Figure P4.41
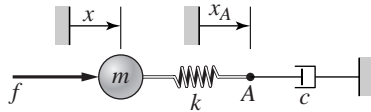
Figure P4.42
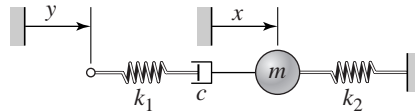
4.42 In the system shown in Figure P4.42, the input is the displacement y and the output is the displacement x. When $x = y = 0$ the springs are at their free lengths. Derive the equation of motion.

4.43 Figure P4.43 shows a rack-and-pinion gear in which a damping force and a spring force act against the rack. Develop the equivalent rotational model of the system with the applied torque T as the input variable and the angular displacement θ is the output variable. Neglect any twist in the shaft.

Figure P4.43
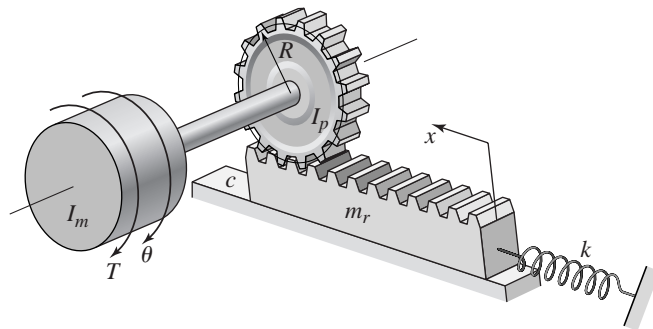
Figure P4.44
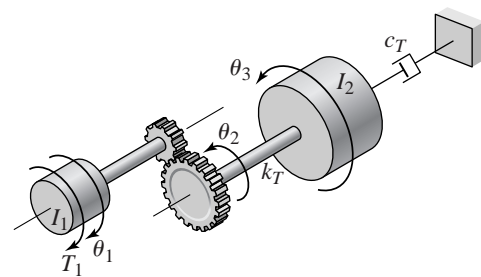
4.44 Figure P4.44 shows a drive train with a spur-gear pair. The first shaft turns N times faster than the second shaft. Develop a model of the system including the elasticity of the second shaft. Assume the first shaft is rigid, and neglect the gear and shaft masses. The input is the applied torque T_1. The outputs are the angles θ_1 and θ_3.

4.45 Assuming that θ is small, derive the equations of motion of the systems shown in parts (a) and (b) of Figure P4.45. When $\theta = 0$ the systems are in equilibrium. Are the systems stable, neutrally stable, or unstable?

4.46 Assuming that θ is small, derive the equation of motion of the pendulum shown in Figure P4.46. The pendulum is in equilibrium when $\theta = 0$. Is the system stable, neutrally stable, or unstable?

4.47 Assuming that θ is small, derive the equation of motion of the pendulum shown in Figure P4.47. The input is $y(t)$ and the output is θ. The equilibrium corresponds to $y = \theta = 0$, when the springs are at their free lengths. The rod inertia about the pivot is I.

4.48 Figure P4.48 shows a quarter-car model that includes the mass of the seats (including passengers). The constants k_3 and c_3 represent the stiffness and damping in the seat supports. Derive the equations of motion of this system.

Figure P4.45

(a)   (b)
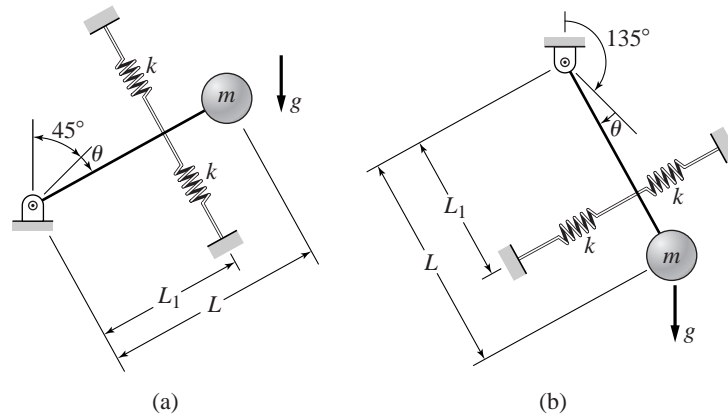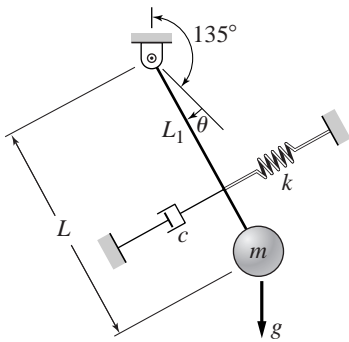
Figure P4.46

Figure P4.47
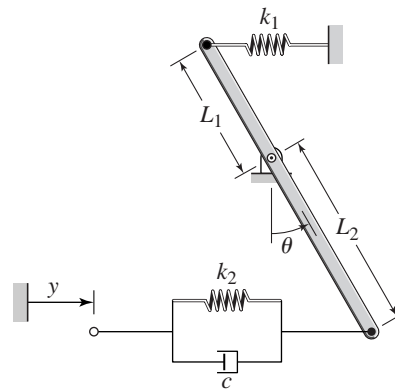
Figure P4.48
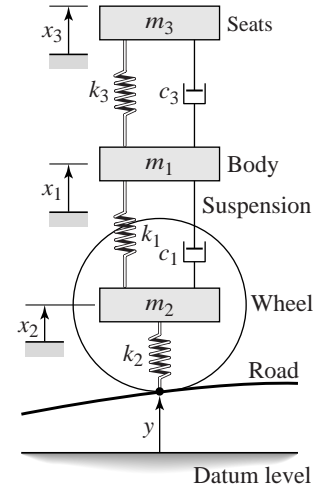
The input is the road displacement $y(t)$. The displacements are measured from equilibrium.

Section 4.6 Collisions and Impulse Response

4.49   Suppose a mass m moving with a speed v_1 becomes embedded in mass m_2 after striking it (Figure 4.6.1). Suppose $m_2 = 5m$. Determine the expression for the displacement $x(t)$ after the collision.

4.50   Consider the system shown in Figure 4.6.3. Suppose that the mass m moving with a speed v_1 rebounds from the mass $m_2 = 5m$ after striking it. Assume that the collision is perfectly elastic. Determine the expression for the displacement $x(t)$ after the collision.

Section 4.7 MATLAB Applications

4.51   (a) Obtain the equations of motion of the system shown in Figure P4.51. The masses are $m_1 = 20$ kg and $m_2 = 60$ kg. The spring constants are $k_1 = 3 \times 10^4$ N/m and $k_2 = 6 \times 10^4$ N/m. (b) Obtain the transfer functions

Figure P4.51
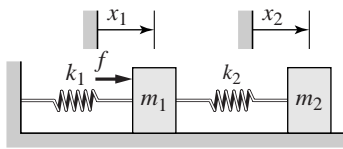
Figure P4.56
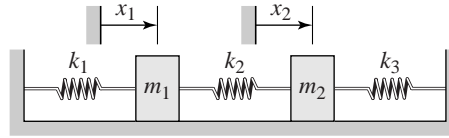
$X_1(s)/F(s)$ and $X_2(s)/F(s)$. (c) Obtain the unit-step response of x_1 for zero initial conditions.

4.52 (a) Obtain the equations of motion of the system shown in Figure P4.19. (b) Suppose the inertias are $I_1 = I$ and $I_2 = 2I$ and the torsional spring constants are $k_1 = k_2 = k_3 = k$. Obtain the transfer functions $\Theta_1(s)/T_2(s)$ and $\Theta_2(s)/T_2(s)$ in terms of I and k. (c) Suppose that $I = 10$ and $k = 60$. Obtain the unit-impulse response of θ_1 for zero initial conditions.

4.53 Refer to part (a) of Problem 4.51. Use MATLAB to obtain the transfer functions $X_1(s)/F(s)$ and $X_2(s)/F(s)$. Compare your answers with those found in part (b) of Problem 4.51.

4.54 Refer to Problem 4.52. Use MATLAB to obtain the transfer functions $\Theta_1(s)/T_2(s)$ and $\Theta_2(s)/T_2(s)$ for the values $I_1 = 10$, $I_2 = 20$, and $k_1 = k_2 = k_3 = 60$. Compare your answers with those found in part (b) of Problem 4.52.

4.55 (a) Obtain the equations of motion of the system shown in Figure P4.20. Assume small angles. The spring is at its free length when $\theta_1 = \theta_2 = 0$. (b) For the values $m_1 = 1$ kg, $m_2 = 4$ kg, $k = 10$ N/m, $L_1 = 2$ m, and $L_2 = 5$ m, use MATLAB to plot the free response of θ_1 if $\theta_1(0) = 0.1$ rad and $\dot{\theta}_1(0) = \dot{\theta}_2(0) = \theta_2(0) = 0$.

4.56 (a) Obtain the equations of motion of the system shown in Figure P4.56. (b) Suppose that the masses are $m_1 = 1$ kg, $m_2 = 2$ kg, and the spring constants are $k_1 = k_2 = k_3 = 1.6 \times 10^4$ N/m. Use MATLAB to obtain the plot of the free response of x_1. Use $x_1(0) = 0.1$ m, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$.

State-Variable Models and Simulation Methods

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Convert a differential equation model into state-variable form.

2. Express a linear state-variable model in the standard vector-matrix form.

3. Apply the `ss`, `ssdata`, `tfdata`, `char`, `eig`, and `initial` functions to analyze linear models.

4. Use the MATLAB `ode` functions to solve differential equations.

5. Use Simulink to create simulations of dynamic models.

D ynamic models derived from basic physical principles can appear in several forms:

1. As a single equation (which is called the *reduced* form),
2. As a set of coupled first-order equations (which is called the *Cauchy* or *state-variable* form), and
3. As a set of coupled higher-order equations.

In Chapters 3 and 4 we analyzed the response of a single equation, such as $m\ddot{x} + c\dot{x} + kx = f(t)$, and sets of coupled first-order equations, such as $\dot{x} = -5x + 7y$, $\dot{y} = 3x - 9y + f(t)$, by first obtaining the transfer functions and then using the transfer functions to obtain a single, but higher-order equation. We also obtained the response of models that consist of a set of coupled higher-order equations.

Each form has its own advantages and disadvantages. We can convert one form into another, with differing degrees of difficulty. In addition, if the model is *linear,* we can convert any of these forms into the transfer function form or a vector-matrix form, each of which has its own advantages.

In Section 5.1, we introduce the state-variable model form. An advantage of the state-variable form is that it enables us to express a linear model of any order in a standard and compact way that is useful for analysis and software applications.

MATLAB has a number of useful functions that are based on the state-variable model form. These functions are covered in Sections 5.2 and 5.3. Section 5.2 deals with linear models. While the analysis methods of the previous chapters are limited to linear models, the state-variable form is also useful for solving nonlinear equations. It is not always possible or convenient to obtain the closed-form solution of a differential equation, and this is usually true for nonlinear equations. Section 5.3 introduces MATLAB functions that are useful for solving nonlinear differential equations.

Section 5.4 introduces Simulink, which provides a graphical user interface for solving differential equations. It is especially useful for solving problems containing nonlinear features such as Coulomb friction, saturation, and dead zones, because these features are very difficult to program with traditional programming methods. In addition, its graphical interface might be preferred by some users to the more traditional programming methods offered by the MATLAB solvers covered in Sections 5.2 and 5.3. In Section 5.4 we begin with solving linear equations so that we can check the results with the analytical solution. Section 5.5 covers Simulink methods for nonlinear equations. ∎

5.1 STATE-VARIABLE MODELS

Models that consist of coupled first-order differential equations are said to be in *state-variable* form. This form, which is also called the *Cauchy* form, has an advantage over the reduced form, which consists of a single, higher-order equation, because it allows a linear model to be expressed in a standard and compact way that is useful for analysis and for software applications. This representation makes use of vector and matrix notation. In this section, we will show how to obtain a model in state-variable form and how to express state-variable models in vector-matrix notation. In Section 5.2 we show how to use this notation with MATLAB.

Consider the second-order equation

$$5\ddot{y} + 7\dot{y} + 4y = f(t)$$

Solve it for the highest derivative:

$$\ddot{y} = \frac{1}{5}f(t) - \frac{4}{5}y - \frac{7}{5}\dot{y}$$

Now define two new variables, x_1 and x_2, as follows: $x_1 = y$ and $x_2 = \dot{y}$. This implies that $\dot{x}_1 = x_2$ and

$$\dot{x}_2 = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2$$

These two equations, called the *state equations,* are the state-variable form of the model, and the variables x_1 and x_2 are called the *state variables.*

The general mass-spring-damper model has the following form:

$$m\ddot{x} + c\dot{x} + kx = f \tag{5.1.1}$$

If we define new variables x_1 and x_2 such that

$$x_1 = x \qquad x_2 = \dot{x}$$

these imply that

$$\dot{x}_1 = x_2 \tag{5.1.2}$$

Then we can write the model (5.1.1) as: $m\dot{x}_2 + cx_2 + kx_1 = f$. Next solve for $\dot{x}_2$:

$$\dot{x}_2 = \frac{1}{m}(f - kx_1 - cx_2) \tag{5.1.3}$$

Equations (5.1.2) and (5.1.3) constitute a state-variable model corresponding to the reduced model (5.1.1). The variables x_1 and x_2 are the *state variables*.

If (5.1.1) represents a mass-spring-damper system, the state-variable x_1 describes the system's potential energy $kx_1^2/2$, which is due to the spring, and the state-variable x_2 describes the system's kinetic energy $mx_2^2/2$, which is due to the mass. Although here we have derived the state variable model from the reduced form, state-variable models can be derived from basic physical principles. Choosing as state variables those variables that describe the types of energy in the system sometimes helps to derive the model (note that k and m are also needed to describe the energies, but these are parameters, not variables).

The choice of state variables is not unique, but the choice must result in a set of first-order differential equations. For example, we could have chosen the state variables to be $z_1 = x$ and $z_2 = m\dot{x}$, which is the system's momentum. In this case the state-variable model would be

$$\dot{z}_1 = \frac{1}{m}z_2$$

$$\dot{z}_2 = f - \frac{c}{m}z_2 - kz_1$$

EXAMPLE 5.1.1 | State-Variable Model of a Two-Mass System

■ Problem

Figure 5.1.1 A two-mass system.
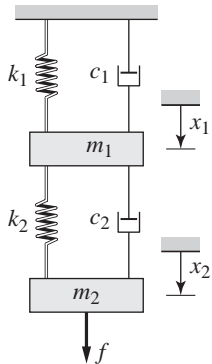
Consider the two-mass system discussed in Example 4.4.5 of Chapter 4 (and shown again in Figure 5.1.1). Suppose the parameter values are $m_1 = 5$, $m_2 = 3$, $c_1 = 4$, $c_2 = 8$, $k_1 = 1$, and $k_2 = 4$. The equations of motion are

$$5\ddot{x}_1 + 12\dot{x}_1 + 5x_1 - 8\dot{x}_2 - 4x_2 = 0 \tag{1}$$

$$3\ddot{x}_2 + 8\dot{x}_2 + 4x_2 - 8\dot{x}_1 - 4x_1 = f(t) \tag{2}$$

Put these equations into state-variable form.

■ Solution

Using the system's potential and kinetic energies as a guide, we see that the displacements x_1 and x_2 describe the system's potential energy and that the velocities $\dot{x}_1$ and $\dot{x}_2$ describe the system's kinetic energy. That is

$$\text{PE} = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2$$

and

$$KE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

This indicates that we need four state variables. (Another way to see that we need four variables is to note that the model consists of two coupled second-order equations, and thus is effectively a fourth-order model.) Thus, we can choose the state variables z_1, z_2, z_3, and z_4 to be

$$z_1 = x_1 \qquad z_2 = \dot{x}_1 \qquad z_3 = x_2 \qquad z_4 = \dot{x}_2 \tag{3}$$

These definitions imply that $\dot{z}_1 = z_2$ and $\dot{z}_3 = z_4$, which are two of the state equations. The remaining two equations can be found by solving equations (1) and (2) for $\ddot{x}_1$ and $\ddot{x}_2$, noting that $\ddot{x}_1 = \dot{z}_2$ and $\ddot{x}_2 = \dot{z}_4$, and using the substitutions given by equation (3).

$$\dot{z}_2 = \frac{1}{5}(-12z_2 - 5z_1 + 8z_4 + 4z_3)$$

$$\dot{z}_4 = \frac{1}{3}[-8z_4 - 4z_3 + 8z_2 + 4z_1 + f(t)]$$

Note that the left-hand sides of the state equations must contain only the first-order derivative of each state variable. This is why we divided by 5 and 3, respectively. Note also that the right-hand sides must not contain any derivatives of the state variables. For example, we should not use the substitution $\dot{z}_1$ for $\dot{x}_1$, but rather should substitute z_2 for $\dot{x}_1$. Failure to observe this restriction is a common mistake.

Now list the four state equations in ascending order according to their left-hand sides, after rearranging the right-hand sides so that the state variables appear in ascending order from left to right.

$$\dot{z}_1 = z_2 \tag{4}$$

$$\dot{z}_2 = \frac{1}{5}(-5z_1 - 12z_2 + 4z_3 + 8z_4) \tag{5}$$

$$\dot{z}_3 = z_4 \tag{6}$$

$$\dot{z}_4 = \frac{1}{3}[4z_1 + 8z_2 - 4z_3 - 8z_4 + f(t)] \tag{7}$$

These are the state equations in standard form.

Note that because the potential energy is a function of the difference $x_1 - x_2$, another possible choice of state variables is $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_1 - x_2$, and $z_4 = \dot{x}_2$.

VECTOR-MATRIX FORM OF STATE-VARIABLE MODELS

Vector-matrix notation enables us to represent multiple equations as a single matrix equation. For example, consider the following set of linear algebraic equations.

$$2x_1 + 9x_2 = 5 \tag{5.1.4}$$

$$3x_1 - 4x_2 = 7 \tag{5.1.5}$$

The term *matrix* refers to an array with more than one column and more than one row. A *column vector* is an array having only one column. A *row vector* has only one row. A matrix is an arrangement of numbers and is not the same as a determinant, which can be reduced to a single number. Multiplication of a matrix having two rows and two columns (a (2×2) matrix) by a column vector having two rows and one column

(a (2×1) vector) is defined as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \tag{5.1.6}$$

This definition is easily extended to matrices having more than two rows or two columns. In general, the result of multiplying an $(n \times n)$ matrix by an $(n \times 1)$ vector is an $(n \times 1)$ vector. This definition of vector-matrix multiplication requires that the matrix have as many columns as the vector has rows. The order of the multiplication cannot be reversed (vector-matrix multiplication does not have the commutative property).

Two vectors are equal if all their respective elements are equal. Thus we can represent the set (5.1.4) and (5.1.5) as follows:

$$\begin{bmatrix} 2 & 9 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{5.1.7}$$

We usually represent matrices and vectors in boldface type, with matrices usually in upper case letters and vectors in lowercase, but this is not required. Thus we can represent the set (5.1.7) in the following compact form.

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{5.1.8}$$

where we have defined the following matrices and vectors:

$$\mathbf{A} = \begin{bmatrix} 2 & 9 \\ 3 & -4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

The matrix $\mathbf{A}$ corresponds in an ordered fashion to the coefficients of x_1 and x_2 in (5.1.4) and (5.1.5). Note that the first row in $\mathbf{A}$ consists of the coefficients of x_1 and x_2 on the left-hand side of (5.1.4), and the second row contains the coefficients on the left-hand side of (5.1.5). The vector $\mathbf{x}$ contains the variables x_1 and x_2, and the vector $\mathbf{b}$ contains the right-hand sides of (5.1.4) and (5.1.5).

EXAMPLE 5.1.2

Vector-Matrix Form of a Single-Mass Model

■ Problem
Express the mass-spring-damper model (5.1.2) and (5.1.3) as a single vector-matrix equation. These equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}f(t) - \frac{k}{m}x_1 - \frac{c}{m}x_2$$

■ Solution
The equations can be written as one equation as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} f(t)$$

In compact form this is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Vector-Matrix Form of the Two-Mass Model | **EXAMPLE 5.1.3**

■ **Problem**

Express the state-variable model of Example 5.1.1 in vector-matrix form. The model is

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \frac{1}{5}(-5z_1 - 12z_2 + 4z_3 + 8z_4)$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = \frac{1}{3}[4z_1 + 8z_2 - 4z_3 - 8z_4 + f(t)]$$

■ **Solution**

In vector-matrix form these equations are

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}f(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\frac{12}{5} & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & \frac{8}{3} & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

STANDARD FORM OF THE STATE EQUATION

We may use any symbols we choose for the state variables and the input function, although the common choice is x_i for the state variables and u_i for the input functions. The standard vector-matrix form of the state equations, where the number of state variables is n and the number of inputs is m, is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (5.1.9)$$

where the vectors $\mathbf{x}$ and $\mathbf{u}$ are column vectors containing the state variables and the inputs, if any. The dimensions are as follows:

■ The *state vector* $\mathbf{x}$ is a column vector having n rows.
■ The *system matrix* $\mathbf{A}$ is a square matrix having n rows and n columns.
■ The *input vector* $\mathbf{u}$ is a column vector having m rows.
■ The *control* or *input matrix* $\mathbf{B}$ has n rows and m columns.

In our examples thus far there has been only one input, and for such cases the input vector **u** reduces to a scalar u. The standard form, however, allows for more than one input function. Such would be the case in the two-mass model if external forces f_1 and f_2 are applied to the masses.

THE OUTPUT EQUATION

Some software packages and some design methods require you to define an *output vector,* usually denoted by **y**. The output vector contains the variables that are of interest for the particular problem at hand. These variables are not necessarily the state variables, but might be some combination of the state variables and the inputs. For example, in the mass-spring model, we might be interested in the total force $f - kx - c\dot{x}$ acting on the mass, and in the momentum $m\dot{x}$. In this case, the output vector has two elements. If the state variables are $x_1 = x$ and $x_2 = \dot{x}$, the output vector is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f - kx - c\dot{x} \\ m\dot{x} \end{bmatrix} = \begin{bmatrix} f - kx_1 - cx_2 \\ mx_2 \end{bmatrix}$$

or

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -k & -c \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f = \mathbf{Cx} + \mathbf{D}f$$

where

$$\mathbf{C} = \begin{bmatrix} -k & -c \\ 0 & m \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is an example of the general form: $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$.

The standard vector-matrix form of the output equation, where the number of outputs is p, the number of state variables is n, and the number of inputs is m, is

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{5.1.10}$$

where the vector **y** contains the output variables. The dimensions are as follows:

- The *output vector* **y** is a column vector having p rows.
- The *state output matrix* **C** has p rows and n columns.
- The *control output matrix* **D** has p rows and m columns.

The matrices **C** and **D** can always be found whenever the chosen output vector **y** is a linear combination of the state variables and the inputs. However, if the output is a nonlinear function, then the standard form (5.1.10) does not apply. This would be the case, for example, if the output is chosen to be the system's kinetic energy: $KE = mx_2^2/2$.

EXAMPLE 5.1.4 | The Output Equation for a Two-Mass Model

■ **Problem**

Consider the two-mass model of Example 5.1.1.

a) Suppose the outputs are x_1 and x_2. Determine the output matrices **C** and **D**. b) Suppose the outputs are $(x_2 - x_1)$, $\dot{x}_2$, and f. Determine the output matrices **C** and **D**.

■ Solution

a. In terms of the **z** vector, $z_1 = x_1$ and $z_3 = x_2$. We can express the output vector **y** as follows.

$$\mathbf{y} = \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f$$

Thus

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b. Here the outputs are $y_1 = x_2 - x_1 = z_3 - z_1$, $y_2 = \dot{x}_2 = z_4$, and $y_3 = f$. Thus we can express the output vector as follows:

$$\mathbf{y} = \begin{bmatrix} z_3 - z_1 \\ z_4 \\ f \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f$$

Thus

$$\mathbf{C} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

MODEL FORMS HAVING NUMERATOR DYNAMICS

Note that if you only need to obtain the free response, then the presence of input derivatives or numerator dynamics in the model is irrelevant. For example, the free response of the model

$$5\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 4\frac{df}{dt} + 9f(t)$$

is identical to the free response of the model

$$5\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 0$$

which does not have any inputs. A state-variable model for this equation is easily found to be

$$x_1 = y \qquad x_2 = \dot{y} \qquad x_3 = \ddot{y}$$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = -\tfrac{6}{5}x_1 - \tfrac{7}{5}x_2 - \tfrac{3}{5}x_3$$

The free response of this model can be easily found with the MATLAB `initial` function to be introduced in the next section.

For some applications you need to obtain a state-variable model in the standard form. However, in the standard state-variable form $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ there is no derivative of the input **u**. When the model has numerator dynamics or input derivatives, the state variables are not so easy to identify. When there are no numerator dynamics you can always obtain a state-variable model in standard form from a transfer-function

or reduced-form model whose dependent variable is x by defining $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, and so forth. This was the procedure followed previously. Note that the initial conditions $x_1(0)$, $x_2(0)$, and $x_3(0)$ are easily obtained from the given conditions $x(0)$, $\dot{x}(0)$, and $\ddot{x}(0)$; that is, $x_1(0) = x(0)$, $x_2(0) = \dot{x}(0)$, and $x_3(0) = \ddot{x}(0)$. However, when numerator dynamics are present, a different technique must be used, and the initial conditions are not as easily related to the state variables.

We now give two examples of how to obtain a state-variable model when numerator dynamics exists.

EXAMPLE 5.1.5

Numerator Dynamics in a First-Order System

■ Problem
Consider the transfer function model

$$\frac{Z(s)}{U(s)} = \frac{5s + 3}{s + 2} \tag{1}$$

This corresponds to the equation

$$\dot{z} + 2z = 5\dot{u} + 3u \tag{2}$$

Note that this equation is not in the standard form $\dot{z} = Az + Bu$ because of the input derivative $\dot{u}$. Demonstrate two ways of converting this model to a state-variable model in standard form.

■ Solution
a. One way of obtaining the state-variable model is to divide the numerator and denominator of equation (1) by s.

$$\frac{Z(s)}{U(s)} = \frac{5 + 3/s}{1 + 2/s} \tag{3}$$

The objective is to obtain a 1 in the denominator, which is then used to isolate $Z(s)$ as follows:

$$Z(s) = -\frac{2}{s}Z(s) + 5U(s) + \frac{3}{s}U(s)$$

$$= \frac{1}{s}[3U(s) - 2Z(s)] + 5U(s)$$

The term within square brackets multiplying $1/s$ is the input to an integrator, and the integrator's output can be selected as a state-variable x. Thus,

$$Z(s) = X(s) + 5U(s)$$

where

$$X(s) = \frac{1}{s}[3U(s) - 2Z(s)] = \frac{1}{s}\{3U(s) - 2[X(s) + 5U(s)]\}$$

$$= \frac{1}{s}[-2X(s) - 7U(s)]$$

This gives

$$\dot{x} = -2x - 7u \tag{4}$$

with the output equation

$$z = x + 5u \tag{5}$$

This fits the standard form (5.1.9) and (5.1.10), with $\mathbf{A} = -2$, $\mathbf{B} = -7$, $\mathbf{y} = z$, $\mathbf{C} = 1$, and $\mathbf{D} = 5$.

Presumably we are given the initial condition $z(0)$, but to solve equation (4) we need $x(0)$. This can be obtained by solving equation (5) for x, $x = z - 5u$, and evaluating it at $t = 0$: $x(0) = z(0) - 5u(0)$. We see that $x(0) = z(0)$ if $u(0) = 0$.

b. Another way is to write equation (1) as

$$Z(s) = (5s + 3)\frac{U(s)}{s + 2} \qquad (6)$$

and define the state-variable x as follows:

$$X(s) = \frac{U(s)}{s + 2} \qquad (7)$$

Thus,

$$sX(s) = -2X(s) + U(s) \qquad (8)$$

and the state equation is

$$\dot{x} = -2x + u \qquad (9)$$

To find the output equation, note that

$$Z(s) = (5s + 3)\frac{U(s)}{s + 2} = (5s + 3)X(s) = 5sX(s) + 3X(s)$$

Using equation (8) we have

$$Z(s) = 5[-2X(s) + U(s)] + 3X(s) = -7X(s) + 5U(s)$$

and thus the output equation is

$$z = -7x + 5u \qquad (10)$$

The initial condition $x(0)$ is found from equation (10) to be $x(0) = [5u(0) - z(0)]/7 = -z(0)/7$ if $u(0) = 0$.

Although the model consisting of equations (9) and (10) looks different than that given by equations (4) and (5), they are both in the standard form and are equivalent, because they were derived from the same transfer function.

This example points out that there is no unique way to derive a state-variable model from a transfer function. It is important to keep this in mind because the state-variable model obtained from the MATLAB `ssdata(sys)` function, to be introduced in the next section, might not be the one you expect. The state-variable model given by MATLAB is $\dot{x} = -2x + 2u$, $z = -3.5x + 5u$. These values correspond to equation (1) being written as

$$Z(s) = \frac{5s + 3}{s + 2}U(s) = (2.5s + 1.5)\left[\frac{2U(s)}{s + 2}\right]$$

and defining x as the term within the square brackets; that is,

$$X(s) = \frac{2U(s)}{s + 2}$$

The order of the system, and therefore the number of state variables required, can be found by examining the denominator of the transfer function. If the denominator polynomial is of order n, then n state variables are required. Frequently a convenient choice is to select the state variables as the outputs of integrations ($1/s$), as was done in Example 5.1.5.

EXAMPLE 5.1.6 | ## Numerator Dynamics in a Second-Order System

■ Problem

Obtain a state-variable model for

$$\frac{X(s)}{U(s)} = \frac{4s + 7}{5s^2 + 4s + 7} \tag{1}$$

Relate the initial conditions for the state variables to the given initial conditions $x(0)$ and $\dot{x}(0)$.

■ Solution

Divide by $5s^2$ to obtain a 1 in the denominator.

$$\frac{X(s)}{U(s)} = \frac{\frac{7}{5}s^{-2} + \frac{4}{5}s^{-1}}{1 + \frac{4}{5}s^{-1} + \frac{7}{5}s^{-2}}$$

Use the 1 in the denominator to solve for $X(s)$.

$$X(s) = \left(\frac{7}{5}s^{-2} + \frac{4}{5}s^{-1}\right)U(s) - \left(\frac{4}{5}s^{-1} + \frac{7}{5}s^{-2}\right)X(s)$$

$$= \frac{1}{s}\left\{-\frac{4}{5}X(s) + \frac{4}{5}U(s) + \frac{1}{s}\left[\frac{7}{5}U(s) - \frac{7}{5}X(s)\right]\right\} \tag{2}$$

This equation shows that $X(s)$ is the output of an integration. Thus x can be chosen as a state-variable x_1. Thus,

$$X_1(s) = X(s)$$

The term within square brackets in (2) is the input to an integration, and thus the second state variable can be chosen as

$$X_2(s) = \frac{1}{s}\left[\frac{7}{5}U(s) - \frac{7}{5}X(s)\right] = \frac{1}{s}\left[\frac{7}{5}U(s) - \frac{7}{5}X_1(s)\right] \tag{3}$$

Then from equation (2)

$$X_1(s) = \frac{1}{s}\left[-\frac{4}{5}X_1(s) + \frac{4}{5}U(s) + X_2(s)\right] \tag{4}$$

The state equations are found from (3) and (4).

$$\dot{x}_1 = -\frac{4}{5}x_1 + x_2 + \frac{4}{5}u \tag{5}$$

$$\dot{x}_2 = -\frac{7}{5}x_1 + \frac{7}{5}u \tag{6}$$

and the output equation is $x = x_1$. The matrices of the standard form are

$$\mathbf{A} = \begin{bmatrix} -\frac{4}{5} & 1 \\ -\frac{7}{5} & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{4}{5} \\ \frac{7}{5} \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0] \qquad \mathbf{D} = [0]$$

Note that the state variables obtained by this technique do not always have straightforward physical interpretations. If the model $m\ddot{x} + c\dot{x} + kx = c\dot{u} + ku$ represents a mass-spring-damper system with a displacement input u with $m = 5, c = 4, k = 7$, the variable x_2 is the integral of the spring force $k(u - x)$, divided by the mass m. Thus, x_2 is the acceleration of the mass due to this force. Sometimes convenient physical interpretations of the state variables are sacrificed to obtain special forms of the state equations that are useful for analytical purposes.

Table 5.1.1 A state-variable form for numerator dynamics.

Transfer function model:

$$\frac{Y(s)}{U(s)} = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

State-variable model:

$$\dot{x}_1 = \gamma_{n-1} u - \alpha_{n-1} x_1 + x_2$$
$$\dot{x}_2 = \gamma_{n-2} u - \alpha_{n-2} x_1 + x_3$$
$$\vdots$$
$$\dot{x}_j = \gamma_{n-j} u - \alpha_{n-j} x_1 + x_{j+1}, \qquad j = 1, 2, \ldots, n-1$$
$$\vdots$$
$$\dot{x}_n = \gamma_0 u - \alpha_0 x_1$$
$$y = \beta_n u + x_1$$

where

$$\gamma_i = \beta_i - \alpha_i \beta_n$$

Usual case:

If $u(0) = \dot{u}(0) = \cdots = 0$, then

$$x_i(0) = y^{(i-1)}(0) + \alpha_{n-1} y^{(i-2)}(0) + \cdots + \alpha_{n-i+1} y(0)$$
$$i = 1, 2, \ldots, n$$

where

$$y^{(i)}(0) = \frac{d^i y}{dt^i}\bigg|_{t=0}$$

Using equations (5) and (6), we need to relate the values of $x_1(0)$ and $x_2(0)$ to $x(0)$ and $\dot{x}(0)$. Because x_1 was defined to be $x_1 = x$, we see that $x_1(0) = x(0)$. To find $x_2(0)$, we solve the first state equation, equation (5), for x_2.

$$x_2 = \dot{x}_1 + \frac{4}{5}(x_1 - u)$$

This gives

$$x_2(0) = \dot{x}_1(0) + \frac{4}{5}[x_1(0) - u(0)] = \dot{x}(0) + \frac{4}{5}[x(0) - u(0)]$$

Thus if $u(0) = 0$,

$$x_2(0) = \dot{x}(0) + \frac{4}{5}x(0)$$

The method of the previous example can be extended to the general case where the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \tag{5.1.11}$$

The results are shown in Table 5.1.1. The details of the derivation are given in [Palm, 1986].

TRANSFER-FUNCTION VERSUS STATE-VARIABLE MODELS

The decision whether to use a reduced-form model (which is equivalent to a transfer-function model) or a state-variable model depends on many factors, including personal preference. In fact, for many applications both models are equally effective and equally easy to use. Application of basic physical principles sometimes directly results in a

state-variable model. An example is the following two-inertia fluid-clutch model derived in Example 4.5.7 in Chapter 4.

$$I_d\dot{\omega}_d = T_d - c(\omega_d - \omega_1)$$

$$I_1\dot{\omega}_1 = -T_1 + c(\omega_d - \omega_1)$$

The state and input vectors are

$$\mathbf{x} = \begin{bmatrix} \omega_d \\ \omega_1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} T_d \\ T_1 \end{bmatrix}$$

The system and input matrices are

$$\mathbf{A} = \begin{bmatrix} -\dfrac{c}{I_d} & \dfrac{c}{I_d} \\ \dfrac{c}{I_1} & -\dfrac{c}{I_1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \dfrac{1}{I_d} & 0 \\ 0 & -\dfrac{1}{I_1} \end{bmatrix}$$

For example, this form of the model is easier to use if you need to obtain only numerical values or a plot of the step response, because you can directly use the MATLAB function `step(A,B,C,D)`, to be introduced in Section 5.2. However, if you need to obtain the step response as a function, it might be easier to convert the model to transfer function form and then use the Laplace transform to obtain the desired function. To obtain the transfer function from the state-variable model, you may use the MATLAB function `tf(sys)`, as shown in Section 5.2.

The MATLAB functions cited require that all the model parameters have specified numerical values. If, however, you need to examine the effects of a system parameter, say the damping coefficient c in the clutch model, then it is perhaps preferable to convert the model to transfer function form. In this form, you can examine the effect of c on system response by examining numerator dynamics and the characteristic equation. You can also use the initial- and final-value theorems to investigate the response.

5.2 STATE-VARIABLE METHODS WITH MATLAB

The MATLAB `step`, `impulse`, and `lsim` functions, treated in Section 3.9, can also be used with state-variable models. However, the `initial` function, which computes the free response, can be used only with a state-variable model. MATLAB also provides functions for converting models between the state-variable and transfer function forms.

Recall that to create an LTI object from the reduced form

$$5\ddot{x} + 7\dot{x} + 4x = f(t) \tag{5.2.1}$$

or the transfer function form

$$\frac{X(s)}{F(s)} = \frac{1}{5s^2 + 7s + 4} \tag{5.2.2}$$

you use the `tf(num,den)` function by typing:

```
≫sys1 = tf(1, [5, 7, 4]);
```

The result, `sys1`, is the LTI object that describes the system in the transfer function form.

The LTI object `sys2` in transfer function form for the equation

$$8\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 4\frac{d^2f}{dt^2} + 3\frac{df}{dt} + 5f \tag{5.2.3}$$

is created by typing

```
≫sys2 = tf([4, 3, 5],[8, -3, 5, 6]);
```

LTI OBJECTS AND THE `ss(A,B,C,D)` FUNCTION

To create an LTI object from a state model, you use the `ss(A,B,C,D)` function, where `ss` stands for *state space*. The matrix arguments of the function are the matrices in the following standard form of a state model:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{5.2.4}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{5.2.5}$$

where **x** is the vector of state variables, **u** is the vector of input functions, and **y** is the vector of output variables. For example, to create an LTI object in state-model form for the system described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2$$

where x_1 is the desired output, you type

```
≫A = [0, 1; -4/5, -7/5];
≫B = [0; 1/5];
≫C = [1, 0];
≫D = 0;
≫sys3 = ss(A,B,C,D);
```

THE `ss(sys)` AND `ssdata(sys)` FUNCTIONS

An LTI object defined using the `tf` function can be used to obtain an equivalent state model description of the system. To create a state model for the system described by the LTI object `sys1` created previously in transfer function form, you type `ss(sys1)`. You will then see the resulting **A**, **B**, **C**, and **D** matrices on the screen. To extract and save the matrices as `A1`, `B1`, `C1`, and `D1` (to avoid overwriting the matrices from the second example here), use the `ssdata` function as follows.

```
≫[A1, B1, C1, D1] = ssdata(sys1);
```

The results are

$$\mathbf{A1} = \begin{bmatrix} -1.4 & -0.8 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{B1} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$\mathbf{C1} = [0 \quad 0.4]$$

$$\mathbf{D1} = [0]$$

which correspond to the state equations:

$$\dot{x}_1 = -1.4x_1 - 0.8x_2 + 0.5f(t)$$

$$\dot{x}_2 = x_1$$

and the output equation $y = 0.4x_2$.

RELATING STATE VARIABLES TO THE ORIGINAL VARIABLES

When using `ssdata` to convert a transfer function form into a state model, note that the output y will be a scalar that is identical to the solution variable of the reduced form; in this case the solution variable of (5.2.1) is the variable x. To interpret the state model, we need to relate its state variables x_1 and x_2 to x. The values of the matrices **C1** and **D1** tell us that the output variable is $y = 0.4x_2$. Because the output y is the same as x, we then see that $x_2 = x/0.4 = 2.5x$. The other state-variable x_1 is related to x_2 by the second state equation $\dot{x}_2 = x_1$. Thus, $x_1 = 2.5\dot{x}$.

THE `tfdata` FUNCTION

To create a transfer function description of the system `sys3`, previously created from the state model, you type `tfsys3 = tf(sys3);`. To extract and save the coefficients of the transfer function, use the `tfdata` function as follows.

≫`[num, den] = tfdata(tfsys3, 'v');`

The optional parameter `'v'` tells MATLAB to return the coefficients as vectors if there is only one transfer function; otherwise, they are returned as cell arrays.

For this example, the vectors returned are `num = [0, 0, 0.2]` and `den = [1, 1.4, 0.8]`. This corresponds to the transfer function

$$\frac{X(s)}{F(s)} = \frac{0.2}{s^2 + 1.4s + 0.8} = \frac{1}{5s^2 + 7s + 4}$$

which is the correct transfer function, as seen from (5.2.2).

EXAMPLE 5.2.1

Transfer Functions of a Two-Mass System

■ Problem
Obtain the transfer functions $X_1(s)/F(s)$ and $X_2(s)/F(s)$ of the state-variable model obtained in Example 5.1.3. The matrices and state vector of the model are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\frac{12}{5} & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & \frac{8}{3} & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

■ **Solution**

Because we want the transfer functions for x_1 and x_2 (which are the same as z_1 and z_3), we must define the **C** and **D** matrices to indicate that z_1 and z_3 are the output variables y_1 and y_2. Thus,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The MATLAB program is as follows.

```
A = [0, 1, 0, 0; -1, -12/5, 4/5, 8/5;...
     0, 0, 0, 1; 4/3, 8/3, -4/3, -8/3];
B = [0; 0; 0; 1/3];
C = [1, 0, 0, 0; 0, 0, 1, 0]; D = [0; 0]
sys4 = ss(A, B, C, D);
tfsys4 = tf(sys4)
```

The results displayed on the screen are labeled #1 and #2. These correspond to the first and second transfer functions in order. The answers are

$$\frac{X_1(s)}{F(s)} = \frac{0.5333s + 0.2667}{s^4 + 5.067s^3 + 4.467s^2 + 1.6s + 0.2667}$$

$$\frac{X_2(s)}{F(s)} = \frac{0.3333s^2 + 0.8s + 0.3333}{s^4 + 5.067s^3 + 4.467s^2 + 1.6s + 0.2667}$$

Table 5.2.1 summarizes these functions.

LINEAR ODE SOLVERS

The Control System Toolbox provides several solvers for linear models. These solvers are categorized by the type of input function they can accept: zero input, impulse input, step input, and a general input function.

Table 5.2.1 LTI object functions.

Command	Description
sys = ss(A, B, C, D)	Creates an LTI object in state-space form, where the matrices A, B, C, and D correspond to those in the model $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$.
[A, B, C, D] = ssdata(sys)	Extracts the matrices A, B, C, and D of the LTI object sys, corresponding to those in the model $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$.
sys = tf(num,den)	Creates an LTI object in transfer function form, where the vector num is the vector of coefficients of the transfer function numerator, arranged in descending order, and den is the vector of coefficients of the denominator, also arranged in descending order.
sys2=tf(sys1)	Creates the transfer function model sys2 from the state model sys1.
sys1=ss(sys2)	Creates the state model sys1 from the transfer function model sys2.
[num, den] = tfdata(sys, 'v')	Extracts the coefficients of the numerator and denominator of the transfer function model sys. When the optional parameter 'v' is used, if there is only one transfer function, the coefficients are returned as vectors rather than as cell arrays.

THE `initial` FUNCTION

The `initial` function computes and plots the free response of a state model. This is sometimes called the *initial condition response* or the *undriven response* in the MATLAB documentation. The basic syntax is `initial(sys,x0)`, where `sys` is the LTI object in state variable form, and `x0` is the initial condition vector. The time span and number of solution points are chosen automatically.

EXAMPLE 5.2.2

Free Response of the Two-Mass Model

■ Problem

Compute the free response $x_1(t)$ of the state model derived in Example 5.1.3, for $x_1(0) = 5$, $\dot{x}_1(0) = -3$, $x_2(0) = 4$, and $\dot{x}_2(0) = 2$. The model is

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \frac{1}{5}(-5z_1 - 12z_2 + 4z_3 + 8z_4)$$

$$\dot{z}_3 = z_4$$

$$\dot{z}_4 = \frac{1}{3}[4z_1 + 8z_2 - 4z_3 - 8z_4 + f(t)]$$

or

$$\dot{z} = \mathbf{A}z + \mathbf{B}f(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\frac{12}{5} & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & \frac{8}{3} & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

■ Solution

We must first relate the initial conditions given in terms of the original variables to the state variables. From the definition of the state vector $\mathbf{z}$, we see that $z_1(0) = x_1(0) = 5$, $z_2(0) = \dot{x}_1(0) = -3$, $z_3(0) = x_2(0) = 4$, and $z_4(0) = \dot{x}_2(0) = 2$. Next we must define the model in state-variable form. The system `sys4` created in Example 5.2.1 specified two outputs, x_1 and x_2. Because we want to obtain only one output here (x_1), we must create a new state model using the same values for the **A** and **B** matrices, but now using

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{D} = [0]$$

The MATLAB program is as follows.

```
A = [0, 1, 0, 0; -1, -12/5, 4/5, 8/5;...
     0, 0, 0, 1; 4/3, 8/3, -4/3, -8/3];
B = [0; 0; 0; 1/3];
C = [1, 0, 0, 0]; D = [0];
```

```
sys5 = ss(A, B, C, D);
initial(sys5, [5, -3, 4, 2])
```

The plot of $x_1(t)$ will be displayed on the screen.

To specify the final time `tfinal`, use the syntax `initial(sys,x0,tfinal)`. To specify a vector of times of the form `t = (0:dt:tfinal)`, at which to obtain the solution, use the syntax `initial(sys,x0,t)`. When called with left-hand arguments, as `[y, t, x] = initial(sys,x0, ...)`, the function returns the output response `y`, the time vector `t` used for the simulation, and the state vector `x` evaluated at those times. The columns of the matrices `y` and `x` are the outputs and the states, respectively. The number of rows in `y` and `x` equals `length(t)`. No plot is drawn. The syntax `initial(sys1,sys2, ...,x0,t)` plots the free response of multiple LTI systems on a single plot. The time vector `t` is optional. You can specify line color, line style, and marker for each system; for example, `initial(sys1,'r',sys2, 'y--',sys3,'gx',x0)`.

THE `impulse`, `step`, AND `lsim` FUNCTIONS

You may use the `impulse`, `step`, and `lsim` functions with state-variable models the same way they are used with transfer function models. However, when used with state-variable models, there are some additional features available, which we illustrate with the `step` function. When called with left-hand arguments, as `[y, t] = step(sys, ...)`, the function returns the output response `y` and the time vector `t` used for the simulation. No plot is drawn. The array `y` is ($p \times q \times m$), where p is `length(t)`, q is the number of outputs, and m is the number of inputs. To obtain the state vector solution for state-space models, use the syntax `[y, t, x] = step(sys, ...)`.

To use the `lsim` function for nonzero initial conditions with a state-space model, use the syntax `lsim(sys,u,t,x0)`. The initial condition vector `x0` is needed only if the initial conditions are nonzero.

OBTAINING THE CHARACTERISTIC POLYNOMIAL

The MATLAB command `poly` can find the characteristic polynomial that corresponds to a specified state matrix **A**. For example, the matrix **A** given in Example 5.1.2 is, for $m = 1, c = 5, k = 6$,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \tag{5.2.6}$$

The coefficients of its characteristic polynomial are found by typing

```
≫A = [0, 1; -6, -5];
≫poly(A)
```

MATLAB returns the answer `[1, 5, 6]`, which corresponds to the polynomial $s^2 + 5s + 6$. The `roots` function can be used to compute the characteristic roots; for example, `roots(poly(A))`, which gives the roots `[-2, -3]`.

MATLAB provides the `eig` function to compute the characteristic roots without obtaining the characteristic polynomial, when the model is given in the state-variable form. Its syntax is `eig(A)`. (The function's name is an abbreviation of *eigenvalue,*

which is another name for characteristic root.) For example, typing `eig([0, 1; -6, -5])` returns the roots `[-2, -3]`.

5.3 THE MATLAB ode FUNCTIONS

Appendix D on the text website is an introduction to numerical methods for solving differential equations. The algorithms presented there are simplified versions of the ones used by MATLAB and Simulink, but an understanding of these methods will improve your understanding of these two programs. However, it is not necessary to have mastered Appendix D to use the MATLAB `ode` functions for many applications.

MATLAB provides functions called *solvers,* that implement several numerical solution methods. The `ode15s` and `ode45` solvers are sufficient to solve the problems encountered in this text. It is recommended that you try `ode45` first. If the equation proves difficult to solve (as indicated by a lengthy solution time or by a warning or error message), then use `ode15s`.

SOLVER SYNTAX

We begin our coverage with several examples of solving first-order equations. Solution of higher-order equations is covered later in this section. When used to solve the equation $\dot{y} = f(t, y)$, the basic syntax is (using `ode45` as the example):

```
[t,y] = ode45(@ydot, tspan, y0)
```

where `ydot` is the name of the function file whose inputs must be t and y, and whose output must be a *column* vector representing dy/dt; that is, $f(t, y)$. The number of rows in this column vector must equal the order of the equation. The syntax for the other solvers is identical. Use the MATLAB Editor to create and save the file `ydot`.

The vector `tspan` contains the starting and ending values of the independent variable t, and optionally, any intermediate values of t where the solution is desired. For example, if no intermediate values are specified, `tspan` is `[t0, tfinal]`, where `t0` and `tfinal` are the desired starting and ending values of the independent parameter t. As another example, using `tspan = [0, 5, 10]` forces MATLAB to find the solution at $t = 5$. You can solve equations backward in time by specifying `t0` to the greater than `tfinal`.

The parameter `y0` is the initial value $y(t_0)$. The function file must have two input arguments, `t` and `y`, even for equations where $f(t, y)$ is *not* an explicit function of t. You need not use array operations in the function file because the ODE solvers call the file with scalar values for the arguments.

As a first example of using a solver, let us solve an equation whose solution is known in closed form, so that we can make sure we are using the method correctly. We can also assess the performance of the `ode45` solver when applied to find an oscillating solution, which can be difficult to obtain numerically.

EXAMPLE 5.3.1   MATLAB Solution of $\dot{y} = \sin t$

■ **Problem**

Use the `ode45` solver for the problem

$$\dot{y} = \sin t \qquad y(0) = 0$$

for $0 \le t \le 4\pi$. The exact solution is $y(t) = 1 - \cos t$.

■ **Solution**

Create and save the following function file. Name it `sinefn.m`.

```
function ydot = sinefn(t,y)
ydot = sin(t);
```

Use the following script file to compute the solution.

```
[t, y] = ode45(@sinefn, [0, 4*pi], 0);
y_exact = 1 - cos(t);
plot(t,y,'o',t,y_exact),xlabel('t'),ylabel('y(t)'),...
    axis([0 4*pi -0.5 2.5])
```

Figure 5.3.1 shows that the solution generated by `ode45` is correct.
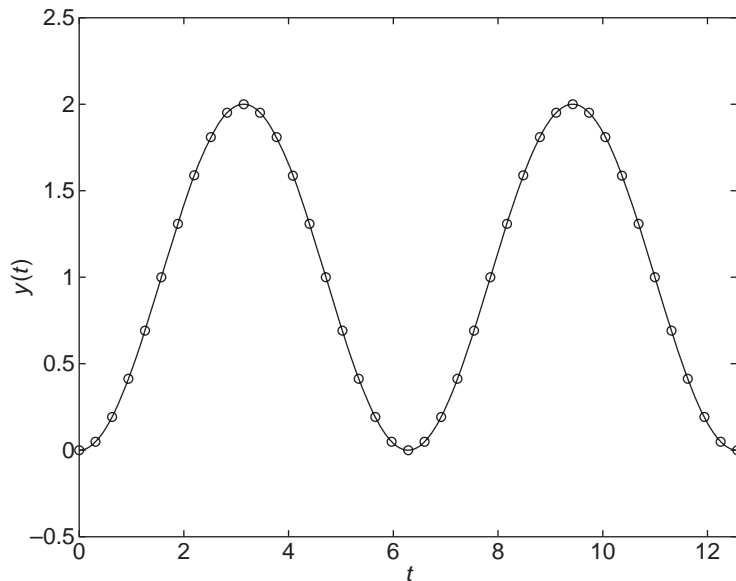
Figure 5.3.1 MATLAB (`ode45`) and exact solutions of $\dot{y} = \sin t$, $y(0) = 0$.

The main application of numerical methods is to solve equations for which a closed-form solution cannot be obtained. The next example shows such an application.

A Rocket-Propelled Sled | **EXAMPLE 5.3.2**

■ **Problem**

A rocket-propelled sled on a track is represented in Figure 5.3.2 as a mass m with an applied force f that represents the rocket thrust. The rocket thrust initially is horizontal, but the engine accidentally pivots during firing and rotates with an angular acceleration of $\ddot{\theta} = \pi/50$ rad/s. Compute the sled's velocity v for $0 \leq t \leq 6$ if $v(0) = 0$. The rocket thrust is 4000 N and the sled mass is 450 kg.

■ **Solution**

The sled's equation of motion is $450\dot{v} = 4000 \cos\theta(t)$. To obtain $\theta(t)$, note that

$$\dot{\theta} = \int_0^t \ddot{\theta} \, dt = \frac{\pi}{50}t$$

Figure 5.3.2 A rocket-propelled sled.
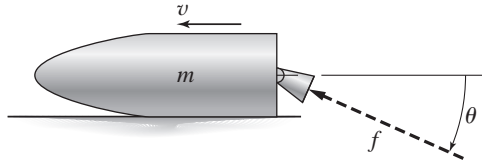
and

$$\theta = \int_0^t \dot{\theta}\, dt = \int_0^t \frac{\pi}{50} t\, dt = \frac{\pi}{100} t^2$$

Thus the equation of motion becomes

$$\dot{v} = \frac{80}{9} \cos\left(\frac{\pi}{100} t^2\right) \tag{1}$$

The solution is formally given by

$$v(t) = \frac{80}{9} \int_0^t \cos\left(\frac{\pi}{100} t^2\right) dt$$

Unfortunately, no closed-form solution is available for the integral, which is called *Fresnel's cosine integral*. The value of the integral has been tabulated numerically, but we will use a MATLAB ODE solver to obtain the solution.

First create the following user-defined function file, which is based on equation (1).

```
function vdot = sled(t,v)
vdot = 80*cos(pi*t^2/100)/9;
```

As a check on our results, we will use the solution for $\theta = 0$ as a comparison. The equation of motion for this case is $\dot{v} = 80/9$, which gives $v(t) = 80t/9$. The following session solves the equation and plots the two solutions, which are shown in Figure 5.3.3.

```
[t,v] = ode45(@sled,[0 6],0);
plot(t,v,t,(80*t/9)),xlabel('t (s)'),...
   ylabel('v (m/s)'),gtext('\theta = 0'),gtext('\theta \neq 0')
```

Figure 5.3.3 Speed response of the sled for $\theta = 0$ and $\theta \neq 0$.
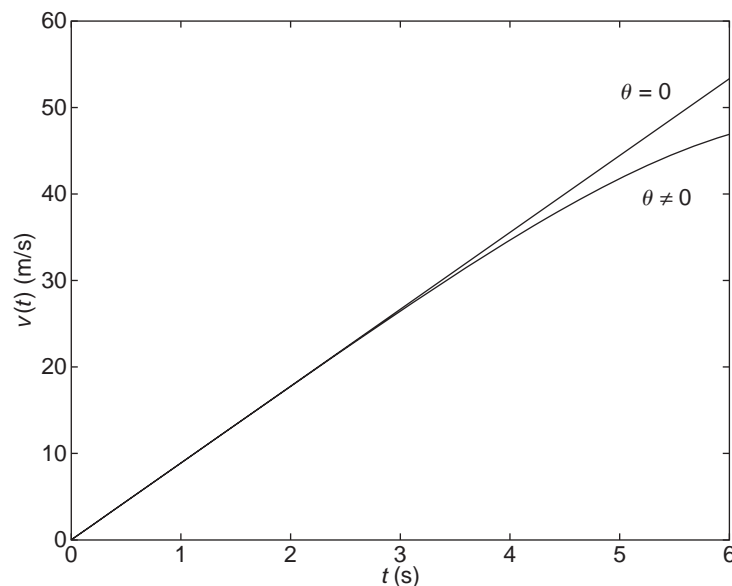
We can make two observations that help us determine whether or not our numerical solution is correct. From the plot we see that the solution for $\theta \neq 0$ is almost identical to the solution for $\theta = 0$, for small values of θ. This is correct because $\cos \theta \approx 1$ for small values of θ. As θ increases, we would expect the velocity to be smaller than the velocity for $\theta = 0$ because the horizontal component of the thrust is smaller. The plot confirms this.

EXTENSION TO HIGHER ORDER EQUATIONS

To use the ODE solvers to solve an equation of order two or greater, you must first write the equation in state-variable form, as a set of first-order equations, and then create a function file that computes the derivatives of the state variables. Consider the second-order equation $5\ddot{y} + 7\dot{y} + 4y = f(t)$. Define the variables, $x_1 = y$ and $x_2 = \dot{y}$. The state-variable form is

$$\dot{x}_1 = x_2 \tag{5.3.1}$$

$$\dot{x}_2 = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2 \tag{5.3.2}$$

Now write a function file that computes the values of $\dot{x}_1$ and $\dot{x}_2$ and stores them in a *column* vector. To do this, we must first have a function specified for $f(t)$. Suppose that $f(t) = \sin t$. Then the required file is

```
function xdot = example1(t,x)
% Computes derivatives of two equations
xdot(1) = x(2);
xdot(2) = (1/5)*(sin(t)-4*x(1)-7*x(2));
xdot = [xdot(1); xdot(2)];
```

Note that xdot(1) represents $\dot{x}_1$, xdot(2) represents $\dot{x}_2$, x(1) represents x_1, and x(2) represents x_2. Once you become familiar with the notation for the state-variable form, you will see that the preceding code could be replaced with the following shorter form.

```
function xdot = example1(t,x)
% Computes derivatives of two equations
xdot = [x(2); (1/5)*(sin(t)-4*x(1)-7*x(2))];
```

Suppose we want to solve (5.3.1) and (5.3.2) for $0 \leq t \leq 6$ with the initial conditions $y(0) = x_1(0) = 3$ and $\dot{y}(0) = x_2(0) = 9$. Then the initial condition for the *vector* x is [3, 9]. To use ode45, you type

```
[t, x] = ode45(@example1, [0, 6], [3, 9]);
```

Each row in the vector x corresponds to a time returned in the column vector t. If you type plot(t,x), you will obtain a plot of both x_1 and x_2 versus t. Note that x is a matrix with two columns; the first column contains the values of x_1 at the various times generated by the solver. The second column contains the values of x_2. Thus, to plot only x_1, type plot(t,x(:,1)).

When solving nonlinear equations, sometimes it is possible to check the numerical results by using an approximation that reduces the equation to a linear one. Example 5.3.3 illustrates such an approach with a second-order equation.

A Nonlinear Pendulum Model

Figure 5.3.4 A pendulum.
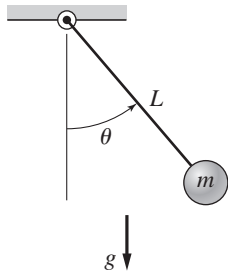
■ Problem

By studying the dynamics of a pendulum like that shown in Figure 5.3.4, we can better understand the dynamics of machines such as a robot arm. The pendulum shown consists of a concentrated mass m attached to a rod whose mass is small compared to m. The rod's length is L. The equation of motion for this pendulum is

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

Suppose that $L = 1$ m and $g = 9.81$ m/s^2. Use MATLAB to solve this equation for $\theta(t)$ for two cases: $\theta(0) = 0.5$ rad, and $\theta(0) = 0.8\pi$ rad. In both cases $\dot{\theta}(0) = 0$. Discuss how to check the accuracy of the results.

■ Solution

If we use the small angle approximation $\sin \approx \theta$, the equation becomes

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{2}$$

which is linear and has the solution:

$$\theta(t) = \theta(0)\cos\sqrt{\frac{g}{L}}t \tag{3}$$

Thus the amplitude of oscillation is $\theta(0)$ and the period is $P = 2\pi\sqrt{L/g} = 2$ s. We can use this information to select a final time, and to check our numerical results.

First rewrite the pendulum equation (1) as two first order equations. To do this, let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Thus

$$\dot{x}_1 = \dot{\theta} = x_2$$
$$\dot{x}_2 = \ddot{\theta} = -\frac{g}{L}\sin x_1 = -9.81\sin x_1$$

The following function file is based on the last two equations. Remember that the output `xdot` must be a *column* vector.

```
function xdot = pendulum(t,x)
xdot = [x(2); -9.81*sin(x(1))];
```

The function file is called as follows. The vectors `ta` and `xa` contain the results for the case where $\theta(0) = 0.5$. The vectors `tb` and `xb` contain the results for $\theta(0) = 0.8\pi$.

```
[ta, xa] = ode45(@pendulum, [0, 5], [0.5, 0]);
[tb, xb] = ode45(@pendulum, [0, 5], [0.8*pi, 0]);
plot(ta,xa(:,1),tb,xb(:,1)),xlabel('Time (s)'),...
   ylabel('Angle (rad)'),gtext('Case 1'),gtext('Case 2')
```

The results are shown in Figure 5.3.5. The amplitude remains constant, as predicted by the small angle analysis, and the period for the case where $\theta(0) = 0.5$ is a little larger than 2 s, the value predicted by the small angle analysis. So we can place some confidence in the numerical procedure. For the case where $\theta(0) = 0.8\pi$, the period of the numerical solution is about 3.3 s. This illustrates an important property of nonlinear differential equations. The free response of a linear equation has the same period for any initial conditions; however, the form of the free response of a nonlinear equation often depends on the particular values of the initial conditions.
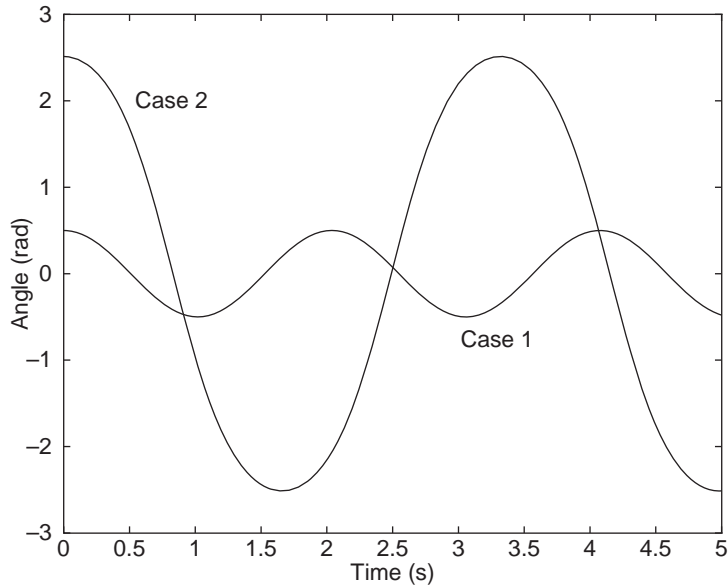
Figure 5.3.5 The pendulum angle as a function of time for two starting positions.

    In the previous example, the values of g and L did not appear in the function `pendulum(t,x)`. Now suppose you want to obtain the pendulum response for different lengths L or different gravitational accelerations g. You could use the `global` command to declare g and L as global variables, or you could pass parameter values through an argument list in the `ode45` function; but starting with MATLAB 7, the preferred method is to use a nested function. Nested functions are discussed in [Palm, 2005] and [Palm, 2009]. The following program shows how this is done.

```
function pendula
g = 9.81; L = 0.75; % First case.
tfinal = 6*pi*sqrt(L/g); % Approximately 3 periods.
[t1, x1] = ode45(@pendulum, [0,tfinal], [0.4, 0]);
%
g = 1.63; L = 2.5; % Second case.
tfinal = 6*pi*sqrt(L/g); % Approximately 3 periods.
[t2, x2] = ode45(@pendulum, [0,tfinal], [0.2, 0]);
plot(t1, x1(:,1), t2, x2(:,1)), ...
    xlabel ('time (s)'), ylabel ('\theta (rad)')
% Nested function.
    function xdot = pendulum(t,x)
    xdot = [x(2);-(g/L)*sin(x(1))];
    end
end
```

MATRIX METHODS

We can use matrix operations to reduce the number of lines to be typed in the derivative function file. For example, consider the mass-spring-damper model $m\ddot{y} + c\dot{y} + ky = f(t)$. This can be put into state-variable form by letting $x_1 = y$ and $x_2 = \dot{y}$. This gives

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m} f(t) - \frac{k}{m}x_1 - \frac{c}{m}x_2$$

This can be written as one matrix equation as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} f(t)$$

In compact form this is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f(t)$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The following function file shows how to use matrix operations. In this example, $m = 1, c = 2, k = 5$, and the applied force f is a constant equal to 10.

```
function xdot = msd(t,x)
% Function file for mass with spring and damping.
% Position is first variable, velocity is second variable.
m = 1; c = 2; k = 5; f = 10;
A = [0, 1;-k/m, -c/m];
B = [0; 1/m];
xdot = A*x+B*f;
```

Note that the output xdot will be a column vector because of the definition of matrix-vector multiplication. The characteristic roots are the roots of $ms^2 + cs + k = s^2 + 2s + 5 = 0$, and are $s = -1 \pm 2j$. The time constant is 1, and the steady state response will thus be reached after $t = 4$. The period of oscillation will be π. Thus if we choose a final time of 5 we will see the entire response. If the initial conditions are $x_1(0) = 0$, $x_2(0) = 0$, the solver is called as follows:

```
[t, x] = ode45(@msd, [0, 5], [0, 0]);
plot(t,x),xlabel('Time (s)'),...
    ylabel('Displacement (m) and Velocity (m/s)'),...
    gtext('Displacement'), gtext('Velocity')
```

The result is shown in Figure 5.3.6.
    Table 5.3.1 summarizes the basic syntax of the ODE solvers.

Figure 5.3.6 Displacement and velocity of the mass as functions of time.
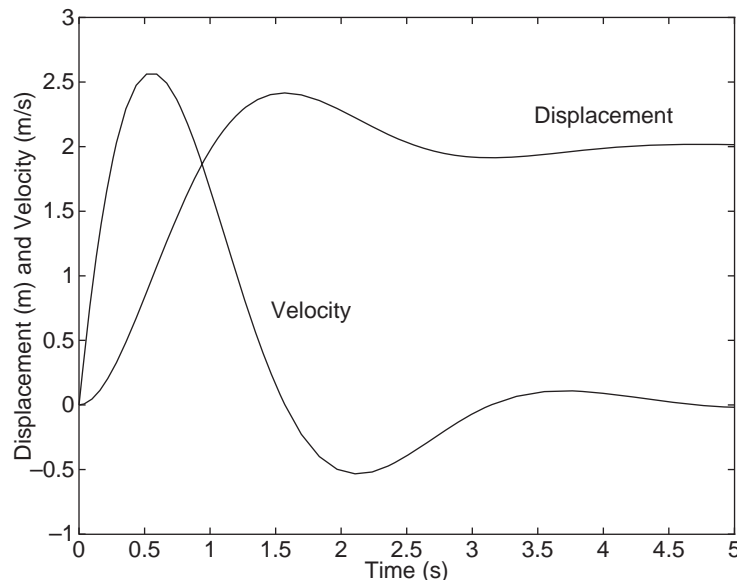
Table 5.3.1 Basic Syntax of ODE solvers

`[t, y] = ode45(@ydot, tspan, y0)`
Solves the vector differential equation $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ specified in the function file `ydot`, whose inputs must be t and $\mathbf{y}$, and whose output must be a *column* vector representing $d\mathbf{y}/dt$; that is, $\mathbf{f}(t, \mathbf{y})$. The number of rows in this column vector must equal the order of the equation. The vector `tspan` contains the starting and ending values of the independent variable t, and optionally, any intermediate values of t where the solution is desired. The vector `y0` contains the initial values $\mathbf{y}(t_0)$. The function file must have the two input arguments, `t` and `y`, even for equations where $\mathbf{f}(t, \mathbf{y})$ is not an explicit function of t.
The basic syntax for the other solvers is identical to that of `ode45`.

5.4 SIMULINK AND LINEAR MODELS

Simulink is built on top of MATLAB, so you must have MATLAB to use Simulink. It is included in the Student Edition of MATLAB and is also available separately from The MathWorks, Inc. It provides a graphical user interface that uses various types of elements called *blocks* to create a simulation of a dynamic system; that is, a system that can be modeled with differential or difference equations whose independent variable is time. For example, one block type is a multiplier, another performs a sum, and another is an integrator. Its graphical interface enables you to position the blocks, resize them, label them, specify block parameters, and interconnect the blocks to describe complicated systems for simulation.

Type `simulink` in the MATLAB Command window to start Simulink. The Simulink Library Browser window opens. See Figure 5.4.1. To create a new model, click on the icon that resembles a clean sheet of paper, or select **New** from the **File** menu in the Browser. A new **Untitled** window opens for you to create the model. To select a block from the Library Browser, double-click on the appropriate library category and a list of blocks within that category then appears, as shown in Figure 5.4.1. Figure 5.4.1 shows the result of double-clicking on the Continuous library, then clicking on the Integrator block. Click on the block name or icon, hold the mouse button down, drag the block to the new model window, and release the button. Note that when you click on the block name in the Library Browser, a brief description of the block's function appears at the bottom of the Browser. You can access help for that block by right-clicking on its name or icon, and selecting **Help** from the drop-down menu.

Simulink model files have the extension `.mdl`. Use the **File** menu in the model window to Open, Close, and Save model files. To print the block diagram of the model, select **Print** on the **File** menu. Use the **Edit** menu to copy, cut and paste blocks. You can also use the mouse for these operations. For example, to delete a block, click on it and press the **Delete** key.

Getting started with Simulink is best done through examples, which we now present.

SIMULATION DIAGRAMS

You construct Simulink models by constructing a diagram that shows the elements of the problem to be solved. Such diagrams are called *simulation diagrams*. Consider the equation $\dot{y} = 10 f(t)$. Its solution can be represented symbolically as

$$y(t) = \int 10 f(t) \, dt$$

which can be thought of as two steps, using an intermediate variable x:

$$x(t) = 10 f(t) \qquad \text{and} \qquad y(t) = \int x(t) \, dt$$

Figure 5.4.1 The Simulink Library Browser.
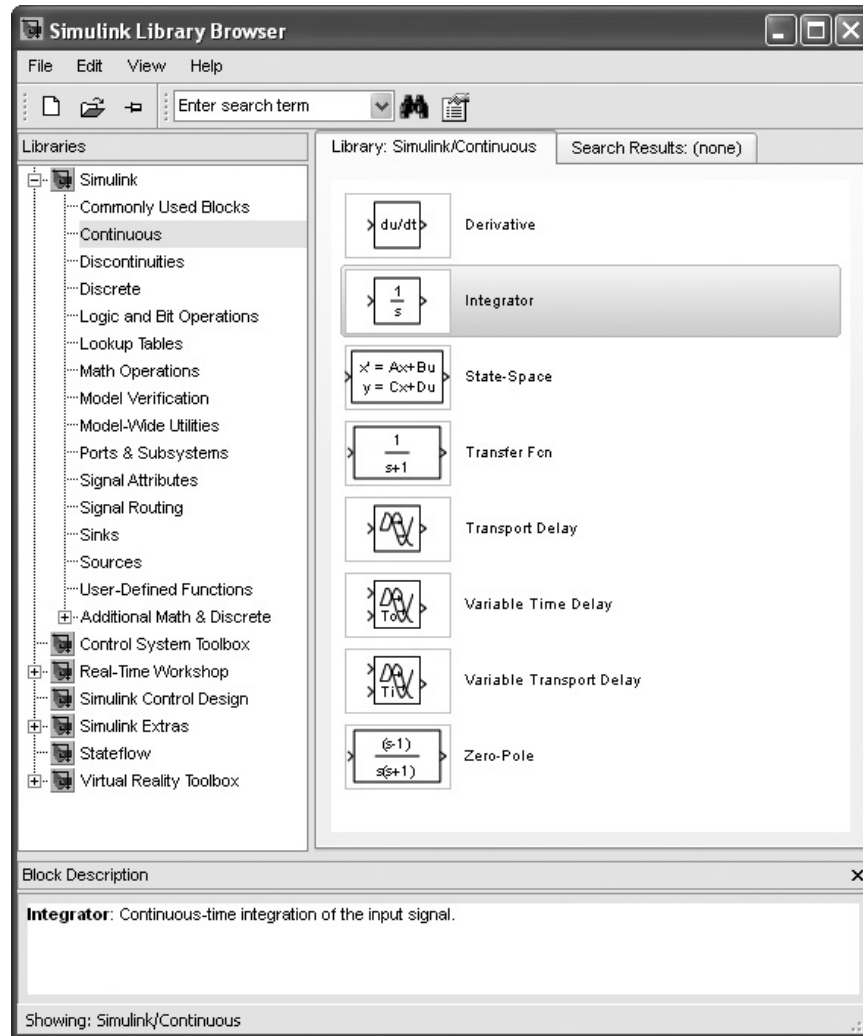
Figure 5.4.2 Simulation diagrams for $\dot{y} = 10f(t)$.
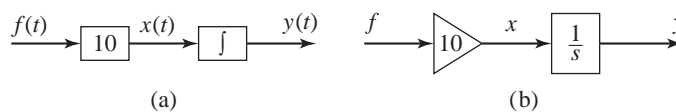
This solution can be represented graphically by the simulation diagram shown in Figure 5.4.2a. The arrows represent the variables y, x, and f. The blocks represent cause-and-effect processes. Thus, the block containing the number 10 represents the process $x(t) = 10f(t)$, where $f(t)$ is the cause (the *input*) and $x(t)$ represents the effect (the *output*). This type of block is called a *multiplier* or *gain* block.

The block containing the integral sign $\int$ represents the integration process $y(t) = \int x(t)\,dt$, where $x(t)$ is the cause (the *input*) and $y(t)$ represents the effect (the *output*). This type of block is called an *integrator* block.

There is some variation in the notation and symbols used in simulation diagrams. Part (b) of Figure 5.4.2 shows one variation. Instead of being represented by a box, the multiplication process is now represented by a triangle like that used to represent an electrical amplifier, hence the name *gain* block.

(a)   (b)
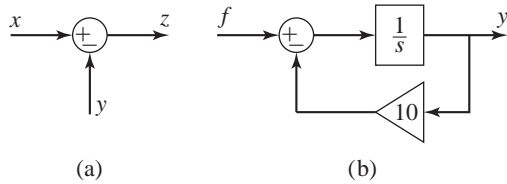
In addition, the integration symbol in the integrator block has been replaced by the symbol $1/s$, which represents integration in Laplace transform notation. Thus the equation $\dot{y} = 10f(t)$ is represented by $sy = 10f$, and the solution is represented as

$$y = \frac{10f}{s}$$

or as the two equations

$$x = 10f \qquad \text{and} \qquad y = \frac{1}{s}x$$

Another element used in simulation diagrams is the *summer* that, despite its name, is used to subtract as well as to sum variables. Its symbol is shown in Figure 5.4.3a. The symbol represents the equation $z = x - y$. Note that a plus or minus sign is required for each input arrow.

The summer symbol can be used to represent the equation $\dot{y} = f(t) - 10y$, which can be expressed as

$$y(t) = \int [f(t) - 10y]\, dt$$

or as

$$y = \frac{1}{s}(f - 10y)$$

You should study the simulation diagram shown in part (b) of Figure 5.4.3 to confirm that it represents this equation.

Figure 5.4.2b forms the basis for developing a Simulink model to solve the equation $\dot{y} = f(t)$.

Simulink Solution of $\dot{y} = 10 \sin t$ | **EXAMPLE 5.4.1**

■ **Problem**

Let us use Simulink to solve the following problem for $0 \le t \le 13$.

$$\frac{dy}{dt} = 10 \sin t \qquad y(0) = 0$$

The exact solution is $y(t) = 10(1 - \cos t)$.

■ **Solution**

To construct the simulation, do the following steps. Refer to Figure 5.4.4.

1. Start Simulink and open a new model window as described previously.
2. Select and place in the new window the Sine Wave block from the Sources category. Double-click on it to open the Block Parameters window, and make sure the Amplitude is set to 1, the Bias to 0, the Frequency to 1, the Phase to 0, and the Sample time to 0. Then click **OK.**
3. Select and place the Gain block from the Math category, double-click on it, and set the Gain Value to 10 in the Block Parameters window. Then click **OK.** Note that the value 10

Figure 5.4.4 Simulink model for $\dot{y} = 10 \sin t$.

Sine Wave   Gain   Integrator   Scope
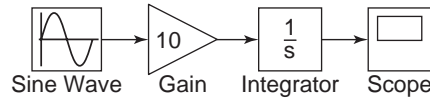
then appears in the triangle. To make the number more visible, click on the block, and drag one of the corners to expand the block so that all the text is visible.

4. Select and place the Integrator block from the Continuous category, double-click on it to obtain the Block Parameters window, and set the Initial Condition to 0 [this is because $y(0) = 0$]. Then click **OK.**

5. Select and place the Scope block from the Sinks category.

6. Once the blocks have been placed as shown in Figure 5.5.4, connect the input port on each block to the outport port on the preceding block. To do this, move the cursor to an input port or an output port; the cursor will change to a cross. Hold the mouse button down, and drag the cursor to a port on another block. When you release the mouse button, Simulink will connect them with an arrow pointing at the input port. Your model should now look like that shown in Figure 5.5.4.

7. Click on the **Simulation** menu, and click the **Configuration Parameters** item. Click on the Solver tab, and enter 13 for the Stop time. Make sure the Start time is 0. Then click **OK.**

8. Run the simulation by clicking on the **Simulation** menu, and then clicking the **Start** item. You can also start the simulation by clicking on the **Start** icon on the toolbar (this is the black triangle).

9. You will hear a bell sound when the simulation is finished. Then double-click on the Scope block and then click on the binoculars icon in the Scope display to enable autoscaling. You should see an oscillating curve with an amplitude of 10 and a period of 2π. The independent variable in the Scope block is time t; the input to the block is the dependent variable y. This completes the simulation.

To have Simulink automatically connect two blocks, select the source block, hold down the Ctrl key, and left-click on the destination block. Simulink provides easy ways to connect multiple blocks and lines; see the help for more information.

Note that blocks have a Block Parameters window that opens when you double-click on the block. This window contains several items, the number and nature of which depend on the specific type of block. In general, you can use the default values of these parameters, except where we have explicitly indicated that they should be changed. You can always click on Help within the Block Parameters window to obtain more information.

When you click Apply in the Block Parameters window, any parameter changes immediately take effect and the window remains open. If you click Close, the changes take effect and the window closes.

Note that most blocks have default labels. You can edit text associated with a block by clicking on the text and making the changes. You can save the Simulink model as an `.mdl` file by selecting **Save** from the **File** menu in Simulink. The model file can then be reloaded at a later time. You can also print the diagram by selecting **Print** on the **File** menu.

The Scope block is useful for examining the solution, but if you want to obtain a labeled and printed plot you can use the To Workspace block, which is described in the next example.

EXAMPLE 5.4.2

Exporting to the MATLAB Workspace

■ Problem
We now demonstrate how to export the results of the simulation to the MATLAB workspace, where they can be plotted or analyzed with any of the MATLAB functions.

■ Solution
Modify the Simulink model constructed in Example 5.4.1 as follows. Refer to Figure 5.4.5.

1. Delete the arrow connecting the Scope block by clicking on it and pressing the Delete key. Delete the Scope block in the same way.
2. Select and place the To Workspace block from the Sinks category and the Clock block from the Sources category.
3. Select and place the Mux block from the Signal Routing category, double-click on it, and set the Number of inputs to 2. Click **OK.** (The name "Mux" is an abbreviation for "multiplexer," which is an electrical device for transmitting several signals.)
4. Connect the top input port of the Mux block to the output port of the Integrator block. Then use the same technique to connect the bottom input port of the Mux block to the outport port of the Clock block. Your model should now look like that shown in Figure 5.4.5.
5. Double-click on the To Workspace block. You can specify any variable name you want as the output; the default is `simout`. Change its name to `y`. The output variable `y` will have as many rows as there are simulation time steps, and as many columns as there are inputs to the block. The second column in our simulation will be time, because of the way we have connected the Clock to the second input port of the Mux. Specify the Save Format as Array. Use the default values for the other parameters (these should be `inf`, 1, and `-1` for Maximum number of rows, Decimation, and Sample Time, respectively). Click on **OK.**
6. After running the simulation, you can use the MATLAB plotting commands from the Command window to plot the columns of `y` (or `simout` in general). To plot $y(t)$, type in the MATLAB Command window:

   ```
   ≫plot(y(:,2),y(:,1)),xlabel('t'),ylabel('y')
   ```
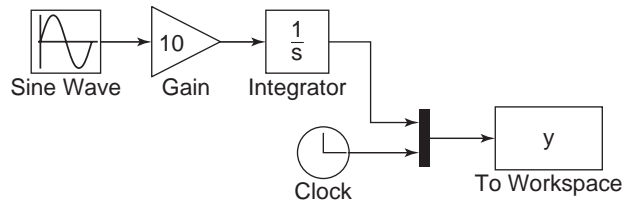

Figure 5.4.5 Simulink model using the Clock and To Workspace block.

Simulink can be configured to put the time variable `tout` into the MATLAB workspace automatically when using the To Workspace block. This is done with the Data Import/Export menu item under Configuration parameters on the Simulation menu. The alternative is to use the Clock block to put `tout` into the workspace. The Clock block has one parameter, Decimation. If this parameter is set to 1, the Clock will output every time step; if set to 10, the Clock will output every 10 time steps, and so on.

EXAMPLE 5.4.3

Simulink Model for $\dot{y} = -10y + f(t)$

■ Problem
Construct a Simulink model to solve

$$\dot{y} = -10y + f(t) \qquad y(0) = 1$$

where $f(t) = 2\sin 4t$, for $0 \leq t \leq 3$.

Figure 5.4.6 Simulink model for $\dot{y} = -10y + f(t)$.

Sine Wave   Integrator   Scope

10
Gain
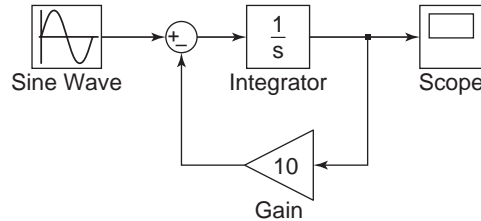
■ **Solution**

To construct the simulation, do the following steps.

1. You can use the model shown in Figure 5.4.4 by rearranging the blocks as shown in Figure 5.4.6. You will need to add a Sum block.
2. Select the Sum block from the Math operations library and place it as shown in the simulation diagram. Its default setting adds two input signals. To change this, double-click on the block, and in the List of Signs window, type | + −. The signs are ordered counterclockwise from the top. The symbol | is a spacer indicating here that the top port is to be empty.
3. To reverse the direction of the gain block, right-click on the block, select **Format** from the pop-up menu, and select **Flip Block.**
4. When you connect the negative input port of the Sum block to the output port of the Gain block, Simulink will attempt to draw the shortest line. To obtain the more standard appearance shown in Figure 5.4.6, first extend the line vertically down from the Sum input port. Release the mouse button and then click on the end of the line and attach it to the Gain block. The result will be a line with a right angle. Do the same to connect the input of the Gain to the arrow connecting the Integrator and the Scope. A small dot appears to indicate that the lines have been successfully connected. This point is called a *takeoff point* because it takes the value of the variable represented by the arrow (here, the variable y) and makes that value available to another block.
5. Select **Configuration Parameters** from the **Simulation** menu, and set the Stop time to 3. Then click **OK.**
6. Run the simulation as before and observe the results in the Scope.

SIMULATING STATE VARIABLE MODELS

State variable models, unlike transfer function models, can have more than one input and more than one output. Simulink has the State-Space block that represents the linear state variable model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$. The vector $\mathbf{u}$ represents the inputs, and the vector $\mathbf{y}$ represents the outputs. Thus when connecting inputs to the State-Space block, care must be taken to connect them in the proper order. Similar care must be taken when connecting the block's outputs to another block. The following example illustrates how this is done.

EXAMPLE 5.4.4   Simulink Model of a Two-Mass System

■ **Problem**

The state-variable model of the two-mass system discussed in Example 5.1.3 is

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}f(t)$$
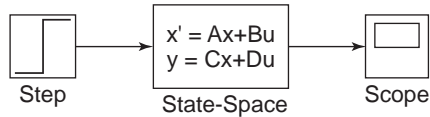
Figure 5.4.7 Simulink model containing the State-Space block and the Step block.

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\frac{12}{5} & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & \frac{8}{3} & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

Develop a Simulink model to plot the unit-step response of the variables x_1 and x_2 with the initial conditions $x_1(0) = 0.2$, $\dot{x}_1(0) = 0$, $x_2(0) = 0.5$, and $\dot{x}_2(0) = 0$.

■ **Solution**

First select appropriate values for the matrices in the output equation $\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}f(t)$. Since we want to plot x_1 and x_2, which are z_1 and z_3, we choose $\mathbf{C}$ and $\mathbf{D}$ as follows.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To create this simulation, obtain a new model window. Then do the following to create the model shown in Figure 5.4.7.

1. Select and place in the new window the Step block from the sources category. Double-click on it to obtain the Block Parameters window, and set the Step time to 0, the Initial and Final values to 0 and 1, and the Sample time to 0. Click **OK.**
2. Select and place the State-Space block. Double-click on it, and enter [0, 1, 0, 0; -1, -12/5, 4/5, 8/5; 0, 0, 0, 1; 4/3, 8/3, -4/3, -8/3] for **A**, [0; 0; 0; 1/3] for **B**, [1, 0, 0, 0; 0, 0, 1, 0] for **C**, and [0; 0] for **D**. Then enter [0.2; 0; 0.5; 0] for the initial conditions. Click **OK.** Note that the dimension of the matrix **B** tells Simulink that there is one input. The dimensions of the matrices **C** and **D** tell Simulink that there are two outputs.
3. Select and place the Scope block.
4. Once the blocks have been placed, connect the input port on each block to the outport port on the preceding block as shown in the figure.
5. Experiment with different values of the Stop time until the Scope shows that the steady-state response has been reached. For this application, a Stop time of 25 is satisfactory. The plots of both x_1 and x_2 will appear in the Scope.

5.5 SIMULINK AND NONLINEAR MODELS

Unlike linear models, closed-form solutions are not available for most nonlinear differential equations, and we must therefore solve such equations numerically. *Piecewise-linear* models are actually nonlinear, although they may appear to be linear. They are

composed of linear models that take effect when certain conditions are satisfied. The effect of switching back and forth between these linear models makes the overall model nonlinear. An example of such a model is a mass attached to a spring and sliding on a horizontal surface with Coulomb friction. The model is

$$m\ddot{x} + kx = f(t) - \mu mg \quad \text{if } \dot{x} \geq 0$$
$$m\ddot{x} + kx = f(t) + \mu mg \quad \text{if } \dot{x} < 0$$

These two linear equations can be expressed as the single, nonlinear equation

$$m\ddot{x} + kx = f(t) - \mu mg \, \text{sign}(\dot{x}) \qquad \text{where} \qquad \text{sign}(\dot{x}) = \begin{cases} +1 & \text{if } \dot{x} \geq 0 \\ -1 & \text{if } \dot{x} < 0 \end{cases}$$

Solution of linear or nonlinear models that contain piecewise-linear functions is very tedious to program. However, Simulink has built-in blocks that represent many of the commonly-found functions such as Coulomb friction. Therefore Simulink is especially useful for such applications.

EXAMPLE 5.5.1   Simulink Model of a Rocket-Propelled Sled

■ **Problem**

A rocket-propelled sled on a track is represented in Figure 5.5.1 as a mass m with an applied force f that represents the rocket thrust. The rocket thrust initially is horizontal, but the engine accidentally pivots during firing and rotates with an angular acceleration of $\ddot{\theta} = \pi/50$ rad/s. Compute the sled's velocity v for $0 \leq t \leq 10$ if $v(0) = 0$. The rocket thrust is 4000 N and the sled mass is 450 kg.

The sled's equation of motion was derived in Example 5.3.2 and is

$$\dot{v} = \frac{80}{9} \cos\left(\frac{\pi}{100}t^2\right) \tag{1}$$

The solution is formally given by

$$v(t) = \frac{80}{9} \int_0^t \cos\left(\frac{\pi}{100}t^2\right) dt$$

Unfortunately, no closed-form solution is available for the integral, which is called *Fresnel's cosine integral*. The value of the integral has been tabulated numerically, but we will use Simulink to obtain the solution.

a.   Create a Simulink model to solve this problem for $0 \leq t \leq 10$ s.
b.   Now suppose that the engine angle is limited by a mechanical stop to 60°, which is $60\pi/180$ rad. Create a Simulink model to solve the problem.

■ **Solution**

a.   There are several ways to create the input function $\theta = (\pi/100)t^2$. Here we note that $\ddot{\theta} = \pi/50$ rad/s and that

$$\dot{\theta} = \int_0^t \ddot{\theta} \, dt = \frac{\pi}{50}t$$

Figure 5.5.1 A rocket-propelled sled.
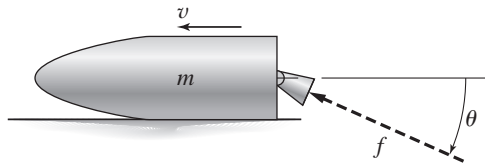
Figure 5.5.2 Simulation model for $v = (80/9)\cos(\pi t^2/100)$.
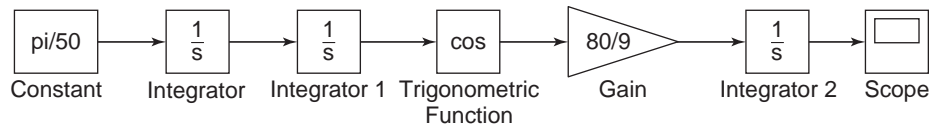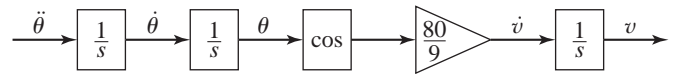
Figure 5.5.3 Simulink model for $v = (80/9)\cos(\pi t^2/100)$.
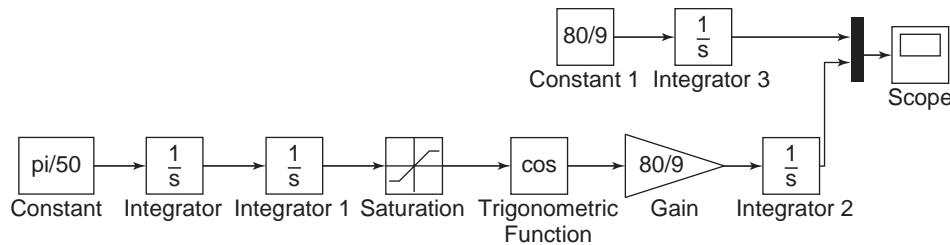
Figure 5.5.4 Simulink model for $v = (80/9)\cos(\pi t^2/100)$ with a Saturation block.

and

$$\theta = \int_0^t \dot{\theta}\, dt = \frac{\pi}{100}t^2$$

Thus we can create $\theta(t)$ by integrating the constant $\ddot{\theta} = \pi/50$ twice. The simulation diagram is shown in Figure 5.5.2. This diagram is used to create the corresponding Simulink model shown in Figure 5.5.3.

There are two new blocks in this model. The Constant block is in the Sources library. After placing it, double click on it and type pi/50 in its Constant Value window.

The Trigonometric function block is in the Math Operations library. After placing it, double click on it and select cos in its Function window.

Set the Stop Time to 10, run the simulation, and examine the results in the Scope.

b. Modify the model in Figure 5.5.3 as follows to obtain the model shown in Figure 5.5.4. We use the Saturation block in the Discontinuities library to limit the range of θ to $60\pi/180$ rad. After placing the block as shown in Figure 5.5.4, double-click on it and type 60*pi/180 in its Upper Limit window. Then type 0 or any negative value in its Lower Limit window.

Enter and connect the remaining elements as shown, and run the simulation. The upper Constant block and Integrator block are used to generate the solution when the engine angle is $\theta = 0$, as a check on our results. [The equation of motion for $\theta = 0$ is $\dot{v} = 80/9$, which gives $v(t) = 80t/9$.]

If you prefer, you can substitute a To Workspace block for the Scope. Then you can plot the results in MATLAB. The resulting plot is shown in Figure 5.5.5.

Figure 5.5.5 Speed response
of the sled for $\theta = 0$ and $\theta \neq 0$.
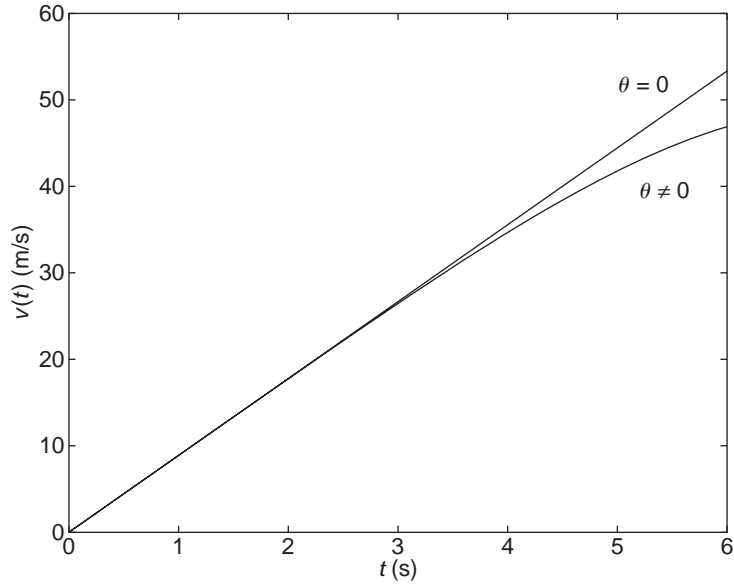
SIMULATING TRANSFER FUNCTION MODELS

The equation of motion of a mass-spring-damper system is

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{5.5.1}$$

Simulink can accept a system description in transfer function form and in state-variable
form. If the mass-spring system is subjected to a sinusoidal forcing function $f(t)$, it is
easy to use the MATLAB commands presented thus far to solve and plot the response
$y(t)$. However, suppose that the force $f(t)$ is created by applying a sinusoidal input
voltage to a hydraulic piston that has a *dead-zone* nonlinearity. This means that the piston
does not generate a force until the input voltage exceeds a certain magnitude, and thus
the system model is piecewise linear. A graph of a particular dead-zone nonlinearity
is shown in Figure 5.5.6. When the input (the independent variable on the graph) is

Figure 5.5.6 A dead-zone
nonlinearity.
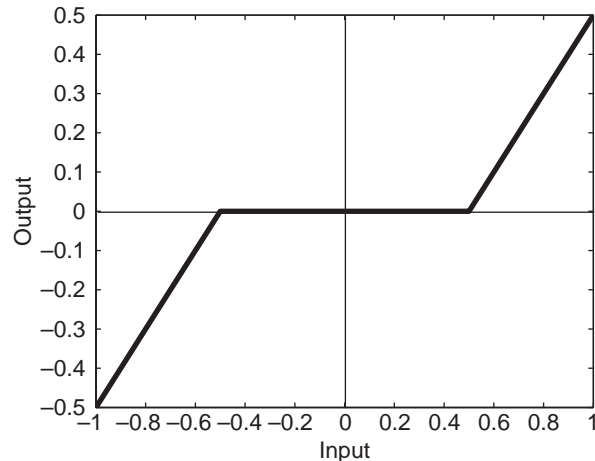
between -0.5 and 0.5, the output is zero. When the input is greater than or equal to the upper limit of 0.5, the output is the input minus the upper limit. When the input is less than or equal to the lower limit of -0.5, the output is the input minus the lower limit. In this example, the dead zone is symmetric about 0, but it need not be in general.

Simulations with dead zone nonlinearities are somewhat tedious to program in MATLAB, but are easily done in Simulink. Example 5.5.2 illustrates how it is done.

A Simulink Model of Response with a Dead Zone | **EXAMPLE 5.5.2**

■ Problem
Create and run a Simulink simulation of a mass-spring-damper system (5.5.1) using the parameter values $m = 1$, $c = 2$, and $k = 4$. The forcing function is the function $f(t) = \sin 1.4t$. The system has the dead-zone nonlinearity shown in Figure 5.5.6.

■ Solution
To construct the simulation, do the following steps.

1. Start Simulink and open a new model window as described previously.
2. Select and place in the new window the Sine Wave block from the Sources category. Double-click on it, and set the Amplitude to 1, the Bias to 0, the Frequency to 1.4, the Phase to 0, and the Sample time to 0. Click **OK.**
3. Select and place the Dead Zone block from the Discontinuities category, double-click on it, and set the Start of dead zone to -0.5 and the End of dead zone to 0.5. Click **OK.**
4. Select and place the Transfer Fcn block from the Continuous category, double-click on it, and set the Numerator to `[1]` and the Denominator to `[1, 2, 4]`. Click **OK.**
5. Select and place the Scope block from the Sinks category.
6. Once the blocks have been placed, connect the input port on each block to the outport port on the preceding block. Your model should now look like that shown in Figure 5.5.7.
7. Click on the **Simulation** menu, then click the **Configuration Parameters** item. Click on the Solver tab, and enter 10 for the Stop time. Make sure the Start time is 0. Then click **OK.**
8. Run the simulation by clicking on the **Simulation** menu, and then clicking the **Start** item.
9. You will hear a bell sound when the simulation is finished. Then double-click on the Scope block and then click on the binoculars icon in the Scope display to enable autoscaling. You should see an oscillating curve.

It is informative to plot both the input and the output of the Transfer Fcn block versus time on the same graph. To do this,

1. Delete the arrow connecting the Scope block to the Transfer Fcn block. Do this by clicking on the arrow line and then pressing the **Delete** key.
2. Select and place the Mux block from the Signal Routing category, double-click on it, and set the Number of inputs to 2. Click **Apply,** then **OK.**
3. Connect the top input port of the Mux block to the output port of the Transfer Fcn block. Then use the same technique to connect the bottom input port of the Mux block to the arrow from the outport port of the Dead Zone block. Just remember to start with the input port. Simulink will sense the arrow automatically and make the connection. Your model should now look like that shown in Figure 5.5.8.

Sine Wave Dead Zone Transfer Fcn Scope
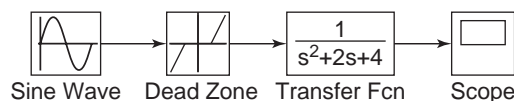
Figure 5.5.7 The Simulink model of dead-zone response.

Figure 5.5.8 Modification of the dead-zone model to include a Mux block.
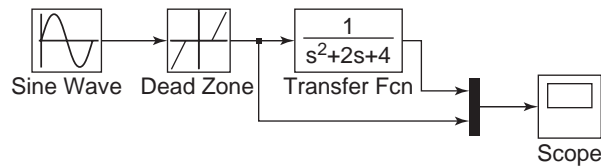
Figure 5.5.9 The response of the dead-zone model.
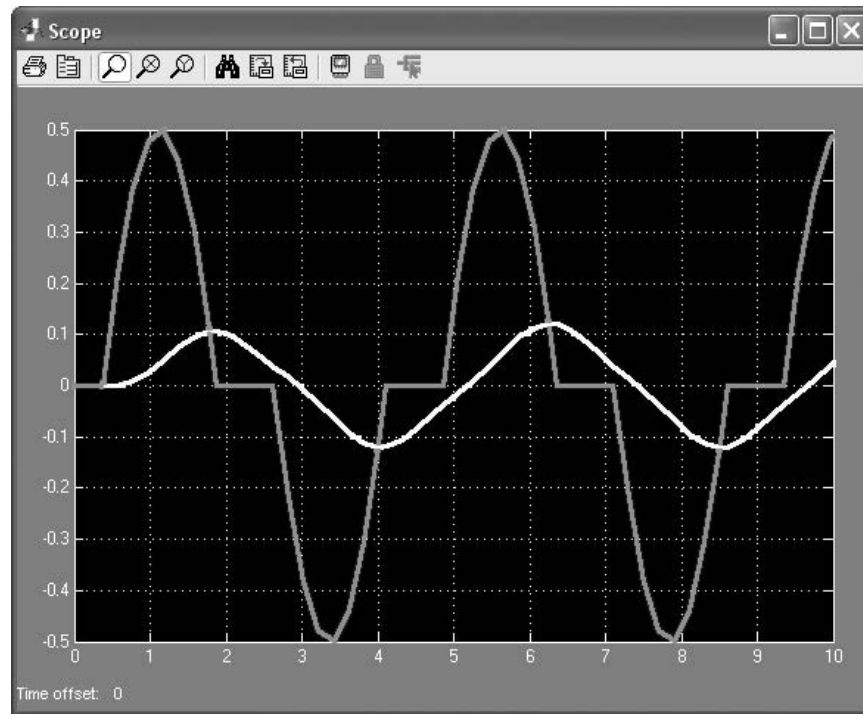
Figure 5.5.10 Modification of the dead-zone model to export variables to the MATLAB workspace.
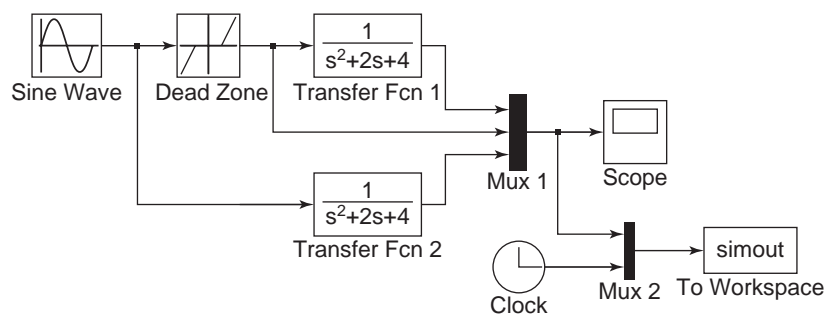
4. Set the Stop time to 10, run the simulation as before, and bring up the Scope display. You should see what is shown in Figure 5.5.9. This plot shows the effect of the dead zone on the sine wave.

   You can bring the simulation results into the MATLAB workspace by using the To Workspace block. For example, suppose we want to examine the effects of the dead zone by comparing the response of the system with and without a dead zone. We can do this with the model shown in Figure 5.5.10. To create this model,

1. Copy the Transfer Fcn block by right-clicking on it, holding down the mouse button, and dragging the block copy to a new location. Then release the button. Copy the Mux block in the same way.

2. Double-click on the first Mux block and change the number of its inputs to 3.

3. In the usual way, select and place the To Workspace block from the Sinks category and the Clock box from the Sources category. Double-click on the To Workspace block. You can specify any variable name you want as the output; the default is `simout`. Change its name to `y`. The output variable `y` will have as many rows as there are simulation time steps, and as many columns as there are inputs to the block. The fourth column in our simulation will be time, because of the way we have connected the Clock to the second Mux. Specify the save format as Array. Use the default values for the other parameters (these should be `inf`, `1`, and `-1` for Maximum number of rows, Decimation, and Sample Time, respectively). Click on **OK.**

4. Connect the blocks as shown, and run the simulation.

5. You can use the MATLAB plotting commands from the Command window to plot the columns of `y`; for example, to plot the response of the two systems and the output of the Dead Zone block versus time, type

```
≫plot(y(:,4),y(:,1),y(:,4),y(:,2),y(:,4),y(:,3))
```

Nonlinear models cannot be put into transfer function form or the state-variable form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$. However, they can be solved in Simulink. Example 5.5.3 shows how this can be done.

Simulink Model of a Nonlinear Pendulum | **EXAMPLE 5.5.3**

■ Problem

The pendulum shown in Figure 5.5.11 has the following nonlinear equation of motion, if there is viscous friction in the pivot and if there is an applied moment $M(t)$ about the pivot.

$$I\ddot{\theta} + c\dot{\theta} + mgL\sin\theta = M(t) \tag{1}$$

Figure 5.5.11 A pendulum.

where I is the mass moment of inertia about the pivot. Create a Simulink model for this system for the case where $I = 4$, $mgL = 10$, $c = 0.8$, and $M(t)$ is a square wave with an amplitude of 3 and a frequency of 0.5 Hz. Assume that the initial conditions are $\theta(0) = \pi/4$ rad and $\dot{\theta}(0) = 0$.

■ Solution

To simulate this model in Simulink, define a set of variables that lets you rewrite the equation as two first-order equations. Thus let $\omega = \dot{\theta}$. Then the model can be written as

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{I}\left[-c\omega - mgL\sin\theta + M(t)\right] = 0.25\left[-0.8\omega - 10\sin\theta + M(t)\right]$$

Integrate both sides of each equation over time to obtain

$$\theta = \int \omega\, dt$$

$$\omega = 0.25\int\left[-0.8\omega - 10\sin\theta + M(t)\right]dt$$

We will introduce four new blocks to create this simulation. Obtain a new model window and do the following.

1. Select and place in the new window the Integrator block from the Continuous category, and change its label to Integrator 1 as shown in Figure 5.5.12. You can edit text associated with a block by clicking on the text and making the changes. Double-click on
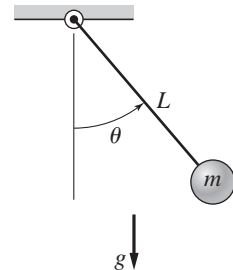
Figure 5.5.12 Simulink model of a nonlinear pendulum.
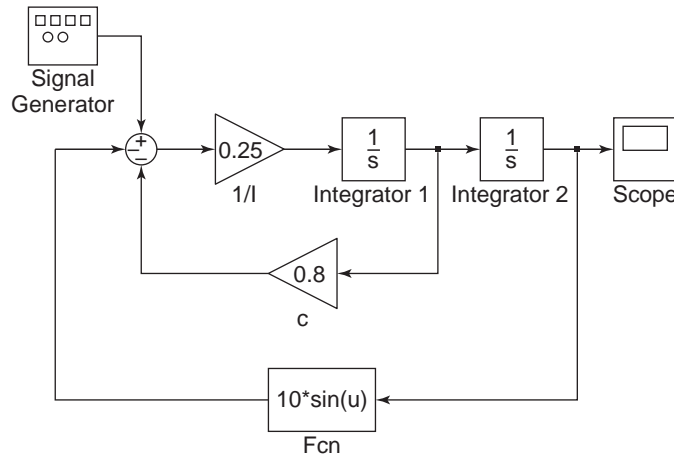
the block to obtain the Block Parameters window, and set the Initial condition to 0 [this is the initial condition $\dot{\theta}(0) = 0$]. Click **OK.**

2. Copy the Integrator block to the location shown and change its label to Integrator 2. Set its initial condition to $\pi/4$ by typing `pi/4` in the Block Parameters window. This is the initial condition $\theta(0) = \pi/4$.

3. Select and place a Gain block from the Math Operations category, double-click on it, and set the Gain value to 0.25. Click **OK.** Change its label to `1/I`. Then click on the block, and drag one of the corners to expand the box so that all the text is visible.

4. Copy the Gain box, change its label to `c`, and place it as shown in Figure 5.5.12. Double-click on it, and set the Gain value to 0.8. Click **OK.** To flip the box left to right, right-click on it, select **Format,** and select **Flip Block.**

5. Select and place the Scope block from the Sinks category.

6. For the term $10 \sin \theta$, we cannot use the Trig function block in the Math Operations category without using a separate gain block to multiply the $\sin \theta$ by 10. Instead we will use the Fcn block under the User-Defined Functions category (Fcn stands for function). Select and place this block as shown. Double-click on it, and type `10*sin(u)` in the expression window. This block uses the variable `u` to represent the input to the block. Click **OK.** Then flip the block.

7. Select and place the Sum block from the Math Operations category. Double-click on it, and select round for the Icon shape. In the List of signs window, type $+ - -$. Click **OK.**

8. Select and place the Signal Generator block from the Sources category. Double-click on it, select square wave for the Wave form, 3 for the Amplitude, and 0.5 for the Frequency, and Hertz for the Units. Click **OK.**

9. Once the blocks have been placed, connect arrows as shown in the figure.

10. Set the Stop time to 10, run the simulation, and examine the plot of $\theta(t)$ in the Scope. This completes the simulation.

In the **Configuration Parameters** submenu under the **Simulation** menu, you can select the ODE solver to use by clicking on the Solver tab. The default is `ode45`.

Problems involving nonlinear functions such as the saturation block are much easier to solve with Simulink. In later chapters we will discover other nonlinear blocks and other advantages to using Simulink. There are menu items in the model window we have not discussed. However, the ones we have discussed are the most important

ones for getting started. We have introduced just a few of the blocks available within Simulink and we will introduce more in later chapters. In addition, some blocks have additional properties that we have not mentioned. However, the examples given here will help you get started in exploring the other features of Simulink. Consult the online help for information about these items.

5.6 CHAPTER REVIEW

The state-variable model form, which can be expressed as a vector-matrix equation, is a concise representation that is useful for analytical purposes and for writing general-purpose computer programs. Section 5.1 shows how to convert models into state-variable form.

In theory it is possible to use the Laplace transform to obtain the closed-form solution of a linear constant-coefficient differential equation if the input function is not too complicated. However, the Laplace transform method cannot be used when the Laplace transform or inverse transform either does not exist or cannot be found easily, as is the case with higher-order models. The reasons include the large amount of algebra required and the need to solve the characteristic equation numerically (closed-form solutions for polynomial roots do not exist for polynomials of order five and higher). Therefore, in this chapter we introduced several types of numerical methods for solving differential equations.

Section 5.2 focuses on MATLAB functions for solving linear state-variable models. These are the `ss`, `ssdata`, `tfdata`, `step`, `impulse`, `lsim`, `initial`, `char`, and `eig` functions. Section 5.3 treats the MATLAB `ode` functions, which are useful for solving both linear and nonlinear equations.

Section 5.4 introduces Simulink, which provides a graphical user interface for solving differential equations. It includes many program blocks and features that enable you to create simulations that are otherwise difficult to program in MATLAB. Section 5.4 treats applications to linear systems, while Section 5.5 treats nonlinear system applications.

Now that you have finished this chapter, you should be able to

1. Convert a differential equation model into state-variable form.
2. Express a linear state-variable model in the standard vector-matrix form.
3. Apply the `ss`, `ssdata`, `tfdata`, `char`, `eig`, and `initial` functions to analyze linear models.
4. Use the MATLAB `ode` functions to solve linear and nonlinear differential equations.
5. Use Simulink to create simulations of linear and nonlinear models expressed either as differential equations or, if linear, as transfer functions.

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PROBLEMS

Section 5.1 State-Variable Models

5.1 Obtain the state model for the reduced-form model $5\ddot{x} + 7\dot{x} + 4x = f(t)$.

5.2 Obtain the state model for the reduced-form model

$$2\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = f(t)$$

5.3 Obtain the state model for the reduced-form model $2\ddot{x} + 5\dot{x} + 4x = 4y(t)$.

5.4 Obtain the state model for the transfer-function model

$$\frac{Y(s)}{F(s)} = \frac{6}{3s^2 + 6s + 10}$$

5.5 Obtain the state model for the two-mass system discussed in Example 4.2.1. The equations of motion are

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = f(t)$$

$$m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2x_2 = 0$$

5.6 Obtain the state model for the two-mass system discussed in Example 4.4.5. The equations of motion for specific values of the spring and damping constants are

$$10\ddot{x}_1 + 8\dot{x}_1 - 5\dot{x}_2 + 40x_1 - 25x_2 = 0$$

$$5\ddot{x}_2 - 25x_1 + 25x_2 - 5\dot{x}_1 + 5\dot{x}_2 = f(t)$$

5.7 Put the following model in standard state-variable form and obtain the expressions for the matrices **A**, **B**, **C**, and **D**. The output is x.

$$2\ddot{x} + 5\dot{x} + 4x = 4y(t)$$

5.8 Given the state-variable model

$$\dot{x}_1 = -5x_1 + 3x_2 + 2u_1$$

$$\dot{x}_2 = -4x_2 + 6u_2$$

and the output equations

$$y_1 = x_1 + 3x_2 + 2u_1$$

$$y_2 = x_2$$

obtain the expressions for the matrices **A**, **B**, **C**, and **D**.

5.9 Given the following state-variable models, obtain the expressions for the matrices **A**, **B**, **C**, and **D** for the given inputs and outputs.

a. The outputs are x_1 and x_2; the input is u.

$$\dot{x}_1 = -5x_1 + 3x_2$$

$$\dot{x}_2 = x_1 - 4x_2 + 5u$$

b. The output is x_1; the inputs are u_1 and u_2.

$$\dot{x}_1 = -5x_1 + 3x_2 + 4u_1$$

$$\dot{x}_2 = x_1 - 4x_2 + 5u_2$$

5.10 Obtain the expressions for the matrices **A**, **B**, **C**, and **D** for the state-variable model you obtained in Problem 5.6. The outputs are x_1 and x_2.

5.11 The transfer function of a certain system is

$$\frac{Y(s)}{F(s)} = \frac{6s + 7}{s + 3}$$

Use two methods to obtain a state-variable model in standard form. For each model, relate the initial value of the state-variable to the given initial value $y(0)$.

5.12 The transfer function of a certain system is

$$\frac{Y(s)}{F(s)} = \frac{s + 2}{s^2 + 4s + 3}$$

Use two methods to obtain a state-variable model in standard form. For each model, relate the initial values of the state variables to the given initial values $y(0)$ and $\dot{y}(0)$.

Section 5.2 State-Variable Methods with MATLAB

5.13 Use MATLAB to create a state-variable model; obtain the expressions for the matrices **A**, **B**, **C**, and **D**, and then find the transfer functions of the following models, for the given inputs and outputs.

a. The outputs are x_1 and x_2; the input is u.

$$\dot{x}_1 = -5x_1 + 3x_2$$

$$\dot{x}_2 = x_1 - 4x_2 + 5u$$

b. The output is x_1; the inputs are u_1 and u_2.

$$\dot{x}_1 = -5x_1 + 3x_2 + 4u_1$$

$$\dot{x}_2 = x_1 - 4x_2 + 5u_2$$

5.14 Use MATLAB to obtain a state model for the following equations; obtain the expressions for the matrices **A**, **B**, **C**, and **D**. In both cases, the input is $f(t)$; the output is y.

a.
$$2\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = f(t)$$

b.
$$\frac{Y(s)}{F(s)} = \frac{6}{3s^2 + 6s + 10}$$

5.15 Use MATLAB to obtain a state-variable model for the following transfer functions.

a.
$$\frac{Y(s)}{F(s)} = \frac{6s + 7}{s + 3}$$

b.
$$\frac{Y(s)}{F(s)} = \frac{s + 2}{s^2 + 4s + 3}$$

5.16 For the following model the output is x_1 and the input is $f(t)$.

$$\dot{x}_1 = -5x_1 + 3x_2$$

$$\dot{x}_2 = x_1 - 4x_2 + 5f(t)$$

a. Use MATLAB to compute and plot the free response for $x_1(0) = 3$, and $x_2(0) = 5$.

b. Use MATLAB to compute and plot the unit-step response for zero initial conditions.

c. Use MATLAB to compute and plot the response for zero initial conditions with the input $f(t) = 3 \sin 10\pi t$, for $0 \le t \le 2$.

5.17 Given the state-variable model

$$\dot{x}_1 = -5x_1 + 3x_2 + 2u_1$$

$$\dot{x}_2 = -4x_2 + 6u_2$$

and the output equations

$$y_1 = x_1 + 3x_2 + 2u_1$$

$$y_2 = x_2$$

Use MATLAB to find the characteristic polynomial and the characteristic roots.

5.18 The equations of motion for the two-mass, quarter-car model of a suspension system given in Example 4.5.9 are

$$m_1\ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) + k_2(y - x_2)$$

Suppose the coefficient values are: $m_1 = 240$ kg, $m_2 = 36$ kg, $k_1 = 1.6 \times 10^4$ N/m, $k_2 = 1.6 \times 10^5$ N/m, $c_1 = 98$ N·s/m.

a. Use MATLAB to create a state model. The input is $y(t)$; the outputs are x_1 and x_2.

b. Use MATLAB to compute and plot the response of x_1 and x_2 if the input $y(t)$ is a unit impulse and the initial conditions are zero.

c. Use MATLAB to find the characteristic polynomial and the characteristic roots.

d. Use MATLAB to obtain the transfer functions $X_1(s)/Y(s)$ and $X_2(s)/Y(s)$.

5.19 A representation of a car's suspension suitable for modeling the bounce and pitch motions is shown in Figure P5.19, which is a side view of the vehicle's body showing the front and rear suspensions. Assume that the car's motion is constrained to a vertical translation x of the mass center and rotation θ about a single axis which is perpendicular to the page. The body's mass is m and its
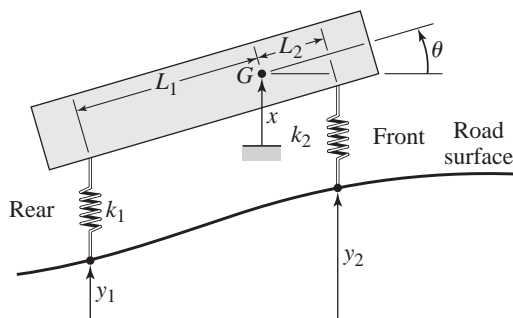
Figure P5.19

moment of inertia about the mass center is I_G. As usual, x and θ are the displacements from the equilibrium position corresponding to $y_1 = y_2 = 0$. The displacements $y_1(t)$ and $y_2(t)$ can be found knowing the vehicle's speed and the road surface profile.

a. Assume that x and θ are small, and derive the equations of motion for the bounce motion x and pitch motion θ.

b. For the values $k_1 = 1100$ lb/ft, $k_2 = 1525$ lb/ft, $c_1 = c_2 = 4$ lb-sec/ft, $L_1 = 4.8$ ft, $L_2 = 3.6$ ft, $m = 50$ slugs, and $I_G = 1000$ slug-ft^2, use MATLAB to obtain a state-variable model in standard form.

c. Use MATLAB to obtain and plot the solution for $x(t)$ and $\theta(t)$ when $y_1 = 0$ and y_2 is a unit impulse. The initial conditions are zero.

Section 5.3 The MATLAB ode Functions

5.20 a. Use a MATLAB ode function to solve the following equation for $0 \le t \le 12$. Plot the solution.
$$\dot{y} = \cos t \qquad y(0) = 6$$

b. Use the closed-form solution to check the accuracy of the numerical method.

5.21 a. Use a MATLAB ode function to solve the following equation for $0 \le t \le 1$. Plot the solution.
$$\dot{y} = 5e^{-4t} \qquad y(0) = 2$$

b. Use the closed-form solution to check the accuracy of the numerical method.

5.22 a. Use a MATLAB ode function to solve the following equation for $0 \le t \le 1$. Plot the solution.
$$\dot{y} + 3y = 5e^{4t} \qquad y(0) = 10$$

b. Use the closed-form solution to check the accuracy of the numerical method.

5.23 a. Use a MATLAB ode function to solve the following nonlinear equation for $0 \le t \le 4$. Plot the solution.
$$\dot{y} + \sin y = 0 \qquad y(0) = 0.1 \qquad (1)$$

b. For small angles, $\sin y \approx y$. Use this fact to obtain a linear equation that approximates equation (1). Use the closed-form solution of this linear equation to check the output of your program.

5.24 Sometimes it is tedious to obtain a solution of a linear equation, especially if all we need is a plot of the solution. In such cases, a numerical method might be preferred. Use a MATLAB ode function to solve the following equation for $0 \le t \le 7$. Plot the solution.
$$\dot{y} + 2y = f(t) \qquad y(0) = 2$$

where

$$f(t) = \begin{cases} 3t & \text{for } 0 \le t \le 2 \\ 6 & \text{for } 2 \le t \le 5 \\ -3(t-5) + 6 & \text{for } 5 \le t \le 7 \end{cases}$$

5.25   A certain jet-powered ground vehicle is subjected to a nonlinear drag force. Its equation of motion, in British units, is

$$50\dot{v} = f - (20v + 0.05v^2)$$

Use a numerical method to solve for and plot the vehicle's speed as a function of time if the jet's force is constant at 8000 lb and the vehicle starts from rest.

5.26   The following model describes a mass supported by a nonlinear, hardening spring. The units are SI. Use $g = 9.81$ m/s^2.

$$5\ddot{y} = 5g - (900y + 1700y^3)$$

Suppose that $\dot{y}(0) = 0$. Use a numerical method to solve for and plot the solution for two different initial conditions: (1) $y(0) = 0.06$ and (2) $y(0) = 0.1$.

5.27   Van der Pol's equation is a nonlinear model for some oscillatory processes. It is

$$\ddot{y} - b(1 - y^2)\dot{y} + y = 0$$

Use a numerical method to solve for and plot the solution for the following cases:
1. $b = 0.1$, $y(0) = \dot{y}(0) = 1$, $0 \leq t \leq 25$
2. $b = 0.1$, $y(0) = \dot{y}(0) = 3$, $0 \leq t \leq 25$
3. $b = 3$, $y(0) = \dot{y}(0) = 1$, $0 \leq t \leq 25$

5.28   Van der Pol's equation is

$$\ddot{y} - b(1 - y^2)\dot{y} + y = 0$$

This equation can be difficult to solve for large values of the parameter b. Compare the performance of a nonstiff solver (such as `ode45`) with that of a more accurate solver (such as `ode15s`). Use $b = 1000$ and $0 \leq t \leq 3000$, with the initial conditions $y(0) = 2$ and $\dot{y}(0) = 0$.

5.29   The equation of motion for a pendulum whose base is accelerating horizontally with an acceleration $a(t)$ is

$$L\ddot{\theta} + g \sin\theta = a(t)\cos\theta$$

Suppose that $g = 9.81$ m/s^2, $L = 1$ m, and $\dot{\theta}(0) = 0$. Solve for and plot $\theta(t)$ for $0 \leq t \leq 10$ s for the following three cases.
a. The acceleration is constant: $a = 5$ m/s^2, and $\theta(0) = 0.5$ rad.
b. The acceleration is constant: $a = 5$ m/s^2, and $\theta(0) = 3$ rad.
c. The acceleration is linear with time: $a = 0.5t$ m/s^2, and $\theta(0) = 3$ rad.

5.30   Suppose the spring in Figure 4.5.14 is nonlinear and is described by the cubic force-displacement relation. The equation of motion is

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k_1(y - x) + k_2(y - x)^3$$

where $m = 100$, $c = 600$, $k_1 = 8000$, and $k_2 = 24000$. Approximate the unit-step input $y(t)$ with $y(t) = 1 - e^{-t/\tau}$, where τ is chosen to be small compared to the period and time constant of the model when the cubic term is neglected. Use MATLAB to plot the forced response $x(t)$.

Section 5.4 Simulink and Linear Models

5.31 Create a Simulink model to plot the solution of the following equation for $0 \le t \le 6$.

$$10\ddot{y} = 7\sin 4t + 5\cos 3t \qquad y(0) = 4 \qquad \dot{y}(0) = 1$$

5.32 A projectile is launched with a velocity of 100 m/s at an angle of $30°$ above the horizontal. Create a Simulink model to solve the projectile's equations of motion, where x and y are the horizontal and vertical displacements of the projectile.

$$\ddot{x} = 0 \qquad x(0) = 0 \qquad \dot{x}(0) = 100\cos 30°$$

$$\ddot{y} = -g \qquad y(0) = 0 \qquad \dot{y}(0) = 100\sin 30°$$

Use the model to plot the projectile's trajectory y versus x for $0 \le t \le 10$ s.

5.33 In Example 3.7.5 in Chapter 3 we obtained an approximate solution of the following problem, which has no analytical solution even though it is linear.

$$\dot{x} + x = \tan t \qquad x(0) = 0$$

The approximate solution, which is less accurate for large values of t, is

$$x(t) = \frac{1}{3}t^3 - t^2 + 3t - 3 + 3e^{-t}$$

Create a Simulink model to solve this problem and compare its solution with the approximate solution over the range $0 \le t \le 1$.

5.34 Construct a Simulink model to plot the solution of the following equation for $0 \le t \le 10$.

$$15\dot{x} + 5x = 4u_s(t) - 4u_s(t-2) \qquad x(0) = 2$$

Section 5.5 Simulink and Nonlinear Models

5.35 Use the Transfer Function block to construct a Simulink model to plot the solution of the following equation for $0 \le t \le 4$.

$$2\ddot{x} + 12\dot{x} + 10x^2 = 5u_s(t) - 5u_s(t-2) \qquad x(0) = \dot{x}(0) = 0$$

5.36 Construct a Simulink model to plot the solution of the following equation for $0 \le t \le 4$.

$$2\ddot{x} + 12\dot{x} + 10x^2 = 5\sin 0.8t \qquad x(0) = \dot{x}(0) = 0$$

5.37 Use the Saturation block to create a Simulink model to plot the solution of the following equation for $0 \le t \le 6$.

$$3\dot{y} + y = f(t) \qquad y(0) = 2$$

where

$$f(t) = \begin{cases} 8 & \text{if } 10\sin 3t > 8 \\ -8 & \text{if } 10\sin 3t < -8 \\ 10\sin 3t & \text{otherwise} \end{cases}$$

5.38 Construct a Simulink model of the following problem.

$$5\dot{x} + \sin x = f(t) \qquad x(0) = 0$$

The forcing function is

$$f(t) = \begin{cases} -5 & \text{if } g(t) \leq -5 \\ g(t) & \text{if } -5 \leq g(t) \leq 5 \\ 5 & \text{if } g(t) \geq 5 \end{cases}$$

where $g(t) = 10 \sin 4t$.

5.39 Create a Simulink model to plot the solution of the following equation for $0 \leq t \leq 3$.

$$\dot{x} + 10x^2 = 2 \sin 4t \qquad x(0) = 1$$

5.40 Construct a Simulink model of the following problem.

$$10\dot{x} + \sin x = f(t) \qquad x(0) = 0$$

The forcing function is $f(t) = \sin 2t$. The system has the dead-zone nonlinearity shown in Figure 5.5.6.

5.41 The following model describes a mass supported by a nonlinear, hardening spring. The units are SI. Use $g = 9.81$ m/s^2.

$$5\ddot{y} = 5g - (900y + 1700y^3) \qquad y(0) = 0.5 \qquad \dot{y}(0) = 0.$$

Create a Simulink model to plot the solution for $0 \leq t \leq 2$.

5.42 Consider the system for lifting a mast, discussed in Example 2.2.4 in Chapter 2 and shown again in Figure P5.42. The 70-ft-long mast weighs 500 lb. The winch applies a force $f = 380$ lb to the cable. The mast is supported initially at an angle of $\theta = 30°$, and the cable at A is initially horizontal. The equation of motion of the mast is

$$25{,}400\ddot{\theta} = -17{,}500 \cos\theta + \frac{626{,}000}{Q} \sin(1.33 + \theta)$$

where

$$Q = \sqrt{2020 + 1650 \cos(1.33 + \theta)}$$

Create and run a Simulink model to solve for and plot $\theta(t)$ for $\theta(t) \leq \pi/2$ rad.

5.43 A certain mass, $m = 2$ kg, moves on a surface inclined at an angle $\phi = 30°$ above the horizontal. Its initial velocity is $v(0) = 3$ m/s up the incline. An external force of $f_1 = 5$ N acts on it parallel to and up the incline. The coefficient of dynamic friction is $\mu_d = 0.5$. Use the Coulomb Friction block or the Sign block and create a Simulink model to solve for the velocity of the mass until the mass comes to rest. Use the model to determine the time at which the mass comes to rest.

Figure P5.42
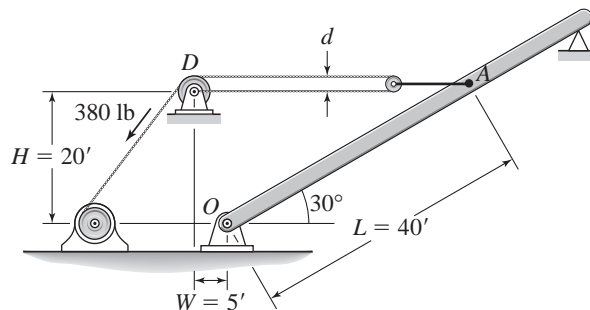
5.44 If a mass-spring system has Coulomb friction on the horizontal surface rather than viscous friction, its equation of motion is

$$m\ddot{y} = -ky + f(t) - \mu mg \quad \text{if } \dot{y} \geq 0$$

$$m\ddot{y} = -ky + f(t) + \mu mg \quad \text{if } \dot{y} < 0$$

where μ is the coefficient of friction. Develop a Simulink model for the case where $m = 1$ kg, $k = 5$ N/m, $\mu = 0.4$, and $g = 9.8$ m/s^2. Run the simulation for two cases: (a) the applied force $f(t)$ is a step function with a magnitude of 10 N, and (b) the applied force is sinusoidal: $f(t) = 10 \sin 2.5t$.

Electrical and Electromechanical Systems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1.   Develop models of electrical circuits. This includes

   Application of Kirchhoff's voltage and current laws, and

   Use of loop analysis.

2.   Obtain circuit models in transfer-function and state-variable form.

3.   Apply impedance methods to obtain models of circuits and amplifiers.

4.   Apply Newton's laws, electrical circuit laws, and electromagnetic principles to develop models of electromechanical systems, including dc motors and sensors.

5.   Assess the performance of motors and amplifiers.

6.   Apply MATLAB and Simulink to analyze models of circuits and electromechanical systems in state-variable and transfer function form.

The majority of engineering systems now have at least one electrical subsystem. This may be a power supply, sensor, motor, controller, or an acoustic device such as a speaker. So an understanding of electrical systems is essential to understanding the behavior of many systems.

Section 6.1 introduces the basic physics, common elements, and terminology of electrical circuits and treats the two main physical laws needed to develop circuit models. These are Kirchhoff's current and voltage laws. Section 6.2 is an extensive collection of circuit examples that emphasize resistance networks and circuits having one or two capacitors or inductors.

Impedance, a generalization of the electrical resistance concept, is covered in Section 6.3. This concept enables you to derive circuit models more easily, especially for more complex circuits, and is especially useful for obtaining models of circuits containing operational amplifiers.

The principles of direct-current (dc) motors are established in Section 6.4, and these principles are used to develop transfer function and state-variable models of motors. Section 6.5 examines some practical considerations in motor modeling, including methods for assessing the performance of motors and amplifiers. In Section 6.6, these principles are extended to other electromechanical devices such as sensors and speakers. Many of the systems treated in this chapter are more easily designed with computer simulation, and Sections 6.7 and 6.8 show how to apply MATLAB and Simulink in electromechanical systems analysis. ∎

6.1 ELECTRICAL ELEMENTS

Voltage and current are the primary variables used to describe a circuit's behavior. *Current* is the flow of electrons. It is the time rate of change of electrons passing through a defined area, such as the cross section of a wire. Because electrons are negatively charged, the positive direction of current flow is opposite to that of the electron flow. The mathematical description of the relation between the number of electrons (called *charge Q*) and current i is

$$i = \frac{dQ}{dt} \qquad \text{or} \qquad Q(t) = \int i \, dt$$

The unit of charge is the *coulomb* (C), and the unit of current is the *ampere* (A), which is one coulomb per second. These units, and the others we will use for electrical systems, are the same in both the SI and FPS systems.

Energy is required to move a charge between two points in a circuit. The work per unit charge required to do this is called *voltage*. The unit of voltage is the *volt* (V), which is defined to be one joule per coulomb. The voltage difference between two points in a circuit is a measure of the energy required to move charge from one point to the other.

The sign of voltage difference is important. Figure 6.1.1a shows a battery connected to a lightbulb. The electrons in the wire are attracted to the battery's positive terminal; thus the positive direction of current is clockwise, as indicated by the arrow. Because the battery supplies energy to move electrons and the lightbulb dissipates energy (through light and heat), the sign of voltage difference across the battery is the opposite of the sign of the voltage difference across the lightbulb. This is indicated by the + and − signs in the diagram. Although the charge flows counterclockwise, we can think of a positive current flowing clockwise. This current picks up energy as it passes through the battery from the negative to the positive terminal. As it flows through the lightbulb, the current loses energy; this is indicated by the + and − signs above and below the bulb.

(a) (b)
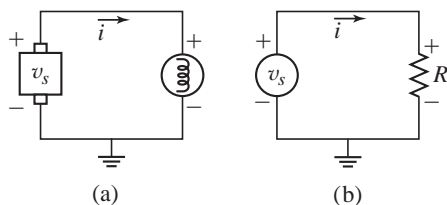
Figure 6.1.1 (a) A battery-lightbulb circuit. (b) Circuit diagram representation of the battery-lightbulb circuit

The bulb current depends on the voltage difference and the bulb's material properties, which resist the current and cause the energy loss. When a current flows through wire or other circuit elements, it encounters resistance. Sometimes this resistance is desirable and intentionally introduced; sometimes not. A *resistor* is an element designed to provide resistance. Most resistors are designed to have a linear relation between the current passing through them and the voltage difference across them. This linear relation is *Ohm's law*. It states that

$$v = iR$$

where i is the current, v is the voltage difference, and R is the *resistance*. The unit of resistance is the *ohm* (Ω), which is one volt per ampere.

Figure 6.1.1b shows a voltage source, such as a battery, connected to a resistor. Because of conservation of energy, the voltage increase v_s supplied by the source must equal the voltage drop iR across the resistor. Thus the model of this circuit is $v_s = iR$. If we know the voltage and the resistance, we can calculate the current that must be supplied by the source as follows: $i = v_s/R$.

ACTIVE AND PASSIVE ELEMENTS

Circuit elements may be classified as *active* or *passive*. Passive elements such as resistors, capacitors, and inductors are not sources of energy, although the latter two can store it temporarily. Elements that provide energy are *sources,* and elements that dissipate energy are *loads*.

The active elements are energy sources that drive the system. There are several types available; for example, chemical (batteries), mechanical (generators), thermal (thermocouples), and optical (solar cells). Active elements are modeled as either *ideal voltage sources* or *ideal current sources*. Their circuit symbols are given in Table 6.1.1.

Table 6.1.1 Electrical quantities.

Quantity	Units	Circuit symbol
Voltage	volt (V)	Voltage Source
Charge	coulomb (C) $= $ N $\cdot$ m/V	
Current	ampere (A) $=$ C/s	Current Source
Resistance	ohm (Ω) $=$ V/A	R
Capacitance	farad (F) $=$ C/V	C
Inductance	henry (H) $=$ V $\cdot$ s/A	L
Battery	—	
Ground	—	
Terminals (input or output)	—	

An ideal voltage source supplies the desired voltage no matter how much current is drawn by the loading circuit. An ideal current source supplies whatever current is needed by the loading circuit. Obviously no real source behaves exactly this way. For example, battery voltage drops because of heat produced as current is drawn from the battery. If the current is small the battery can be treated as an ideal voltage source. If not, the battery is frequently modeled as an ideal voltage source plus an internal resistance whose value can be obtained from the voltage-current curve for the battery.

Power P is work done per unit time, so the power generated by an active element, or the power dissipated or stored by a passive element, can be calculated as follows:

$$\text{power} = \frac{\text{work}}{\text{time}} = \frac{\text{work}}{\text{unit charge}} \frac{\text{charge}}{\text{time}} = \text{voltage} \times \text{current}$$

Thus the power generated, dissipated, or stored by a circuit element equals the product of the voltage difference across the element and the current flowing through the element. That is, $P = iv$. The unit of power in the SI system is one joule per second, which is defined to be a *watt* (W).

If the element is a linear resistor, the power dissipated is given by

$$P = iv = i^2 R = \frac{v^2}{R}$$

The appropriate form to use depends on which two of the three quantities i, v, and R are known.

MODELING CIRCUITS

The dynamics of physical systems result from the transfer, loss, and storage of mass or energy. A basic law used to model electrical systems is conservation of charge, also known as *Kirchhoff's current law*. Another basic law is conservation of energy. In electrical systems, conservation of energy is commonly known as *Kirchhoff's voltage law,* which states that the algebraic sum of the voltages around a closed circuit or *loop* must be zero. These physical laws alone do not provide enough information to write the equations that describe the system. Three more types of information must be provided; the four requirements are

1. The appropriate physical laws, such as conservation of charge and energy.
2. Empirically based descriptions, called *constitutive relations,* for some or all of the system elements.
3. The specific way the system elements are arranged or connected.
4. Any relationships due to integral causality, such as the relation between charge and current, $Q = \int i \, dt$.

The voltage-current relation for a resistor, $v = iR$, is an example of an empirically based description of a system element, the third type of required information. This type of description is an algebraic relation not derivable from a basic physical law. Rather, the relation is obtained from a series of measurements. For example, if we apply a range of currents to a resistor, then measure the resulting voltage difference for each current, we would find that the voltage is directly proportional to the applied current. The constant of proportionality is the resistance R, and we can determine its value from the test results.

Knowledge of the elements' arrangement is important because it is possible to connect two elements in more than one way. For example, two resistors can be connected

Figure 6.1.2 Series resistors.
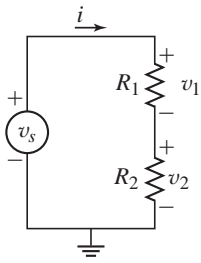
Figure 6.1.3 Parallel resistors.
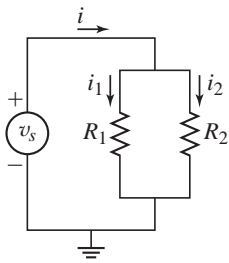
differently to form two different circuits (Figures 6.1.2 and 6.1.3); the models are different for each circuit. We can use the voltage-current relation for a resistor along with conservation of charge and conservation of energy to obtain the models.

SERIES RESISTANCES

For Figure 6.1.2, conservation of charge implies that the current is the same through each resistor. When traversing the loop clockwise, a voltage *increase* occurs when we traverse the source v_s, and a voltage *decrease* occurs when we traverse each resistor. Assign a positive sign to a voltage increase, and a negative sign to a voltage decrease. Then Kirchhoff's voltage law gives

$$v_s - v_1 - v_2 = v_s - iR_1 - iR_2 = 0$$

or

$$v_s = (R_1 + R_2)i \tag{6.1.1}$$

Thus, the supply voltage v_s must equal the sum of the voltages across the two resistors.

Equation (6.1.1) is an illustration of the *series resistance law*. If the same current passes through two or more electrical elements, those elements are said to be in *series,* and the series resistance law states that they are equivalent to a single resistance R that is the sum of the individual resistances. Thus the circuit in Figure 6.1.2 can be modeled as the simpler circuit in Figure 6.1.1b.

Because $i = v_s/(R_1 + R_2)$ in Figure 6.1.2,

$$v_1 = R_1 i = \left(\frac{R_1}{R_1 + R_2} \right) v_s \tag{6.1.2}$$

$$v_2 = R_2 i = \left(\frac{R_2}{R_1 + R_2} \right) v_s \tag{6.1.3}$$

These equations express the *voltage-divider rule*. They show that

$$\frac{v_1}{v_2} = \frac{R_1}{R_2} \tag{6.1.4}$$

This rule is useful for reducing a circuit with many resistors to an equivalent circuit containing one resistor.

PARALLEL RESISTANCES

For Figure 6.1.3, conservation of charge gives $i = i_1 + i_2$. From Kirchhoff's voltage law we see that the voltage across each resistor must be v_s, and thus, $i_1 = v_s/R_1$ and $i_2 = v_s/R_2$. Combining these three relations gives

$$i = i_1 + i_2 = \frac{v_s}{R_1} + \frac{v_s}{R_2}$$

Solve for i to obtain

$$i = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) v_s \tag{6.1.5}$$

Equation (6.1.5) is an illustration of the *parallel resistance law*. If the same voltage difference exists across two or more electrical elements, those elements are said to be in *parallel*. For two resistors in parallel, the parallel resistance law states that they are

equivalent to a single resistance R given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6.1.6}$$

Thus, the circuit in Figure 6.1.3 can be modeled as the simpler circuit in Figure 6.1.1b. This formula can be extended to more than two resistors.

Note that

$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i \tag{6.1.7}$$

$$i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i \tag{6.1.8}$$

These equations express the *current-divider rule*. They show that

$$\frac{i_1}{i_2} = \frac{R_2}{R_1} \tag{6.1.9}$$

The current-divider rule can be used to find the equivalent resistance of a circuit with many resistors.

NONLINEAR RESISTANCES

Not all resistance elements have the linear voltage-current relation $v = iR$. An example of a specific diode's voltage-current relationship found from experiments is

$$i = 0.16(e^{0.12v} - 1)$$

For low voltages, we can approximate this curve with a straight line whose slope equals the derivative di/dv at $v = 0$.

$$\left.\frac{di}{dv}\right|_{v=0} = 0.16\big(0.12e^{0.12v}\big)\big|_{v=0} = 0.0192$$

Thus, for small voltages, $i = 0.0192v$, and the resistance is $R = 1/0.0192 = 52\ \Omega$.

CAPACITANCE

A *capacitor* is designed to store charge. The *capacitance C* of a capacitor is a measure of how much charge can be stored for a given voltage difference across the element. Capacitance thus has the units of charge per volt. This unit is named the *farad* (F).

For a capacitor, $Q = \int i\, dt$, where Q is the charge on the capacitor, and i is the current passing through the capacitor. The constitutive relation for a capacitor is $v = Q/C$, where v is the voltage across the capacitor. Combining these two relations gives

$$v = \frac{1}{C}\int i\, dt = \frac{1}{C}\int_0^t i\, dt + \frac{Q_0}{C}$$

where Q_0 is the initial charge on the capacitor at time $t = 0$. In derivative form, this relation is expressed as

$$i = C\frac{dv}{dt}$$

INDUCTANCE

A magnetic field (a *flux*) surrounds a moving charge or current. If the conductor of the current is coiled, the flux created by the current in one loop affects the adjacent loops. This flux is proportional to the time integral of the applied voltage, and the current is proportional to the flux. The constitutive relation for an inductor is $\phi = Li$, where L is the *inductance* and ϕ is the flux across the inductor.

The integral causality relation between flux and voltage is

$$\phi = \int v \, dt$$

Combining the two preceding expressions for ϕ gives the voltage-current relation for the inductor

$$i = \frac{1}{L} \int v \, dt$$

which is equivalent to

$$v = L \frac{di}{dt}$$

The unit of inductance is the *henry* (H), which is one volt-second per ampere.

POWER AND ENERGY

The power dissipated by or stored by an electrical element is the product of its current and the voltage across it: $P = iv$. Capacitors and inductors store electrical energy as stored charge and in a magnetic field, respectively. The energy E stored in a capacitor can be found as follows:

$$E = \int P \, dt = \int iv \, dt = \int \left(C \frac{dv}{dt} \right) v \, dt = C \int v \, dv = \frac{1}{2} C v^2$$

Similarly, the energy E stored in an inductor is

$$E = \int P \, dt = \int iv \, dt = \int i \left(L \frac{di}{dt} \right) dt = L \int i \, di = \frac{1}{2} L i^2$$

Table 6.1.2 summarizes the voltage-current relations and the energy expressions for resistance, capacitance, and inductance elements.

Table 6.1.2 Voltage-current and energy relations for circuit elements.

Resistance:	$v = iR$	$P = Ri^2 = \dfrac{v^2}{R}$
Capacitance:	$v = \dfrac{1}{C} \displaystyle\int_0^t i \, dt + \dfrac{Q_0}{C}$	$E = \dfrac{1}{2} C v^2$
Inductance:	$v = L \dfrac{di}{dt}$	$E = \dfrac{1}{2} L i^2$

6.2 CIRCUIT EXAMPLES

The examples in this section illustrate how to apply the basic circuit principles introduced in Section 6.1.

| Current and Power in a Resistance Circuit | **EXAMPLE 6.2.1** |

■ **Problem**

For the circuit shown in Figure 6.2.1, the applied voltage is $v_s = 6$ V and the resistance is $R = 10\,\Omega$. Determine the current and the power that must be produced by the power supply.

Figure 6.2.1 A simple resistance circuit.

■ **Solution**

The current is found from $i = v_s/R = 6/10 = 0.6$ A. The power is computed from $P = v_s^2/R = 6^2/10 = 3.6$ W. Note that we can also compute the power from $P = i v_s$.
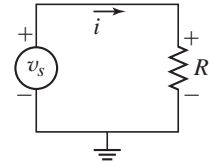
| A Summing Circuit | **EXAMPLE 6.2.2** |

■ **Problem**

Figure 6.2.2 shows a circuit for summing the voltages v_1 and v_2 to produce v_3. Derive the expression for v_3 as a function of v_1 and v_2, for the case where $R_1 = R_2 = 10\,\Omega$, and $R_3 = 20\,\Omega$.

Figure 6.2.2 A summing circuit.

■ **Solution**

Define the currents shown in the diagram. The voltage-current relation for each resistor gives

$$i_1 = \frac{v_1 - v_3}{R_1}$$

$$i_2 = \frac{v_2 - v_3}{R_2}$$

$$i_3 = \frac{v_3}{R_3}$$

There is no capacitance in the circuit. Therefore, no charge can be stored anywhere in the circuit. Thus conservation of charge gives $i_3 = i_1 + i_2$. Substituting the expressions for the currents into this equation, we obtain

$$\frac{v_3}{R_3} = \frac{v_1 - v_3}{R_1} + \frac{v_2 - v_3}{R_2}$$

which gives

$$v_3 = 0.4(v_1 + v_2)$$

Thus, v_3 is proportional to the sum of v_1 and v_2. This is true in general only if $R_1 = R_2$.

| Application of the Voltage-Divider Rule | **EXAMPLE 6.2.3** |

■ **Problem**

Consider the circuit shown in Figure 6.2.3. Obtain the voltage v_o as a function of the applied voltage v_s by applying the voltage-divider rule. Use the values $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 6\,\Omega$, and $R_4 = 2\,\Omega$.

Figure 6.2.3 A resistance
network with two loops.
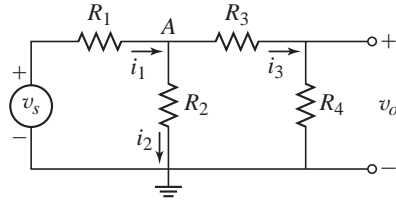
Figure 6.2.4 Application of the
voltage-divider rule.

(a)   (b)   (c)
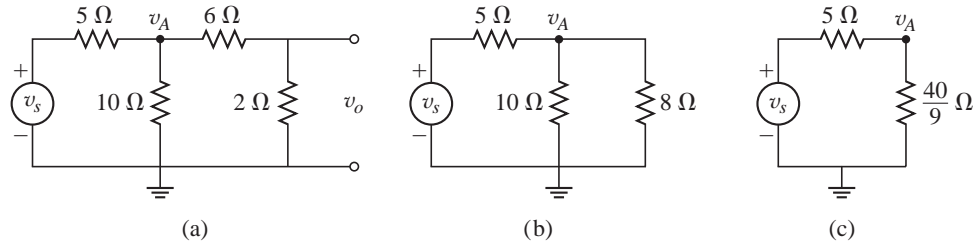
■ **Solution**

Let v_A be the voltage at the node shown in Figure 6.2.4a. The voltage-divider rule applied to
resistors R_3 and R_4 gives

$$v_o = \frac{R_4}{R_3 + R_4} v_A = \frac{2}{6+2} v_A = \frac{1}{4} v_A \tag{1}$$

Because resistors R_3 and R_4 are in series, we can add their values to obtain their equivalent
resistance $R_s = 2 + 6 = 8 \ \Omega$. The equivalent circuit is shown in Figure 6.2.4b.

Resistors R_s and R_2 are parallel, so we can combine their values to obtain their equivalent
resistance R_p as follows:

$$\frac{1}{R_p} = \frac{1}{10} + \frac{1}{8} = \frac{9}{40}$$

Thus, $R_p = 40/9$. The equivalent circuit is shown in Figure 6.2.4c.

Finally, we apply the voltage-divider rule again to obtain

$$v_A = \frac{40/9}{5 + 40/9} v_s = \frac{8}{17} v_s \tag{2}$$

Using (1) and (2) we obtain

$$v_o = \frac{1}{4} v_A = \frac{1}{4} \left(\frac{8}{17} \right) v_s = \frac{2}{17} v_s$$

A *potentiometer* is a resistance with a sliding electrical pick-off (Figure 6.2.5a).
Thus the resistance R_1 between the sliding contact and ground is a function of the
distance x of the contact from the end of the potentiometer. Potentiometers, commonly
called *pots,* are used as linear and angular position sensors. The volume control knob
on some radios is a potentiometer that is used to adjust the voltage to the speakers.
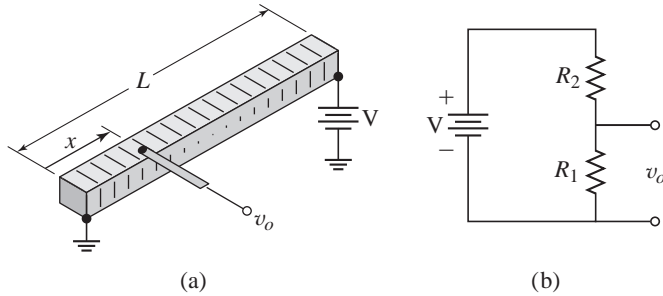
Figure 6.2.5 A translational (linear) potentiometer.

Potentiometers | **EXAMPLE 6.2.4**

■ **Problem**

Assuming the potentiometer resistance R_1 is proportional to x, derive the expression for the output voltage v_o as a function of x.

■ **Solution**

The length of the pot is L and its total resistance is $R_1 + R_2$. Figure 6.2.5b shows the circuit diagram of the system. From the voltage-divider rule,

$$v_o = \frac{R_1}{R_1 + R_2} V \tag{1}$$

Because the resistance R_1 proportional to x,

$$R_1 = (R_1 + R_2)\left(\frac{x}{L}\right)$$

Substituting this into equation (1) gives

$$v_o = \frac{x}{L} V = Kx$$

where $K = V/L$ is the gain of the pot.

The voltage source for the pot can be a battery or it can be a power supply.

A *rotational potentiometer* is shown in Figure 6.2.6. Following a similar procedure we can show that if the resistance is proportional to θ, then

$$v_o = \frac{\theta}{\theta_{\max}} V = K\theta$$

where $K = V/\theta_{\max}$.

Figure 6.2.6 A rotational potentiometer.
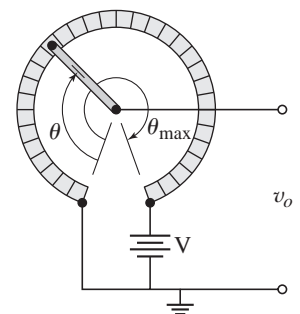
Maximum Power Transfer in a Speaker-Amplifier System | **EXAMPLE 6.2.5**

■ **Problem**

A common example of an electrical system is an amplifier and a speaker. The load is the speaker, which requires current from the amplifier in order to produce sound. In Figure 6.2.7a the resistance R_L is that of the load. Part (b) of the figure shows the circuit representation of the system. The source supplies a voltage v_S and a current i_S, and has its own internal resistance

Figure 6.2.7 (a) An amplifier-speaker system. (b) Circuit representation with a voltage source and a resistive load.

(a)

(b)
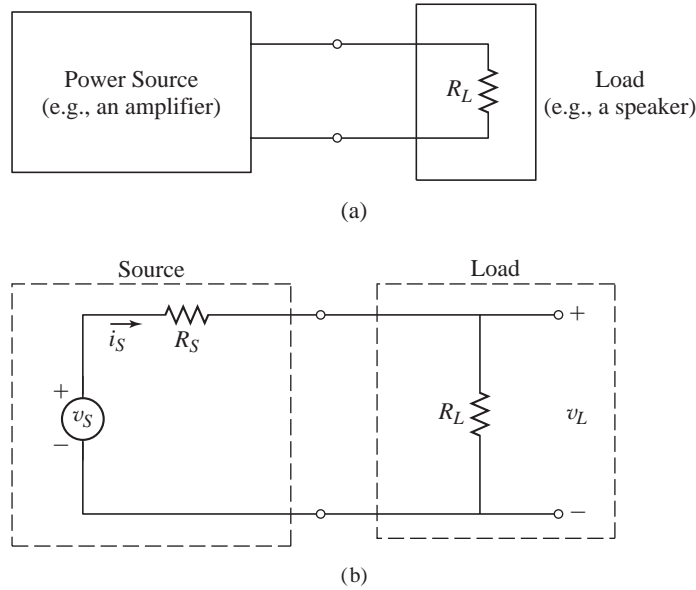
R_S. For optimum efficiency, we want to maximize the power supplied to the speaker, for given values of v_S and R_S. Determine the value of R_L to maximize the power transfer to the load.

■ Solution

The required model should describe the power supplied to the speaker in terms of v_S, R_S, and R_L. From Kirchhoff's voltage law,

$$v_s - i_S R_S - i_S R_L = 0$$

We want to find v_L in terms of v_S. From the voltage-divider rule,

$$v_L = \frac{R_L}{R_S + R_L} v_S$$

The power consumed by the load is $P_L = i_S^2 R_L = v_L^2 / R_L$. Using the relation between v_L and v_S we can express P_L in terms of v_S as

$$P_L = \frac{R_L}{(R_S + R_L)^2} v_S^2$$

To maximize P_L for a fixed value of v_S, we must choose R_L to maximize the ratio

$$r = \frac{R_L}{(R_S + R_L)^2}$$

The maximum of r occurs where $dr/dR_L = 0$.

$$\frac{dr}{dR_L} = \frac{(R_S + R_L)^2 - 2R_L(R_S + R_L)}{(R_S + R_L)^4} = 0$$

This is true if $R_L = R_S$. Thus to maximize the power to the load we should choose the load resistance R_L to be equal to the source resistance R_S. This result for a resistance circuit is a special case of the more general result known as *impedance matching*.

A Feedback Amplifier

EXAMPLE 6.2.6

■ **Problem**

Early in the twentieth century engineers struggled to design vacuum-tube amplifiers whose gain remained constant at a predictable value. The gain is the ratio of the output voltage to the input voltage. The vacuum-tube gain G can be made large but is somewhat unpredictable and unreliable due to heat effects and manufacturing variations. A solution to the problem is shown in Figure 6.2.8. Derive the input-output relation for v_o as a function of v_i. Investigate the case where the gain G is very large.

Figure 6.2.8 A feedback amplifier.

■ **Solution**

Part of the voltage drop across the resistors is used to raise the ground level at the amplifier input, so the input voltage to the amplifier is $v_i - R_2 v_o / (R_1 + R_2)$. Thus the amplifier's output is

$$v_o = G \left(v_i - \frac{R_2}{R_1 + R_2} v_o \right)$$

Solve for v_o:

$$v_o = \frac{G}{1 + G R_2 / (R_1 + R_2)} v_i$$

If $G R_2 / (R_1 + R_2) \gg 1$, then

$$v_o \approx \frac{R_1 + R_2}{R_2} v_i$$

Presumably, the resistor values are sufficiently accurate and constant enough to allow the gain $(R_1 + R_2)/R_2$ to be predictable and reliable.

LOOP CURRENTS

Sometimes the circuit equations can be simplified by using the concept of a *loop current,* which is a current identified with a specific loop in the circuit. A loop current is not necessarily an actual current. If an element is part of two or more loops, the actual current through the element is the algebraic sum of the loop currents. Use of loop currents usually reduces the number of unknowns to be found, although when deriving the circuit equations you must be careful to use the proper algebraic sum for each element.

Analysis with Loop Currents

EXAMPLE 6.2.7

■ **Problem**

We are given the values of the voltages and the resistances for the circuit in Figure 6.2.9a. (a) Solve for the currents i_1, i_2, and i_3 passing through the three resistors. (b) Use the loop-current method to solve for the currents.
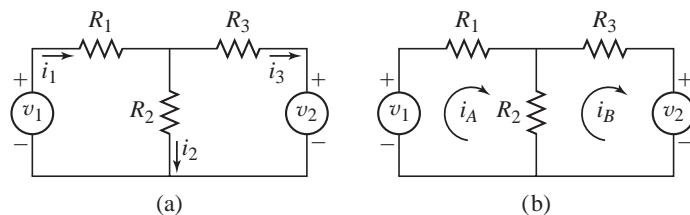
Figure 6.2.9 Example of loop analysis.

■ **Solution**

a.   Applying Kirchhoff's voltage law to the left-hand loop gives

$$v_1 - R_1 i_1 - R_2 i_2 = 0$$

For the right-hand loop,

$$v_2 - R_2 i_2 + i_3 R_3 = 0$$

From conservation of charge, $i_1 = i_2 + i_3$. These are three equations in three unknowns. Their solution is

$$i_1 = \frac{(R_2 + R_3)v_1 - R_2 v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$i_2 = \frac{R_3 v_1 + R_1 v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{1}$$

$$i_3 = \frac{R_2 v_1 - (R_1 + R_2)v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

b.   Define the loop currents i_A and i_B positive clockwise, as shown in Figure 6.2.9b. Note that there is a voltage *drop* $R_2 i_A$ across R_2 due to i_A and a voltage *increase* $R_2 i_B$ due to i_B. Applying Kirchhoff's voltage law to the left-hand loop gives

$$v_1 - R_1 i_A - R_2 i_A + R_2 i_B = 0$$

For the right-hand loop,

$$v_2 + R_3 i_B + R_2 i_B - R_2 i_A = 0$$

Now we have only two equations to solve. Their solution is

$$i_A = \frac{(R_2 + R_3)v_1 - R_2 v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{2}$$

$$i_B = \frac{R_2 v_1 - (R_1 + R_2)v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{3}$$

The current i_1 through R_1 is the same as i_A, and the current i_3 through R_3 is the same as i_B. The current i_2 through R_2 is $i_2 = i_A - i_B$, which gives expression (1) when expressions (2) and (3) are substituted.

CAPACITANCE AND INDUCTANCE IN CIRCUITS

Examples 6.2.8 through 6.2.13 illustrate how models are developed for circuits containing capacitors or inductors.

EXAMPLE 6.2.8

Series *RC* Circuit

■ **Problem**

The resistor and capacitor in the circuit shown in Figure 6.2.10 are said to be in series because the same current flows through them. Obtain the model of the capacitor voltage v_1. Assume that the supply voltage v_s is known.

■ Solution

From Kirchhoff's voltage law, $v_s - Ri - v_1 = 0$. This gives

$$i = \frac{1}{R}(v_s - v_1) \tag{1}$$

For the capacitor,

$$v_1 = \frac{1}{C}\int_0^t i\, dt + \frac{Q_0}{C}$$

Differentiate this with respect to t to obtain

$$\frac{dv_1}{dt} = \frac{1}{C}i$$

Then substitute for i from (1):

$$\frac{dv_1}{dt} = \frac{1}{RC}(v_s - v_1)$$

This the required model. It is often expressed in the following rearranged form:

$$RC\frac{dv_1}{dt} + v_1 = v_s \tag{2}$$

Figure 6.2.10 A series *RC* Circuit.
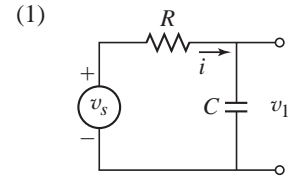
Pulse Response of a Series *RC* Circuit | **EXAMPLE 6.2.9**

■ Problem

A rectangular pulse input is a positive step function that lasts a duration D. One way of producing a step voltage input is to use a switch like that shown in Figure 6.2.11a. The battery voltage V is constant and the switch is initially closed at point B. At $t = 0$ the switch is suddenly moved from point B to point A. Then at $t = D$ the switch is suddenly moved back to point B. Obtain the expression for the capacitor voltage $v_1(t)$ assuming that $v_1(0) = 0$.

■ Solution

When at $t = 0$ the switch is suddenly moved from point B to point A, the circuit is identical to that shown in Figure 6.2.10 with $v_s = V$, and its model is

$$RC\frac{dv_1}{dt} + v_1 = V \tag{1}$$

The input voltage is a step function of magnitude V. Using the methods of Chapter 3, we can obtain the following solution for v_1 as a function of time. Since $v_1(0) = 0$, the solution is the forced response.

$$v_1(t) = V\left(1 - e^{-t/RC}\right) \tag{2}$$

When the switch is moved back to point B at time $t = D$, the circuit is equivalent to that shown in Figure 6.2.11b, whose model is equation (1) with $V = 0$.

(a) (b)
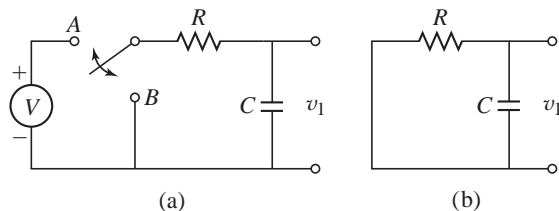
Figure 6.2.11 (a) Series *RC* circuit with a switch. (b) Circuit with switch in position B.

Figure 6.2.12 Pulse response of a series *RC* circuit.
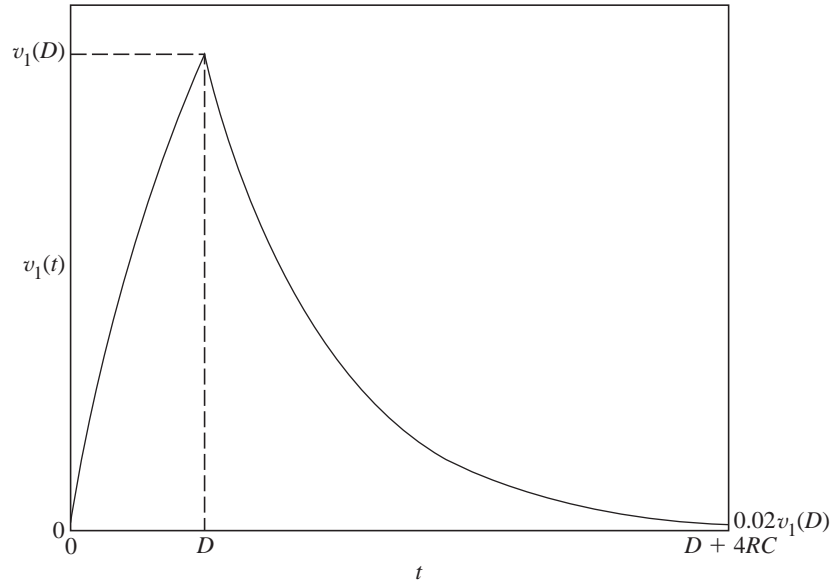
The solution of this equation for $t \geq D$ is simply the free response with the initial condition $v_1(D)$, whose expression can be obtained from equation (2).

$$v_1(t) = v_1(D)e^{-(t-D)/RC} = V\left(1 - e^{-D/RC}\right)e^{-(t-D)/RC}$$

The solution is sketched in Figure 6.2.12.

EXAMPLE 6.2.10

Series *RCL* Circuit

■ Problem

The resistor, inductor, and capacitor in the circuit shown in Figure 6.2.13 are in series because the same current flows through them. Obtain the model of the capacitor voltage v_1 with the supply voltage v_s as the input.

Figure 6.2.13 A series *RCL* circuit.

■ Solution

From Kirchhoff's voltage law,

$$v_s - Ri - L\frac{di}{dt} - v_1 = 0 \tag{1}$$

For the capacitor,

$$v_1 = \frac{1}{C}\int_0^t i\,dt$$

Differentiate this with respect to t to obtain

$$i = C\frac{dv_1}{dt}$$

and substitute this for i in (1):

$$v_s - RC\frac{dv_1}{dt} - LC\frac{d^2v_1}{dt^2} - v_1 = 0$$

This is the required model. It can be expressed in the following form:

$$LC\frac{d^2v_1}{dt^2} + RC\frac{dv_1}{dt} + v_1 = v_s \tag{2}$$

| Parallel *RL* Circuit | **EXAMPLE 6.2.11** |

■ Problem

The resistor and inductor in the circuit shown in Figure 6.2.14 are said to be in parallel because they have the same voltage v_1 across them. Obtain the model of the current i_2 passing through the inductor. Assume that the supply current i_s is known.

Figure 6.2.14 A parallel *RL* circuit.

■ Solution

The currents i_1 and i_2 are defined in the figure. Then,

$$v_1 = L\frac{di_2}{dt} = Ri_1 \tag{1}$$

From conservation of charge, $i_1 + i_2 = i_s$. Thus, $i_1 = i_s - i_2$. Substitute this expression into (1) to obtain

$$L\frac{di_2}{dt} = R(i_s - i_2)$$

This is the required model. It can be rearranged as follows:

$$\frac{L}{R}\frac{di_2}{dt} + i_2 = i_s \tag{2}$$
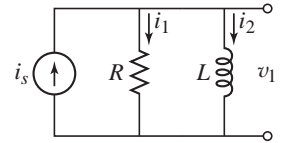
| Analysis of a Telegraph Line | **EXAMPLE 6.2.12** |

■ Problem

Figure 6.2.15 shows a circuit representation of a telegraph line. The resistance R is the line resistance and L is the inductance of the solenoid that activates the receiver's clicker. The switch represents the operator's key. Assume that when sending a "dot," the key is closed for 0.1 s. Using the values $R = 20\ \Omega$ and $L = 4$ H, obtain the expression for the current $i(t)$ passing through the solenoid.

Figure 6.2.15 Circuit representation of a telegraph line.

■ Solution

From the voltage law we have

$$L\frac{di}{dt} + Ri = v_i(t) \tag{1}$$

where $v_i(t)$ represents the input voltage due to the switch and the 12-V supply. We could model $v_i(t)$ as a rectangular pulse of height 12 V and duration 0.1 s, but the differential equation (1) is easier to solve if we model $v_i(t)$ as an impulsive input of strength $12(0.1) = 1.2$ V · s. This model can be justified by the fact that the circuit time constant, $L/R = 4/20 = 0.2$, is greater than the duration of $v_i(t)$. Thus we model $v_i(t)$ as $v_i(t) = 1.2\delta(t)$. The Laplace transform of equation (1) with $i(0) = 0$ gives

$$(4s + 20)I(s) = 1.2 \quad \text{or} \quad I(s) = \frac{1.2}{4s + 20} = \frac{0.3}{s + 5}$$

This gives the solution

$$i(t) = 0.3e^{-5t}\ \text{A}$$

Note that this solution gives $i(0+) = 0.3$, whereas $i(0) = 0$. The difference is due to the impulsive input.

EXAMPLE 6.2.13

An *RLC* Circuit with Two Input Voltages

■ Problem

Figure 6.2.16 An *RLC* circuit with two voltage sources.
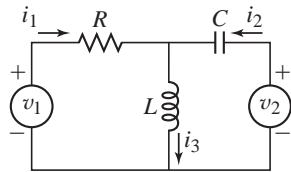
The *RLC* circuit shown in Figure 6.2.16 has two input voltages. Obtain the differential equation model for the current i_3.

■ Solution

Applying Kirchhoff's voltage law to the left-hand loop gives

$$v_1 - Ri_1 - L\frac{di_3}{dt} = 0 \tag{1}$$

For the right-hand loop,

$$v_2 - \frac{1}{C}\int i_2\, dt - L\frac{di_3}{dt} = 0$$

Differentiate this equation with respect to t:

$$\frac{dv_2}{dt} - \frac{1}{C}i_2 - L\frac{d^2i_3}{dt^2} = 0 \tag{2}$$

From conservation of charge,

$$i_3 = i_1 + i_2 \tag{3}$$

These are three equations in the three unknowns i_1, i_2, and i_3. To eliminate i_1 and i_2, solve equation (1) for i_1

$$i_1 = \frac{1}{R}\left(v_1 - L\frac{di_3}{dt}\right) \tag{4}$$

In equation (2), substitute for i_2 from equation (3):

$$\frac{dv_2}{dt} - \frac{1}{C}(i_3 - i_1) - L\frac{d^2i_3}{dt^2} = 0$$

Now substitute for i_1 from equation (4):

$$\frac{dv_2}{dt} - \frac{1}{C}i_3 + \frac{1}{C}\left[\frac{1}{R}\left(v_1 - L\frac{di_3}{dt}\right)\right] - L\frac{d^2i_3}{dt^2} = 0$$

Rearrange this equation to obtain the answer:

$$LRC\frac{d^2i_3}{dt^2} + L\frac{di_3}{dt} + Ri_3 = v_1 + RC\frac{dv_2}{dt} \tag{5}$$

STATE-VARIABLE MODELS OF CIRCUITS

The presence of several current and voltage variables in a circuit can sometimes lead to difficulty in identifying the appropriate variables to use for expressing the circuit model. Use of state variables can often reduce this confusion. To choose a proper set of state variables, identify where the energy is stored in the system. The variables that describe the stored energy are appropriate choices for state variables.

State-Variable Model of a Series *RLC* Circuit | **EXAMPLE 6.2.14**

■ **Problem**

Consider the series *RLC* circuit shown in Figure 6.2.17. Choose a suitable set of state variables, and obtain the state variable model of the circuit in matrix form. The input is the voltage v_s.
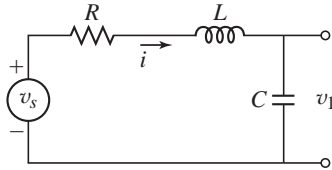
Figure 6.2.17 Series *RLC* circuit.

■ **Solution**

In this circuit the energy is stored in the capacitor and in the inductor. The energy stored in the capacitor is $Cv_1^2/2$ and the energy stored in the inductor is $Li^2/2$. Thus a suitable choice of state variables is v_1 and i.

From Kirchhoff's voltage law,

$$v_s - Ri - L\frac{di}{dt} - v_1 = 0$$

Solve this for di/dt:

$$\frac{di}{dt} = \frac{1}{L}v_s - \frac{1}{L}v_1 - \frac{R}{L}i$$

This is the first state equation.

Now find an equation for dv_1/dt. From the capacitor relation,

$$v_1 = \frac{1}{C}\int i\, dt$$

Differentiating gives the second state equation.

$$\frac{dv_1}{dt} = \frac{1}{C}i$$

The two state equations can be expressed in matrix form as follows.

$$\begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dv_1}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v_1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} v_s$$

6.3 IMPEDANCE AND AMPLIFIERS

The Laplace transform and the transfer function concept enable us to deal with algebraic equations rather than differential equations, and thus ease the task of analyzing circuit models. This section illustrates why this is so, and introduces a related concept, impedance, which is simply the transfer function between voltage and current.

Coupled *RC* Loops | **EXAMPLE 6.3.1**

■ **Problem**

Determine the transfer function $V_o(s)/V_s(s)$ of the circuit shown in Figure 6.3.1.
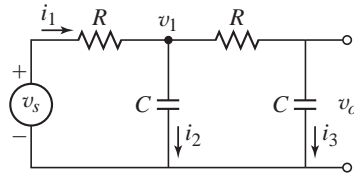
■ Solution

The energy in this circuit is stored in the two capacitors. Because the energy stored in a capacitor is expressed by $Cv^2/2$, appropriate choices for the state variables are the voltages v_1 and v_o. The capacitance relations are

$$\frac{dv_o}{dt} = \frac{i_3}{C} \qquad \frac{dv_1}{dt} = \frac{i_2}{C} \tag{1}$$

For the right-hand loop,

$$i_3 = \frac{v_1 - v_o}{R} \tag{2}$$

From equation (1)

$$\frac{dv_o}{dt} = \frac{1}{RC}(v_1 - v_o) \tag{3}$$

For the left-hand loop,

$$i_1 = \frac{v_s - v_1}{R} \tag{4}$$

From conservation of charge and equations (2) and (4),

$$i_2 = i_1 - i_3 = \frac{v_s - v_1}{R} - \frac{v_1 - v_o}{R} = \frac{1}{R}(v_s - 2v_1 + v_o) \tag{5}$$

Using this with equation (2) gives

$$\frac{dv_1}{dt} = \frac{1}{RC}(v_s - 2v_1 + v_o) \tag{6}$$

Equations (3) and (6) are the state equations. To obtain the transfer function $V_o(s)/V_s(s)$, transform these equations for zero initial conditions to obtain

$$sV_o(s) = \frac{1}{RC}[V_1(s) - V_o(s)]$$

$$sV_1(s) = \frac{1}{RC}[V_s(s) - 2V_1(s) + V_o(s)]$$

Eliminating $V_1(s)$ from these two equations gives the transfer function

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1} \tag{7}$$

IMPEDANCE

We have seen that a resistance resists or "impedes" the flow of current. The corresponding relation is $v/i = R$. Capacitance and inductance elements also impede the flow of current. In electrical systems an *impedance* is a generalization of the resistance concept and is defined as the ratio of a voltage transform $V(s)$ to a current transform $I(s)$ and thus implies a current source. A standard symbol for impedance is $Z(s)$. Thus

$$Z(s) = \frac{V(s)}{I(s)} \tag{6.3.1}$$

The impedance of a resistor is its resistance R. The impedances of the other two common electrical elements are found as follows. For a capacitor,

$$v(t) = \frac{1}{C} \int_0^t i \, dt$$

or $V(s) = I(s)/C(s)$. The impedance of a capacitor is thus

$$Z(s) = \frac{1}{Cs} \qquad (6.3.2)$$

For an inductor,

$$v(t) = L\frac{di}{dt}$$

or $V(s) = LsI(s)$. Thus the impedance of an inductor is

$$Z(s) = Ls \qquad (6.3.3)$$

SERIES AND PARALLEL IMPEDANCES

The concept of impedance is useful because the impedances of individual elements can be combined with series and parallel laws to find the impedance at any point in the system. The laws for combining series or parallel impedances are extensions to the dynamic case of the laws governing series and parallel resistance elements. Two impedances are in *series* if they have the *same current*. If so, the total impedance is the sum of the individual impedances.

$$Z(s) = Z_1(s) + Z_2(s)$$

For example, a resistor R and capacitor C in series, as shown in Figure 6.3.2a, have the equivalent impedance

$$Z(s) = R + \frac{1}{Cs} = \frac{RCs + 1}{Cs}$$

Thus the relation between the current i flowing through them and the total voltage drop v across them is

$$\frac{V(s)}{I(s)} = Z(s) = \frac{RCs + 1}{Cs}$$

and the differential equation model is

$$C\frac{dv}{dt} = RC\frac{di}{dt} + i(t)$$

If the impedances have the *same voltage difference* across them, they are in *parallel,* and their impedances combine by the reciprocal rule

$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}$$
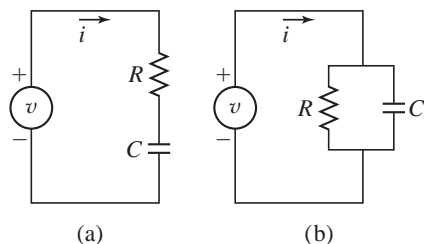
Figure 6.3.2 Series and parallel *RC* circuits.

where $Z(s)$ is the total equivalent impedance. If a resistor R and capacitor C are in parallel, as shown in Figure 6.3.2b, their equivalent total impedance $Z(s)$ is found from

$$\frac{1}{Z(s)} = \frac{1}{1/Cs} + \frac{1}{R}$$

or

$$Z(s) = \frac{R}{RCs + 1}$$

Thus the relation between the total current i and the voltage drop v across them is

$$\frac{V(s)}{I(s)} = Z(s) = \frac{R}{RCs + 1}$$

and the differential equation model is

$$RC\frac{dv}{dt} + v = Ri(t)$$

EXAMPLE 6.3.2

Circuit Analysis Using Impedance

■ Problem

For the circuit shown in Figure 6.3.3a, determine the transfer function between the input voltage v_s and the output voltage v_o.

■ Solution

Note that R and C are in parallel. Therefore their equivalent impedance $Z(s)$ is found from

$$\frac{1}{Z(s)} = \frac{1}{1/Cs} + \frac{1}{R}$$

or

$$Z(s) = \frac{R}{RCs + 1}$$

An impedance representation of the equivalent circuit is shown in Figure 6.3.3b. In this representation we may think of the impedance as a simple resistance, provided we express the relations in Laplace transform notation. Kirchhoff's voltage law gives

$$V_s(s) - R_1 I(s) - Z(s)I(s) = 0$$

The output voltage is related to the current by $V_o(s) = Z(s)I(s)$. Eliminating $I(s)$ from these two relations gives

$$V_s(s) - R_1\frac{V_o(s)}{Z(s)} - V_o(s) = 0$$

Figure 6.3.3 Circuit analysis using impedance.

(a)   (b)
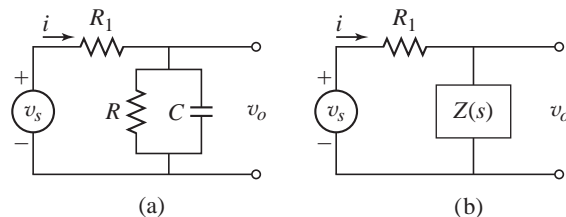
which yields the desired transfer function:

$$\frac{V_o(s)}{V_s(s)} = \frac{Z(s)}{R_1 + Z(s)} = \frac{R}{RR_1Cs + R + R_1}$$

This network is a first-order system whose time constant is

$$\tau = \frac{RR_1}{R + R_1}$$

If the voltage output is measured at the terminals to which the driving current is applied, the impedance so obtained is the *driving-point* or *input* impedance. If the voltage is measured at another place in the circuit, the impedance obtained is a *transfer* impedance (because the effect of the input current has been transferred to another point). Sometimes the term *admittance* is used. This is the reciprocal of impedance, and it is an indication of to what extent a circuit "admits" current flow.

ISOLATION AMPLIFIERS

A *voltage-isolation amplifier* is designed to produce an output voltage that is proportional to the input voltage. It is intended to boost the electrical signal from a low-power source. Therefore, the amplifier requires an external power source, which is not usually shown in the circuit diagrams. We will not be concerned with the internal design details of amplifiers, but we need to understand their effects on any circuit in which they are used.

Such an amplifier may be considered to be a voltage source if it does not affect the behavior of the source circuit that is attached to the amplifier input terminals, and if the amplifier is capable of providing the voltage independently of the particular circuit (the "load") attached to amplifier output terminals.

Consider the system in Figure 6.3.4. The internal impedances of the amplifier at its input and output terminals are $Z_i(s)$ and $Z_o(s)$, respectively. The impedance of the source circuit is $Z_s(s)$ and the impedance of the load is $Z_L(s)$. A simple circuit analysis will reveal that

$$V_s(s) - I_1(s)Z_s(s) - V_i(s) = 0$$

and $I_1(s) = V_i(s)/Z_i(s)$. Thus, if $Z_i(s)$ is large, the current i_1 drawn by the amplifier will be small. Therefore, if the input impedance $Z_i(s)$ is large, the amplifier does not affect the current i_1, and thus the amplifier does not affect the behavior of the source circuit. In addition, if $Z_i(s)$ is large,

$$V_i(s) = \frac{Z_i(s)}{Z_i(s) + Z_s(s)} V_s(s) \approx V_s(s)$$

So we conclude that a voltage-isolation amplifier must have a high input impedance.
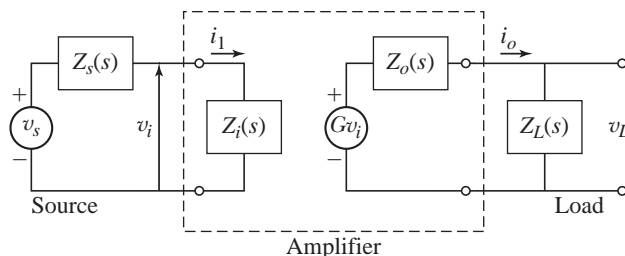
Figure 6.3.4 Input and output impedance in an amplifier.

Figure 6.3.5 An op-amp
multiplier.

(a)   (b)
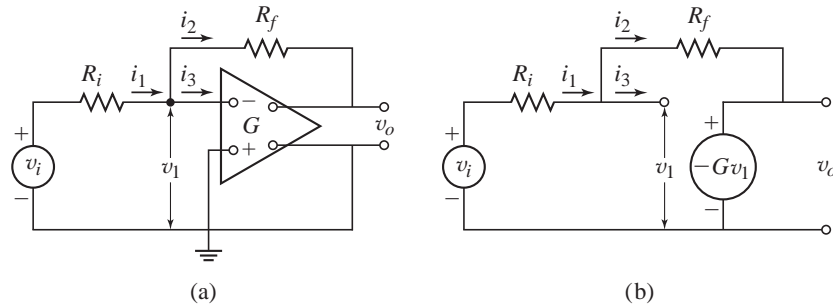
Denote the amplifier's voltage gain as G. This means that the amplifier's output
voltage v_o is $v_o = Gv_i$. For the load circuit,

$$GV_i(s) - I_o(s)Z_o(s) - I_o(s)Z_L(s) = 0$$

Thus

$$I_o(s) = \frac{GV_i(s)}{Z_o(s) + Z_L(s)}$$

and

$$V_L(s) = Z_L(s)I_o(s) = \frac{Z_L(s)}{Z_o(s) + Z_L(s)}GV_o(s)$$

Thus, if $Z_o(s)$ is small, $v_L \approx Gv_o$. So if the amplifier output impedance is small the
voltage v_L delivered to the load is independent of the load.

A similar analysis applies to a *current amplifier,* which provides a current propor-
tional to its input signal regardless of the load. Thus such an amplifier acts as a current
source.

OPERATIONAL AMPLIFIERS

A modern version of the feedback amplifier discussed in Example 6.2.6 is the *opera-
tional amplifier* (op amp), which is a voltage amplifier with a very large gain G (greater
than 10^5). The op amp has a large input impedance so it draws a negligible current.
The op amp is an integrated circuit chip that contains many transistors, capacitors, and
resistors and has several external terminals. We can attach two resistors in series with
and parallel to the op amp, as shown in Figure 6.3.5a. This circuit diagram does not
show all of the op amp's external terminals; for example, some terminals are needed
to power the device and to provide constant bias voltages. Our diagram shows only
two pairs of terminals: the input terminals intended for time-varying input signals and
the output terminals. A plus sign or a minus sign on an input terminal denotes it as a
noninverting terminal or an *inverting* terminal, respectively.

Op amps are widely used in instruments and control systems for multiplying,
integrating, and differentiating signals.

EXAMPLE 6.3.3   Op-Amp Multiplier

■ **Problem**

Determine the relation between the input voltage v_i and the output voltage v_o of the op-amp
circuit shown in Figure 6.3.5a. Assume that the op amp has the following properties:

1.   The op-amp gain G is very large,

2. $v_o = -Gv_1$; and
3. The op-amp input impedance is very large, and thus the current i_3 drawn by the op amp is very small.

■ Solution

Because the current i_3 drawn by the op amp is very small, the input terminal pair can be represented as an open circuit, as in Figure 6.3.5b. The voltage-current relation for each resistor gives

$$i_1 = \frac{v_i - v_1}{R_1}$$

and

$$i_2 = \frac{v_1 - v_o}{R_2}$$

From conservation of charge, $i_1 = i_2 + i_3$. However, from property 3, $i_3 \approx 0$, which implies that $i_1 \approx i_2$. Thus,

$$\frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_2}$$

From property 1, $v_1 = -v_o/G$. Substitute this into the preceeding equation:

$$\frac{v_i}{R_1} + \frac{v_o}{R_1 G} = -\frac{v_o}{R_1 G} - \frac{v_o}{R_2}$$

Because G is very large, the terms containing G in the denominator are very small, and we obtain

$$\frac{v_i}{R_1} = -\frac{v_o}{R_2}$$

Solve for v_o:

$$v_o = -\frac{R_2}{R_1} v_i \tag{1}$$

This circuit can be used to multiply a voltage by the factor R_2/R_1, and is called an op-amp *multiplier*. Note that this circuit inverts the sign of the input voltage.

Resistors usually can be made so that their resistance values are known with sufficient accuracy and are constant enough to allow the gain R_2/R_1 to be predictable and reliable.

GENERAL OP-AMP INPUT-OUTPUT RELATION

We can use the impedance concept to simplify the process of obtaining a model of an op-amp circuit. A circuit diagram of the op amp with general feedback and input elements $Z_f(s)$ and $Z_i(s)$ is shown in Figure 6.3.6a. A similar but simplified form is given in part (b). The impedance $Z_i(s)$ of the input elements is defined such that

$$V_i(s) - V_1(s) = Z_i(s)I_1(s)$$

For the feedback elements,

$$V_1(s) - V_o(s) = Z_f(s)I_2(s)$$

The high internal impedance of the op amp implies that $i_1 \approx 0$, and thus $i_1 \approx i_2$. The final relation we need is the amplifier relation $v_o = -Gv_1$ or

$$V_o(s) = -GV_1(s) \tag{6.3.4}$$

Figure 6.3.6 General circuit representation of an op-amp system.

(a)   (b)
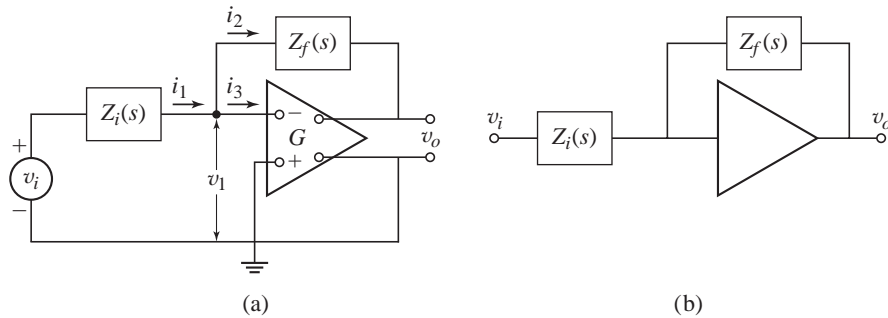
Figure 6.3.7 An op-amp multiplier with an inverter.
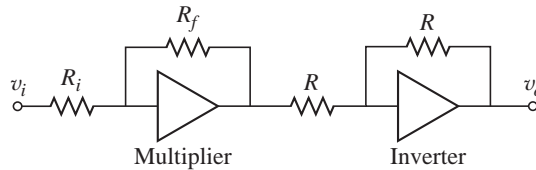
When the preceding relations are used to eliminate $I_1(s)$ and $I_2(s)$, the result is

$$V_1(s) = \frac{Z_f(s)}{Z_f(s) + Z_i(s)} V_i(s) + \frac{Z_i(s)}{Z_f(s) + Z_i(s)} V_o(s)$$

Using (6.3.4) to eliminate $V_1(s)$, the transfer function between $V_i(s)$ and $V_o(s)$ is found to be

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_f(s) + Z_i(s)} \frac{G}{1 + GH(s)}$$

where

$$H(s) = \frac{Z_i(s)}{Z_f(s) + Z_i(s)}$$

Because G is a very large number, $|GH(s)| \gg 1$, and we obtain

$$\frac{V_o(s)}{V_i(s)} \approx -\frac{Z_f(s)}{Z_i(s)} \tag{6.3.5}$$

This is the transfer function model for op-amp circuits.

An op-amp multiplier is created by using two resistors as shown in Figure 6.3.5a, where $Z_i(s) = R_i$ and $Z_f(s) = R_f$. Thus,

$$\frac{V_o(s)}{V_i(s)} \approx -\frac{R_f}{R_i}$$

The gain of this multiplier is R_f/R_i, with a sign reversal. In some applications, we want to eliminate the sign reversal. We can do this by using an *inverter,* which is a multiplier having equal resistances. Using an inverter in series with the multiplier, as shown in Figure 6.3.7, eliminates the overall sign reversal.

EXAMPLE 6.3.4

Integration with Op Amps

■ **Problem**
Determine the transfer function $V_o(s)/V_i(s)$ of the circuit shown in Figure 6.3.8.

■ Solution

The impedance of a capacitor is $1/Cs$. Thus, the transfer function of the circuit is found from (6.3.5) with $Z_i(s) = R$ and $Z_f(s) = 1/Cs$. It is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -\frac{1}{RCs}$$

Thus in the time domain, the circuit model is

$$v_o = -\frac{1}{RC} \int_0^t v_i \, dt \qquad (1)$$

Thus the circuit integrates the input voltage, reverses its sign, and divides it by RC. It is called an op-amp *integrator,* and is used in many devices for computing, signal generation, and control, some of which will be analyzed in later chapters.
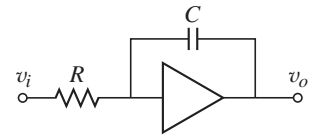
| Differentiation with Op Amps | **EXAMPLE 6.3.5** |

■ Problem

Design an op-amp circuit that differentiates the input voltage.

■ Solution

In theory, a differentiator can be created by interchanging the resistance and capacitance in the integrator circuit. The result is shown in Figure 6.3.9, where $Z_i(s) = 1/Cs$ and $Z_f(s) = R$. The input-output relation for this ideal differentiator is

Figure 6.3.9 An op-amp differentiator.

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -RCs$$

The model in the time domain is

$$v_o(t) = -RC\frac{dv_i(t)}{dt}$$

The difficulty with this design is that no electrical signal is "pure." Contamination always exists as a result of voltage spikes, ripple, and other transients generally categorized as "noise." These high-frequency signals have large slopes compared with the more slowly varying primary signal, and thus they will dominate the output of the differentiator. Example 8.1.5 in Chapter 8 shows an improved differentiator design that does not have this limitation.

6.4 ELECTRIC MOTORS

Electromechanical systems consist of an electrical subsystem and a mechanical subsystem with mass and possibly elasticity and damping. In some devices, such as motors and speakers, the mass is driven by a force generated by the electrical subsystem. In other devices, such as microphones, the motion of the mass generates a voltage or current in the electrical subsystem. Thus, we must apply electrical principles and Newton's laws to develop a model of an electromechanical system. Often the forces and torques are generated electromagnetically, but other methods are used as well; for example, *piezoelectric* devices contain crystals that generate forces when a voltage is applied to them.

In this section and Section 6.5 we treat electric motors. In Section 6.6 we treat other electromechanical devices.

MAGNETIC COUPLING

The majority of electromechanical devices utilize a magnetic field. Two basic principles apply to a conductor, such as a wire, carrying a current within a magnetic field: (1) a force is exerted on the conductor by the field and (2) if the conductor moves relative to the field, the field induces a voltage in the conductor that opposes the voltage producing the current. In many applications, the following model relates the force f to the current i:

$$f = BLi \tag{6.4.1}$$

where B is the flux density of the field and L is the length of the conductor. In SI the units of B are webers per square meter (Wb/m^2). The direction of the force, which is perpendicular to the conductor and the field, can be found with the right-hand rule. Sweep the fingers from the positive current direction to the positive field direction; the thumb will point in the positive force direction. Equation (6.4.1) is a special case of the more general physical principle. It applies to two commonly found situations: (1) straight conductors that are perpendicular to a uniform magnetic field and (2) circular conductors in a radial field.

When the directions of the field, the conductor, and its velocity are mutually perpendicular, the second principle can be expressed as

$$v_b = BLv \tag{6.4.2}$$

where v_b is the voltage induced in the conductor by its velocity v in the field. Again using the right-hand rule, we find the positive direction of the current induced by v_b by sweeping the fingers from the positive direction of v to the positive direction of the field.

The two principles and the expressions (6.4.1) and (6.4.2) can be represented graphically as in Figure 6.4.1. The circuit represents the electrical behavior of the conductor and the mass m represents the mass of the conductor and any attached mass. The power generated by the circuit is $v_b i$. The power applied to the mass m by the force f is fv. Neglecting any energy loss due to resistance in the conductor or friction or damping acting on the mass, we see that no power will be lost between the electrical subsystem and the mechanical subsystem, and thus,

$$v_b i = fv = BLiv$$

from which we obtain (6.4.2). In addition, from Newton's law,

$$m\dot{v} = f = BLi$$

The basic principles underlying the operation of electric motors can be best understood by first considering a simpler device called the *D'Arsonval meter,* named after its inventor. The device is also known as a *galvanometer*.

Figure 6.4.1 Electromagnetic interaction between electrical and mechanical subsystems.

THE D'ARSONVAL METER

A D'Arsonval meter can be used to measure current (Figure 6.4.2a). The current to be measured is passed through a coil to which a pointer is attached. The coil is positioned within a magnetic field and is wrapped around an iron core to strengthen the effects of the field. The core thus acts like an inductor. The interaction between the current and the field produces a torque that tends to rotate the coil and pointer. This rotation is opposed by a torsional spring of stiffness k_T.
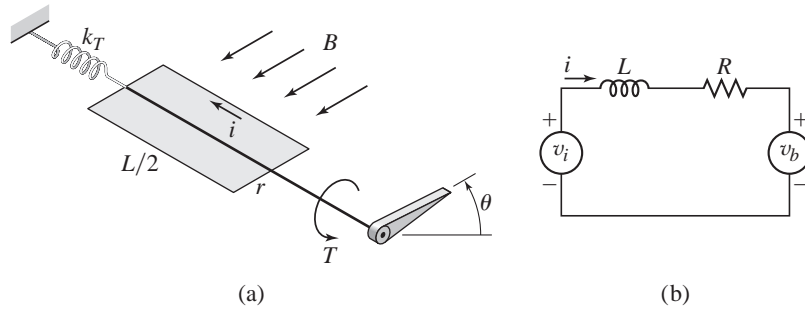
A Model of the D'Arsonval Meter

■ Problem

Derive a model of a D'Arsonval meter in terms of the coil angular displacement θ and the coil current i. The input is the applied voltage v_i. Discuss the case where there are n coils around the core.

■ Solution

Let the length of one side of the coil be $L/2$ and its radius be r. Then the torque T acting on both sides of the coil due to the magnetic field B is

$$T = fr = \left(2B\frac{L}{2}i\right)r = (BLr)i$$

If a torsional viscous damping torque $c\dot{\theta}$, for example, due to air resistance or damping in the bearings, also acts on the core shaft as it rotates, the equation of motion of the core/coil unit is

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k_T\theta = T = (BLr)i \tag{1}$$

where I is the inertia of the core/coil unit.

The rotation of the coil induces a voltage v_b in the coil that is proportional to the coil's linear velocity v such that $v_b = BLv$. The linear velocity is related to the coil's angular velocity $\dot{\theta}$ by $v = r\dot{\theta}$. Thus,

$$v_b = BLv = BLr\frac{d\theta}{dt}$$

The coil circuit is represented in part (b) of Figure 6.4.2, where R represents the resistance of the wire in the coil. Kirchhoff's voltage law applied to the coil circuit gives

$$v_i - L\frac{di}{dt} - Ri - v_b = 0$$

or

$$L\frac{di}{dt} + Ri + BLr\frac{d\theta}{dt} = v_i \tag{2}$$

The model consists of equations (1) and (2). Note that the system model is third order.

If there are n coils, the resulting torque expression is $T = n(BLr)i$ and the induced voltage expression is $v_b = nBLr\dot{\theta}$. Thus equations (1) and (2) become

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k_T\theta = n(BLr)i \tag{3}$$

$$L\frac{di}{dt} + Ri + nBLr\frac{d\theta}{dt} = v_i \tag{4}$$

Note that if the applied voltage v_i is constant, the system will reach a steady-state in which the pointer comes to rest. At steady-state, $\dot{\theta} = di/dt = 0$, and equation (4) gives

$$i = \frac{v_i}{R}$$

and equation (3) gives

$$\theta = \frac{nBLri}{k_T} = \frac{nBLrv_i}{Rk_T}$$

This equation can be used to calibrate the device by relating the pointer displacement θ to either the measured current i or the measured voltage v_i.

DC MOTORS

There are many types of electric motors, but the two main categories are *direct current (dc)* motors and *alternating current (ac)* motors. Within the dc motor category there are the armature-controlled motor and the field-controlled motor.

The basic elements of a motor, like that shown in Figure 6.4.3, are the *stator,* the *rotor,* the *armature,* and the *commutator*. The stator is stationary and provides the magnetic field. The rotor is an iron core that is supported by bearings and is free to rotate. The coils are attached to the rotor, and the combined unit is called the armature. A dc motor operates on the same principles as a D'Arsonval meter, but the design of a practical dc motor requires the solution of the problems caused by the fact that the coils must be free to rotate continually. As a coil rotates through 180° the torque will reverse direction unless the current can be made to reverse direction also. In addition, a means must be found to maintain electrical contact between the rotating coil and the power supply leads. A solution is provided by the commutator, which is a pair of electrically conducting, spring-loaded carbon sticks (called *brushes*) that slide on the armature and transfer power to the coil contacts.

The stator may be a permanent magnet or an electromagnet with its own separate power supply, which creates additional cost. It is now possible to manufacture permanent magnets of high field intensity and armatures of low inertia so that permanent-magnet motors with a high torque-to-inertia ratio are now available.

Figure 6.4.3 Cutaway view of a permanent magnet motor.
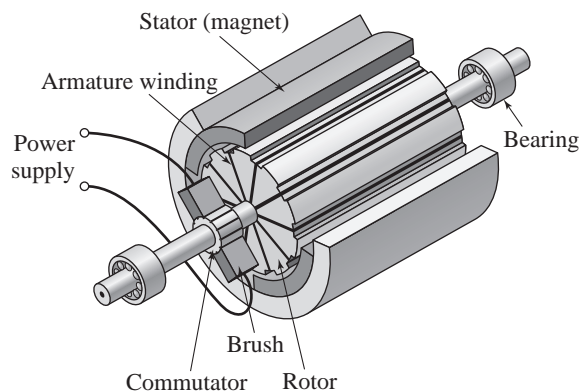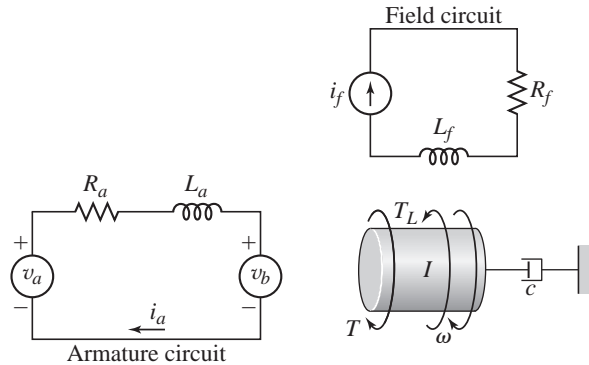
Figure 6.4.4 Diagram of an armature-controlled dc motor.

MODEL OF AN ARMATURE-CONTROLLED DC MOTOR

We now develop a model for the armature-controlled motor shown in Figure 6.4.4. The armature voltage v_a is the input, and the armature current i_a and motor speed ω are the outputs.

The electrical subsystems of the motor can be represented by the armature circuit and the field circuit in Figure 6.4.4. In a permanent-magnet motor, the field circuit is replaced by the magnet. The mechanical subsystem consists of the inertia I and the damping c. The inertia is due to the load inertia as well as the armature inertia. Damping can be present because of shaft bearings or load damping, such as with a fan or pump. The external torque T_L represents an additional torque acting on the load, other than the damping torque. The load torque T_L opposes the motor torque in most applications, so we have shown it acting in the direction opposite that of T. However, sometimes the load torque assists the motor. For example, if the load is the wheel of a vehicle, then T_L could be the torque produced by gravity as the vehicle ascends or descends a hill. When descending, the load torque assists the motor, and in such a case we would reverse the direction of T_L shown in Figure 6.4.4.

The motor produces a torque T that is proportional to the armature current i_a. This relation can be derived by noting that the force on the armature due to the magnetic field is, from (6.4.1), $f = nBLi_a$, where n is the number of armature coils. If the armature radius is r, then the torque on the armature is

$$T = (nBLi_a)r = (nBLr)i_a = K_T i_a \tag{6.4.3}$$

where $K_T = nBLr$ is the motor's *torque constant*. This relation can be used by motor designers to determine the effect of changing the number of coils, the field strength, or the armature geometry. The user of such motors (as opposed to the motor's designer) can obtain values of K_T for a specific motor from the manufacturer's literature.

As we have seen, the motion of a current-carrying conductor in a field produces a voltage in the conductor that opposes the current. This voltage in the armature is called the *back emf* (for *electromotive force,* an older term for voltage). Its magnitude is proportional to the speed. The coils' linear velocity v is related to their angular velocity by $v = r\omega$. Thus, from (6.4.2),

$$v_b = nBLv = (nBLr)\omega = K_b\omega \tag{6.4.4}$$

where $K_b = nBLr$ is the motor's *back emf constant,* and is sometimes called the *voltage constant*. Note that the expressions for K_T and K_b are identical and thus, K_T

and K_b have the same numerical value if expressed in the same units. For this reason, motor manufacturers usually do not give values for K_b.

The back emf is a voltage drop in the armature circuit. Thus, Kirchhoff's voltage law gives

$$v_a - R_a i_a - L_a \frac{di_a}{dt} - K_b \omega = 0 \tag{6.4.5}$$

From Newton's law applied to the inertia I,

$$I \frac{d\omega}{dt} = T - c\omega - T_L = K_T i_a - c\omega - T_L \tag{6.4.6}$$

Equations (6.4.5) and (6.4.6) constitute the system model.

Motor Transfer Functions   Normally we are interested in both the motor speed ω and the current i_a. The two inputs are the applied voltage v_a and the load torque T_L. Thus there are four transfer functions for the motor, one transfer function for each input-output pair. We can obtain these transfer functions by transforming (6.4.5) and (6.4.6) and solving for $I_a(s)$ and $\Omega(s)$. The result for the output $I_a(s)$ is

$$\frac{I_a(s)}{V_a(s)} = \frac{Is + c}{L_a I s^2 + (R_a I + c L_a) s + c R_a + K_b K_T} \tag{6.4.7}$$

$$\frac{I_a(s)}{T_L(s)} = \frac{K_b}{L_a I s^2 + (R_a I + c L_a) s + c R_a + K_b K_T} \tag{6.4.8}$$

For the output $\Omega(s)$,

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a I s^2 + (R_a I + c L_a) s + c R_a + K_b K_T} \tag{6.4.9}$$

$$\frac{\Omega(s)}{T_L(s)} = -\frac{L_a s + R_a}{L_a I s^2 + (R_a I + c L_a) s + c R_a + K_b K_T} \tag{6.4.10}$$

The denominator is the same in each of the motor's four transfer functions. It is the characteristic polynomial and it gives the characteristic equation:

$$L_a I s^2 + (R_a I + c L_a) s + c R_a + K_b K_T = 0 \tag{6.4.11}$$

Note that $I_a(s)/V_a(s)$ and $\Omega(s)/T_L(s)$ have numerator dynamics. This can cause a large overshoot in i_a if v_a is a step function, and a large overshoot in ω if T_L is a step function.

State-Variable Form of the Motor Model   Equations (6.4.5) and (6.4.6) can be put into state variable form by isolating the derivatives of the state variables i_a and ω. The state equations thus obtained are the following.

$$\frac{di_a}{dt} = \frac{1}{L_a} (v_a - R_a i_a - K_b \omega) \tag{6.4.12}$$

$$\frac{d\omega}{dt} = \frac{1}{I} (K_T i_a - c\omega - T_L) \tag{6.4.13}$$

Note that these state variables describe the energies $L i_a^2/2$ and $I \omega^2/2$ stored in the system.
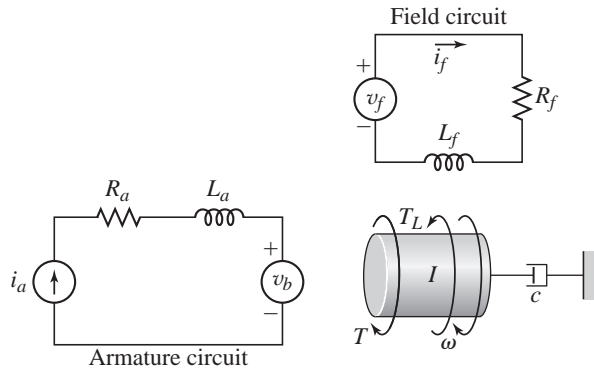
Figure 6.4.5 Diagram of a field-controlled dc motor.

FIELD-CONTROLLED MOTORS

Another way to control a dc motor is to keep the armature current constant while adjusting the voltage applied to the field windings to vary the intensity of the magnetic field surrounding the armature (see Figure 6.4.5). Thus, unlike permanent-magnet motors, field-controlled motors require two power supplies, one for the armature circuit and one for the field circuit. They also require a control circuit to maintain a constant armature current in the presence of the back emf, which varies with motor speed and field strength.

In general, the field strength B is a nonlinear function of the field current i_f and can be expressed as $B(i_f)$. Thus, if the armature radius is r, the torque on the armature is

$$T = nB(i_f)Li_ar = (nLri_a)B(i_f) = T(i_f)$$

and we see that the motor torque is also a nonlinear function of i_f. Often the linear approximation $T - T_r = K_T(i_f - i_{fr})$ is used, where T_r and i_{fr} are the torque and current values at a reference operating equilibrium, and the torque constant K_T is the slope of the $T(i_f)$ curve at the reference condition. In the rest of our development, we will assume that $T_r = i_{fr} = 0$ to simplify the discussion. Thus we will use the relation $T = K_T i_f$.

Model of a Field-Controlled dc Motor | **EXAMPLE 6.4.2**

■ Problem
Develop a model of the field-controlled motor shown in Figure 6.4.5.

■ Solution
The voltage v_f is applied to the field circuit, whose inductance and resistance are L_f and R_f. No back emf exists in the field circuit because it does not move within the field, and Kirchhoff's voltage law applied to the field circuit gives

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \tag{1}$$

For the inertia I,

$$I\frac{d\omega}{dt} = T - c\omega - T_L = K_T i_f - c\omega - T_L \tag{2}$$

where T_L is the load torque. These two equations form the motor model.

6.5 ANALYSIS OF MOTOR PERFORMANCE

We now use the transfer function model of an armature-controlled dc motor to investigate the performance of such motors. The transfer functions given by (6.4.7) through (6.4.10) are repeated here.

$$\frac{I_a(s)}{V_a(s)} = \frac{Is + c}{L_aIs^2 + (R_aI + cL_a)s + cR_a + K_bK_T} \tag{6.5.1}$$

$$\frac{I_a(s)}{T_L(s)} = \frac{K_b}{L_aIs^2 + (R_aI + cL_a)s + cR_a + K_bK_T} \tag{6.5.2}$$

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_aIs^2 + (R_aI + cL_a)s + cR_a + K_bK_T} \tag{6.5.3}$$

$$\frac{\Omega(s)}{T_L(s)} = -\frac{L_as + R_a}{L_aIs^2 + (R_aI + cL_a)s + cR_a + K_bK_T} \tag{6.5.4}$$

STEADY-STATE MOTOR RESPONSE

The steady-state operating conditions can be obtained by applying the final value theorem to the transfer functions. If v_a and T_L are step functions of magnitude V_a and T_L, respectively, then the steady-state current and speed are

$$i_a = \frac{cV_a + K_bT_L}{cR_a + K_bK_T} \tag{6.5.5}$$

$$\omega = \frac{K_TV_a - R_aT_L}{cR_a + K_bK_T} \tag{6.5.6}$$

Thus an increased load torque leads to an increased current and a decreased speed, as would be expected. From equation (6.5.6) the steady-state speed is often plotted versus T_L for different values of the applied voltage V_a. This plot is known as the load-speed curve of the motor. For a given value of V_a, it gives the maximum load torque the motor can handle at a specified speed.

The *no-load speed* is the motor speed when the load torque is zero. Setting $T_L = 0$ in (6.5.6) gives $\omega = K_TV_a/(cR_a + K_bK_T)$. This is the highest motor speed for a given applied voltage. The corresponding no-load current required can be found by setting $T_L = 0$ in (6.5.5). It is $i_a = cV_a/(cR_a + K_bK_T)$.

The *stall torque* is the value of the load torque that produces zero motor speed. Setting $\omega = 0$ in (6.5.6) gives the stall torque: $T_L = K_TV_a/R_a$. The corresponding *stall current* can be found by substituting this value into (6.5.5).

EXAMPLE 6.5.1 | No-Load Speed and Stall Torque

■ **Problem**
The parameter values for a certain motor are

$$K_T = K_b = 0.05 \text{ N} \cdot \text{m/A}$$

$$c = 10^{-4} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.5 \, \Omega$$

The manufacturer's data states that the motor's maximum speed is 3000 rpm, and the maximum armature current it can withstand without demagnetizing is 30 A.

Compute the no-load speed, the no-load current, and the stall torque. Determine whether the motor can be used with an applied voltage of $v_a = 10$ V.

■ **Solution**

For $v_a = 10$ V, (6.5.5) and (6.5.6) give

$$i_a = 0.392 + 19.61 T_L \text{ A} \qquad \omega = 196.1 - 196.1 T_L \text{ rad/s}$$

The no-load speed is found from the second equation with $T_L = 0$. It is 196.1 rad/s, or 1872 rpm, which is less than the maximum speed of 3000 rpm. The corresponding no-load current is $i_a = 0.392$ A, which is less than the maximum allowable current of 30 A. The no-load current is required to provide a motor torque $K_T i_a$ to cancel the damping torque $c\omega$.

The stall torque is found by setting $\omega = 0$. It is $T_L = 1$ N · m. The corresponding stall current is $i_a = 20$ A, which is less than the maximum allowable current.

MOTOR DYNAMIC RESPONSE

The steady-state relations are often used because they are algebraic relations and thus are easier to use than the motor differential equations. However, they can be misleading. Because $I_a(s)/V_a(s)$ and $\Omega(s)/T_L(s)$ have numerator dynamics, the actual maximum current required and the actual maximum speed attained might be quite different than their steady-state values. Example 6.5.2 illustrates this effect.

Response of an Armature-Controlled dc Motor | **EXAMPLE 6.5.2**

■ **Problem**

The parameter values for a certain motor are

$$K_T = K_b = 0.05 \text{ N · m/A}$$

$$c = 10^{-4} \text{ N · m · s/rad} \qquad R_a = 0.5 \text{ } \Omega$$

$$L_a = 2 \times 10^{-3} \text{ H} \qquad I = 9 \times 10^{-5} \text{ kg · m}^2$$

where I includes the inertia of the armature and that of the load. The load torque T_L is zero.

Obtain the step response of $i_a(t)$ and $\omega(t)$ if the applied voltage is $v_a = 10$ V.

■ **Solution**

Substituting the given parameter values into (6.5.1) and (6.5.3), gives

$$\frac{I_a(s)}{V_a(s)} = \frac{9 \times 10^{-5}s + 10^{-4}}{18 \times 10^{-8}s^2 + 4.52 \times 10^{-5}s + 2.55 \times 10^{-3}}$$

$$\frac{\Omega(s)}{V_a(s)} = \frac{0.05}{18 \times 10^{-8}s^2 + 4.52 \times 10^{-5}s + 2.55 \times 10^{-3}}$$

If v_a is a step function of magnitude 10 V,

$$I_a(s) = \frac{5 \times 10^3 s + 5.555 \times 10^4}{s(s + 165.52)(s + 85.59)} = \frac{C_1}{s} + \frac{C_2}{s + 165.52} + \frac{C_3}{s + 85.59}$$

$$\Omega(s) = \frac{2.777 \times 10^6}{s(s + 165.52)(s + 85.59)} = \frac{D_1}{s} + \frac{D_2}{s + 165.52} + \frac{D_3}{s + 85.59}$$

Figure 6.5.1 Step response of an armature-controlled dc motor.
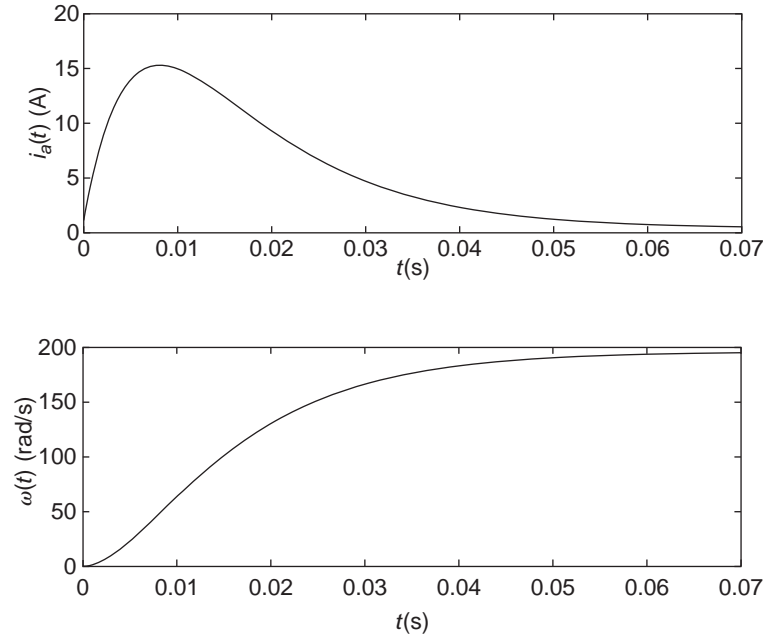
Evaluating the partial-fraction coefficients by hand or with MATLAB, as described in Chapter 3, we obtain

$$i_a(t) = 0.39 - 61e^{-165.52t} + 61.74e^{-85.59t}$$

$$\omega(t) = 196.1 + 210^{-165.52t} - 406e^{-85.59t}$$

The plots are shown in Figure 6.5.1. Note the large overshoot in i_a, which is caused by the numerator dynamics. The plot shows that the steady-state calculation of $i_a = 0.39$ A greatly underestimates the maximum required current, which is approximately 15 A.

In practice, of course, a pure step input is impossible, and thus the required current will not be as high as 15 A. The real input would take some time to reach 10 V. The response to such an input is more easily investigated by computer simulation, so we will return to this topic in Section 6.7.

THE EFFECT OF ARMATURE INDUCTANCE

If we set $L_a = 0$, the second-order motor model reduces to a first-order model, which is easier to use. For this reason, even though L_a must be nonzero for physical reasons, you often see L_a treated as negligible. Another reason L_a is sometimes neglected is that it is difficult to calculate or to measure. The following discussion shows why you must be careful in using this approximation.

Consider the motor of Example 6.5.2. Suppose the combined inertia of the armature and load is $I = 3 \times 10^{-5}$ kg · m^2. The roots of the characteristic equation are the complex pair $s = -126.7 \pm 162.3j$, which correspond to an *oscillatory* response that reaches steady state after approximately $4/126.7 = 0.032$ s. If we neglect the inductance and set $L_a = 0$ in (6.4.11), we obtain the first-order equation $1.5 \times 10^{-5}s + 0.00255 = 0$, which has the single root $s = -170$. Thus, the $L_a = 0$ approximation incorrectly predicts a *nonoscillatory* response that reaches steady state after approximately $4/170 = 0.024$ s, which differs by 25% from the correct value.

If instead the inertia is larger, say $I = 9 \times 10^{-5}$, (6.4.11) gives the real roots $s = -165.5$ and -85.6, which corresponds to a nonoscillatory response that reaches steady state after approximately $4/85.6 = 0.047$ s. Setting $L_a = 0$ in (6.4.11) gives the single root $s = -56.7$, which correctly predicts a nonoscillatory response but implies that steady state is reached after approximately $4/56.7 = 0.071$ s, which differs by 51% from the correct value.

We conclude from this example that you should be careful in using the approximation $L_a = 0$, although one sees it in common use. With $L_a \neq 0$ the characteristic equation and the motor differential equations are still only second order and thus are manageable. So this approximation really is not needed here. However, models of some types of control systems are third order or higher if the $L_a = 0$ approximation is not used, as we will see in Chapter 10. In such cases the mathematics becomes much more difficult, and so the approximation is used to reduce the order of the equations. In such cases, the correct, nonzero value of L_a is used in computer simulation studies to assess accuracy of the predictions obtained from the lower-order model.

DETERMINING MOTOR PARAMETERS

Motor parameter values can often be obtained from the manufacturer. If not, they must be either calculated or measured. Calculating K_T and K_b for an existing motor from the formula $nBLr$ is not always practical because the value of the magnetic field parameter B might be difficult to determine. An approximate value of the armature inertia I_a can be calculated from the formula for the inertia of a cylinder using the density of iron, assuming that the length and radius are available. The inertia can be measured by suspending it with a metal wire and measuring the torsional oscillation frequency f_n Hz as the armature twists on the wire. The inertia can be calculated from $I_a = k_T/(2\pi f_n)^2$, where k_T is the torsional spring constant of the wire.

Some parameters can be measured with static (steady-state) tests. By slowly increasing the load torque T_L until the motor stalls and measuring the resulting stall current, we can compute K_T from $K_T = T_L/i_a$. Knowing the voltage V_a, we can compute the armature resistance from $R_a = V_a/i_a$. By measuring the no-load speed ω and the resulting current i_a, and knowing V_a, R_a, and K_T, we can compute c from the steady-state relations (6.5.5) and (6.5.6) with $T_L = 0$.

Much of the viscous damping in the motor is due to air drag as the armature rotates. Drag force is a nonlinear function of speed, and so the linear relation $c\omega$ is an approximation. Therefore the value of c might be different at lower speeds. However, most of the damping in a given application might be due to whatever load the motor is driving (examples include pumps and fans), and so the motor's damping might be small enough to be ignored. Then the damping constant c will need to be measured at the load or calculated from a model of the load.

Because c is difficult to determine precisely, its value is rarely reported by motor manufacturers. However, in motors with good bearings the damping can be slight and is often taken to be zero (perhaps this is why its value is rarely reported!).

The inductance L_a can be difficult to determine because it involves the rate di_a/dt and thus requires a dynamic test. Special instruments such as an *impedance meter* can be used to measure L_a. As we have discussed, the inductance L_a is often assumed to be very small and therefore is often taken to be zero. This is sometimes a good approximation, but not always.

Units for K_T and K_b The units of K_T and K_b can be a source of confusion. In manufacturer's data the units of K_T are often different than those of K_b. We have seen that the formula for K_T ($K_T = nBLr$) is identical to that for K_b. Therefore, the units and numerical values will be identical if a consistent set of units is used in the formula. From the definition of K_T, we see the units in SI to be:

$$[K_T] = \frac{[T]}{[i_a]} = \frac{\text{N} \cdot \text{m}}{\text{A}}$$

Similarly, from the definition of K_b, we see the units in SI to be:

$$[K_b] = \frac{[v_b]}{[\omega]} = \frac{\text{V}}{1/\text{s}} = \text{V} \cdot \text{s}$$

These units are equivalent to one another because $1 \text{ V} = 1 \text{ W/A}$. Thus, $1 \text{ V} \cdot \text{s} = 1$ (W/A) $\cdot$ s $= 1 \text{ N} \cdot \text{m/A}$. So K_T and K_b have the same units and numerical values when SI is used.

However, while values of K_b are usually reported in SI as $\text{V} \cdot \text{s}$, values of K_T are sometimes reported in non-SI units. For example, K_T is often reported in FPS units as lb-ft/A. If so, the numerical values of K_b and K_T will not be the same. The difference corresponds to using SI units in the formula for K_b and non-SI units in the formula for K_T.

If no value for K_b is reported and K_T is reported in FPS units as lb-ft/A, the value of K_b in SI units can be obtained from the relation $K_b = 1.3558 K_T$. This relation is derived in one of the homework problems.

THE TRAPEZOIDAL PROFILE AND MOTION CONTROL

In many motion-control applications we want to accelerate the system to a desired speed and run it at that speed for some time before decelerating to a stop, as illustrated by the *trapezoidal* speed profile shown in Figure 6.5.2. The constant-speed phase of the profile is called the *slew* phase. An example requiring a slew phase of specified duration and speed is a conveyor system in a manufacturing line, in which the item being worked on must move at a constant speed for a specified time while some process is applied, such as painting or welding. The speed profile can be specified in terms of rotational motor speed ω or in terms of the translational speed v of a load.

In other applications, the duration and speed of the slew phase is not specified, but we want the system to move (or rotate) a specified distance (or angle) by the end of one cycle. An example of this is a robot arm link that must move through a specified angle. The acceleration and deceleration times, and the duration and speed of the slew phase must be computed to achieve the desired distance or angle. Often the profile is followed by a zero-speed phase to enable the motor to cool before beginning the next cycle.

Figure 6.5.2 Trapezoidal speed profile.
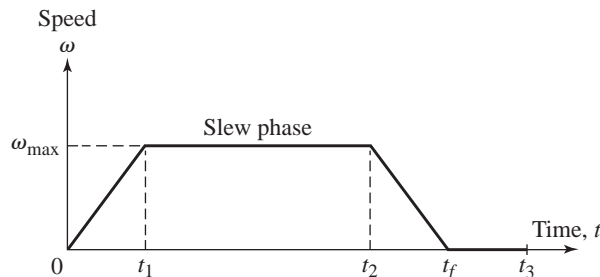
The profile in Figure 6.5.2 assumes that the system begins and ends at rest. Note also that $t_f - t_2$ must equal t_1.

MOTOR AND AMPLIFIER PERFORMANCE

We will treat motor control in Chapters 10, 11, and 12, but for now we develop methods to determine whether or not a specified motor and amplifier are capable of delivering the required performance. In evaluating the performance of a motion-control system, the following are important:

■ Energy consumption per cycle, E. This is the sum of the energy loss in the motor resistance and in the damping. Thus the energy loss per cycle is

$$E = \int_0^{t_f} Ri^2(t)\, dt + \int_0^{t_f} c\omega^2(t)\, dt \tag{6.5.7}$$

where t_f is the duration of the cycle and ω is the speed of the system at the location of the damping.

■ Maximum required current and motor torque, i_{max} and T_{max}. Motor current must be limited to prevent damage to the motor. In addition, the amplifier has an upper limit on the current it can supply. Because $T = i/K_T$, the motor torque is limited by the available current. The profile cannot require more current or torque than is available.

■ Maximum required motor speed, ω_{max}. Motor manufacturers state a limit on motor speed to prevent damage to the motor.

■ Maximum required voltage, v_{max}. Amplifiers have an upper limit on the voltage they can deliver.

■ Average required current and motor torque, i_{rms} and T_{rms}. Most amplifiers and motors have a *rated continuous current* and a *rated continuous torque*. These specify the current and torque that can be supplied for a long period of time. If the profile requires an average current or average torque greater than the rated values, the amplifier or motor can be damaged, often by overheating. The rated values are *less* than the maximum values. The average usually stated is the *rms average,* which stands for *root mean square*. For torque, it is calculated as follows:

$$T_{rms} = \sqrt{\frac{1}{t_f} \int_0^{t_f} T^2(t)\, dt} \tag{6.5.8}$$

with a similar expression used for i_{rms}. Thus the rms average gives equal weight to positive and negative values. Note that $i_{rms} = T_{rms}/K_T$.

■ Maximum speed error: This is the maximum difference between the desired speed given by the profile and the actual speed.

■ Average speed error: The average speed error is commonly computed as the rms value.

■ Displacement error: For applications requiring motion through a specified displacement (distance or angle), this error is the difference between the specified displacement and the actual displacement.

■ System response time: The system must be able to respond fast enough to follow the profile. For a quick check, if the system's largest time constant is greater than one-fourth of the profile's ramp time t_1, then the system is not fast enough.

Not all these criteria are important for every application. For example, the speed errors probably are not relevant for applications requiring motion through a specified distance or angle. Energy consumption may not be important where a single cycle is to be performed, but for applications where the cycle is performed repeatedly, the energy consumption per cycle will probably be important.

In the following we assume that the damping constant c is zero. This is often a good assumption for well-designed electromechanical systems used for motion control. By modifying (6.4.5) and (6.4.6) to account for a speed reduction ratio of N, we obtain the following motor model. See Figure 6.4.4.

$$v = Ri + L\frac{di}{dt} + K_b\omega \tag{6.5.9}$$

$$I\frac{d\omega}{dt} = K_T i - T_d \tag{6.5.10}$$

We have omitted the subscripts on v, i, R, and L to simplify the notation. The speed ω is the *motor* speed. The *load* speed is $\omega_L = \omega/N$. The torque T_d opposing the motor torque T is due to the torque T_L acting on the load, where $T_d = T_L/N$. The inertia I includes the motor inertia I_m and the load inertia I_L reflected to the motor shaft. Thus $I = I_m + I_L/N^2$.

Energy Loss   With $c = 0$, the expression for the energy loss per cycle becomes

$$E = \int_0^{t_f} Ri^2(t)\, dt = R\int_0^{t_f} \left(\frac{I\dot{\omega} + T_d}{K_T}\right)^2 dt$$

where (6.5.10) has been used to substitute for i. Assuming that T_d is constant, and expanding this expression we obtain

$$E = \frac{RI^2}{K_T^2}\int_0^{t_f} \dot{\omega}^2\, dt + \frac{2RIT_d}{K_T^2}\int_0^{t_f} \dot{\omega}\, dt + \frac{RT_d^2}{K_T^2}\int_0^{t_f} dt$$

With the assumption that the motion begins and ends at rest so that $\omega(0) = \omega(t_f)$, the second integral is zero. Thus

$$E = \frac{RI^2}{K_T^2}\int_0^{t_f} \dot{\omega}^2\, dt + \frac{RT_d^2 t_f}{K_T^2} \tag{6.5.11}$$

Note that the first term on the right depends on the profile, while the second term depends on the disturbance torque T_d.

Maximum Motor Speed   We now evaluate the integral in (6.5.11) for the trapezoidal profile, which is specified in terms of the *motor* speed $\omega(t)$. If the load speed $\omega_L(t)$ is specified instead, we can find $\omega(t)$ from $N\omega_L(t)$. The total angular displacement is the area under the trapezoidal profile shown in Figure 6.5.2. Assuming that $\theta(0) = 0$, and because $t_f - t_2 = t_1$, we have

$$\theta(t_f) = \theta_f = \int_0^{t_f} \omega\, dt = 2\left(\frac{1}{2}\omega_{\max}t_1\right) + \omega_{\max}(t_2 - t_1) = \omega_{\max}t_2$$

Thus

$$\omega_{\max} = \frac{\theta_f}{t_2} \tag{6.5.12}$$

So if we specify the displacement θ_f and the time t_2, the maximum required speed is determined from this equation.

For $0 \leq t \leq t_1$, the acceleration $\dot{\omega}$ is $\omega_{max}/t_1 = \theta_f/t_1 t_2$, after using (6.5.12). For $t_1 < t < t_2$, $\dot{\omega} = 0$, and for $t_2 \leq t \leq t_f$, $\dot{\omega} = -\omega_{max}/t_1 = -\theta_f/t_1 t_2$. Therefore,

$$\int_0^{t_f} \dot{\omega}^2 \, dt = \int_0^{t_1} \left(\frac{\theta_f}{t_1 t_2} \right)^2 dt + \int_{t_1}^{t_2} 0 \, dt + \int_{t_2}^{t_f} \left(-\frac{\theta_f}{t_1 t_2} \right)^2 dt$$

$$= \left(\frac{\theta_f}{t_1 t_2} \right)^2 (t_1 + t_f - t_2) = \left(\frac{\theta_f}{t_1 t_2} \right)^2 2 t_1$$

since $t_f - t_2 = t_1$. Therefore, expression (6.5.11) becomes

$$E = \frac{R I^2}{K_T^2} \left[\left(\frac{\theta_f}{t_1 t_2} \right)^2 2 t_1 \right] + \frac{R T_d^2 t_f}{K_T^2}$$

or

$$E = \frac{R}{K_T^2} \left(\frac{2 I^2 \theta_f^2}{t_1 t_2^2} + T_d^2 t_f \right) \tag{6.5.13}$$

Maximum Motor Torque The maximum required acceleration for the trapezoidal profile is

$$\alpha_{max} = \frac{\omega_{max}}{t_1} = \frac{\theta_f}{t_1 t_2} \tag{6.5.14}$$

Using (6.5.10) and the fact that the motor torque is $T = K_T i$, we have

$$T = K_T i = I \frac{d\omega}{dt} + T_d = I\alpha + T_d \tag{6.5.15}$$

Thus the maximum required motor torque is

$$T_{max} = I\alpha_{max} + T_d = I \frac{\omega_{max}}{t_1} + T_d = I \frac{\theta_f}{t_1 t_2} + T_d \tag{6.5.16}$$

RMS Motor Torque The rms torque is calculated from (6.5.8), using (6.5.15):

$$T_{rms}^2 = \frac{1}{t_f} \int_0^{t_f} T^2(t) \, dt = \frac{1}{t_f} \int_0^{t_f} (I\alpha + T_d)^2 \, dt$$

or

$$T_{rms}^2 = \frac{1}{t_f} \left[\int_0^{t_1} (I\alpha_{max} + T_d)^2 \, dt + \int_{t_1}^{t_2} (0 + T_d)^2 \, dt + \int_{t_2}^{t_f} (-I\alpha_{max} + T_d)^2 \, dt \right]$$

This reduces to

$$T_{rms}^2 = \frac{1}{t_f} \left[2 I^2 \alpha_{max}^2 t_1 + T_d^2 (t_1 + t_2) \right]$$

or, since $\alpha_{max} = \theta_f/t_1 t_2$ and $t_1 + t_2 = t_f$,

$$T_{rms} = \sqrt{\frac{2 I^2 \theta_f^2}{t_f t_1 t_2^2} + T_d^2} \tag{6.5.17}$$

These equations are summarized in Table 6.5.1. They are used to compute the motor requirements for the trapezoidal speed profile. To determine whether a given motor will be satisfactory, the calculated values of ω_{max}, T_{max}, and T_{rms} are compared with the motor manufacturer's data on maximum speed, peak torque, and rated continuous torque.

Table 6.5.1 Motor/amplifier requirements for trapezoidal speed profile.

Profile times t_1, t_2, t_f	See Figure 6.5.2
Motor displacement	θ_f = area under speed profile
Load torque felt at motor	T_d
Motor requirements	
Energy consumption/cycle	$E = \dfrac{R}{K_T^2}\left(\dfrac{2I^2\theta_f^2}{t_1 t_2^2} + T_d^2 t_f\right)$
Maximum speed	$\omega_{\max} = \dfrac{\theta_f}{t_2}$
Maximum torque	$T_{\max} = I\dfrac{\theta_f}{t_1 t_2} + T_d$
rms torque	$T_{\mathrm{rms}} = \sqrt{\dfrac{2I^2\theta_f^2}{t_f t_1 t_2^2} + T_d^2}$
Amplifier requirements	
Maximum current	$i_{\max} = \dfrac{T_{\max}}{K_T}$
rms current	$i_{\mathrm{rms}} = \dfrac{T_{\mathrm{rms}}}{K_T}$
Maximum voltage	$v_{\max} = Ri_{\max} + K_b\omega_{\max}$

Amplifier Requirements   We now derive expressions for the amplifier requirements to drive a specific motor through a given profile. Using the motor current equation $i = T/K_T$, we see that the maximum current and the rms current required are

$$i_{\max} = \frac{T_{\max}}{K_T} \tag{6.5.18}$$

$$i_{\mathrm{rms}} = \frac{T_{\mathrm{rms}}}{K_T} \tag{6.5.19}$$

The motor voltage equation is

$$v = Ri + L\frac{di}{dt} + K_b\omega$$

Divide by R:

$$\frac{v}{R} = i + \frac{L}{R}\frac{di}{dt} + \frac{K_b}{R}\omega$$

If the electrical time constant L/R is very small, we can neglect the second term on the right to obtain

$$\frac{v}{R} = i + \frac{K_b}{R}\omega$$

which gives $v = Ri + K_b\omega$. Thus the maximum voltage required is given approximately by

$$v_{\max} = Ri_{\max} + K_b\omega_{\max} \tag{6.5.20}$$

These equations are summarized in Table 6.5.1. They are used to compute the amplifier requirements for the trapezoidal speed profile. To determine whether a given amplifier will be satisfactory, the calculated values of $i_{\max}$, i_{rms}, and $v_{\max}$ are compared with the amplifier manufacturer's data on peak current, rated continuous current, and maximum voltage.

| | Calculating Motor-Amplifier Requirements | | **EXAMPLE 6.5.3** |

■ Problem

The trapezoidal profile requirements for a specific application are given in the following table, along with the load and motor data. Determine the motor and amplifier requirements.

Profile data		
	Cycle times	$t_1 = 0.2$ s, $t_2 = 0.4$ s, $t_f = 0.6$ s
Load data		
	Inertia $I_L = 4 \times 10^{-3}$ kg $\cdot$ m^2	Displacement $\theta_{Lf} = 10\pi$ rad
	Torque $T_L = 0.1$ N $\cdot$ m	Reduction ratio $N = 2$
Motor data		
	Resistance $R = 2\ \Omega$	Torque constant $K_T = 0.3$ N $\cdot$ m/A
	Inductance $L = 3 \times 10^{-3}$ H	Damping $c = 0$
	Inertia $I_m = 10^{-3}$ kg $\cdot$ m^2	
	Time constants	1.56×10^{-3} s and 0.043 s

■ Solution

The total inertia I is the sum of the motor inertia and the reflected load inertia. Thus,

$$I = I_m + \frac{I_L}{N^2} = 10^{-3} + \frac{4 \times 10^{-3}}{2^2} = 2 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

Because the reduction ratio is $N = 2$, the required motor displacement is $N\theta_{Lf} = 2(10\pi) = 20\pi$ rad, and the load torque as felt at the motor shaft is $T_d = T_L/N = 0.1/2 = 0.05$ N $\cdot$ m.

The motor's energy consumption per cycle is found from (6.5.13).

$$E = \frac{2}{(0.3)^2} \left[\frac{2(4 \times 10^{-6})(20\pi)^2}{0.2(0.4)^2} + (0.05)^2 0.6 \right] = 22\ \text{J/cycle}$$

The power consumption is $22/t_f = 37$ J/s, or 37 W.

Equation (6.5.12) shows that the maximum speed for the trapezoidal profile is

$$\omega_{\text{max}} = \frac{\theta_f}{t_2} = 50\pi\ \text{rad/s}$$

which is $50\pi(60)/(2\pi) = 1500$ rpm. So the motor's maximum permissible speed must be greater than 1500 rpm.

The rms torque is found from (6.5.17) to be

$$T_{\text{rms}} = \sqrt{\frac{2(4 \times 10^{-6})(20\pi)^2}{0.6(0.2)(0.4)^2} + (0.05)^2} = 1.28\ \text{N} \cdot \text{m}$$

Use (6.5.16) to compute the maximum required torque.

$$T_{\text{max}} = \frac{2 \times 10^{-3}(20\pi)}{0.2(0.4)} + 0.05 = 1.57 + 0.05 = 1.62\ \text{N} \cdot \text{m}$$

Note that the load torque contributes little to T_{max}. Most of the required torque is needed to accelerate the inertia.

The system time constants are obtained from the roots of (6.4.11), which are $s = -643$ and $s = -23.3$. The system must be fast enough to respond to the profile command. Its largest time constant, $1/23.3 = 0.043$ s, is less than one-fourth of the ramp time $t_1 = 0.2$, so the system is fast enough.

The amplifier requirements are calculated as follows. Note that because the motor data is given in SI units, $K_b = K_T = 0.3$. From (6.5.18), (6.5.19), and (6.5.20),

$$i_{\max} = \frac{1.62}{0.3} = 5.4 \text{ A}$$

$$i_{\text{rms}} = \frac{1.28}{0.3} = 4.27 \text{ A}$$

$$v_{\max} = 2(5.4) + 0.3(50\pi) = 10.8 + 47.1 = 57.9 \text{ V}$$

Note that most of the required voltage is needed to oppose the back emf.

The preceding analysis neglected the damping constant c and assumed that the term $(L/R)di/dt$ is very small. If these conditions are not satisfied in a given application, the performance evaluation is best done by computer solution. A MATLAB example is given in Section 6.7.

6.6 SENSORS AND ELECTROACOUSTIC DEVICES

In this section, we consider two common sensors: the tachometer for measuring velocity and the accelerometer, which can be used to measure either acceleration or displacement. In addition, some electroacoustic devices, such as microphones and speakers, are based on the electromagnetic principles explained in Section 6.4.

A TACHOMETER

There are many devices available to measure linear and rotational velocity. One such device can be constructed in a manner similar to a motor. However, instead of applying an input voltage, we use the load torque as the input. Consider the circuit equation for an armature-controlled motor:

$$v_a - i_a R_a - L_a \frac{di_a}{dt} - K_b \omega = 0$$

With the tachometer there is no applied voltage v_a. Thus, with $v_a = 0$, at steady-state, when the derivative $di_a/dt = 0$, this equation becomes

$$-i_a R_a - K_b \omega = 0$$

The voltage $i_a R_a$ across the resistor is thus given by $i_a R_a = K_b \omega$. If we denote this voltage by v_t, we see that

$$v_t = K_b \omega$$

If we measure the voltage v_t we can use it to determine the velocity ω.

AN ACCELEROMETER

Figure 6.6.1 illustrates the construction of an electromechanical accelerometer or a *seismograph*. A mass m, often called the *seismic mass* or the *proof mass,* is supported in a case by two springs. Its motion, which is damped by a fluid within the case, is measured by a potentiometer and amplifier. The displacement z of the case is its displacement relative to an inertial reference. With proper selection of m, c, and k, the device can be used either as a *vibrometer* to measure the amplitude of a sinusoidal displacement $z = A \sin \omega t$, or as an *accelerometer* to measure the amplitude of the acceleration $\ddot{z} = -A\omega^2 \sin \omega t$. When used to measure ground motion from an earthquake, for example, the instrument is commonly referred to as a *seismograph*.

The mass displacement x is defined relative to an inertial reference, with $x = 0$ corresponding to the equilibrium position of m when $z = 0$. With the potentiometer
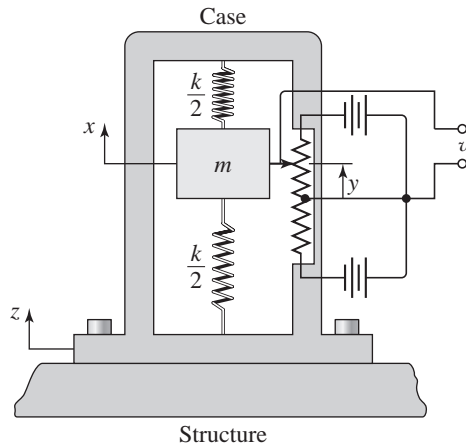
Figure 6.6.1
An accelerometer.

arrangement shown, the voltage v is proportional to the relative displacement y between the case and the mass m, where $y = x - z$. So the measured voltage is $v = Ky$.

We model the system as a mass-spring-damper system. Newton's law gives

$$m\ddot{x} = -c(\dot{x} - \dot{z}) - \frac{k}{2}(x - z) - \frac{k}{2}(x - z)$$

Substituting y for $x - z$, we obtain

$$m\ddot{y} + c\dot{y} + ky = -m\ddot{z} \qquad (6.6.1)$$

The transfer function between the input z and the output y is

$$\frac{Y(s)}{Z(s)} = \frac{-ms^2}{ms^2 + cs + k} \qquad (6.6.2)$$

In Chapter 8 we will investigate the response of such a system to a sinusoidal input. We will see that depending of the selection of m, c, and k, the displacement y, and therefore the output voltage v, can be used to indicate either the acceleration $\ddot{z}$ or the displacement z. A relatively smaller mass is used for an accelerometer, and relatively stiffer springs are used for a seismograph.

Other principles can be used to construct an accelerometer. For example, some accelerometers utilize piezoelectric crystals, whose electrical output is a function of pressure. A coil and magnet can be used instead of the potentiometer to measure the displacement of the mass.

STRAIN GAGE ACCELEROMETERS

Figure 6.6.1 illustrates the general principle of an accelerometer, but there are many types of accelerometers based on that principle but that use different technology. An example is an accelerometer based on a strain gage. In this case, the spring element in Figure 6.6.1 is replaced by a cantilever beam. The strain gage measures the beam strain, which is proportional to the inertia force, which depends on the acceleration. Since the voltage output from the gage is proportional to the strain, we see that the voltage is also proportional to the acceleration.

PIEZOELECTRIC DEVICES

The piezoelectric effect refers to the property of crystals and some ceramics to generate an electric potential in response to an applied force. In crystals, the force causes an

electric charge across the crystal lattice. This charge produces a voltage across the crystal. This effect is reversible so that a force is produced when a voltage is applied. Thus a piezoelectric element can be used as a force *sensor* or as an *actuator*.

Many sensors and actuators are based on the piezoelectric effect. They are used to measure force, pressure, acceleration, or strain. Their large modulus of elasticity means that they show almost no deflection under load, are very rugged, and are very linear over a wide range of input values. They are also not affected by electromagnetic fields and radiation. However, their small deflection means they cannot be used to measure forces that are truly static.

In piezoelectric accelerometers, a seismic mass like the mass *m* shown in Figure 6.6.1 is attached to the piezoelectric element. The piezoelectric element acts like the spring element in Figure 6.6.1, exerting a voltage comparable to the voltage *v* shown in that figure.

Piezoelectric elements are becoming more widely used because of their low cost and small size, not only as accelerometers, but also as pressure and force sensors, gyroscopes, and positioning systems. Some applications include: motion stabilization systems for cameras and orientation and acceleration measurement in game controllers. Because of their small deflection, piezoelectric elements are also used in nano-positioning systems.

ELECTROACOUSTIC DEVICES

Speakers and microphones are common examples of a class of electromechanical devices called *electroacoustic*. There are other speakers and microphones that use different principles, such as a capacitance microphone, but we will focus on those devices that utilize a magnet, a coil, and a cone. A speaker converts electrical energy into mechanical energy (sound waves) by causing the coil to move the cone. On the other hand, a microphone converts the mechanical energy in sound into electrical energy by moving the cone, thus producing a voltage and current in the coil. Here we consider the operation of a speaker. The model for a microphone is treated in the problems at the end of the chapter.

The operation of a speaker is illustrated by Figure 6.6.2. A stereo or radio amplifier produces a current in a coil that is attached to a diaphragm in the cone. This causes

Figure 6.6.2 A speaker.
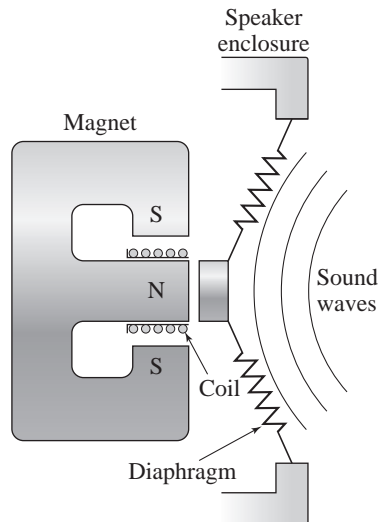
the coil and diaphragm to move relative to the permanent magnet. The motion of the diaphragm produces air pressure waves, which is sound.

An Electromagnetic Speaker | **EXAMPLE 6.6.1**

■ Problem
Develop a model of the electromagnetic speaker shown in Figure 6.6.2, and obtain the transfer function relating the diaphragm displacement x to the applied voltage v.

■ Solution
Figure 6.6.3a shows a simplified model of the mechanical subsystem, along with its free body diagram. The mass m represents the combined mass of the diaphragm and the coil. The spring constant k and damping constant c depend on the material properties of the diaphragm. The force f is the magnetic force, which is related to the coil current i by (6.4.1), $f = nBLi$, where n is the number of turns in the coil. Let $K_f = nBL$. From Newton's law

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx + K_f i \tag{1}$$

Figure 6.6.3b shows the electrical subsystem. The coil's inductance and resistance are L and R. The coil experiences a back emf because it is a current conductor moving in a magnetic field. This back emf is given by $K_b\dot{x}$. The voltage v is the signal from the amplifier. From Kirchhoff's voltage law,

$$v = L\frac{di}{dt} + Ri + K_b\frac{dx}{dt} \tag{2}$$

The speaker model consists of equations (1) and (2).

Transforming equation (1) and solving for $X(s)$ gives

$$X(s) = \frac{K_f}{ms^2 + cs + k}I(s)$$

Transforming equation (2) and solving for $I(s)$ gives

$$I(s) = \frac{1}{Ls + R}[V(s) - K_b s X(s)]$$

Eliminating $I(s)$ from the previous two equations, we obtain the desired transfer function.

$$\frac{X(s)}{V(s)} = \frac{K_f}{mLs^3 + (cL + mR)s^2 + (kL + cR + K_f K_b)s + kR} \tag{3}$$
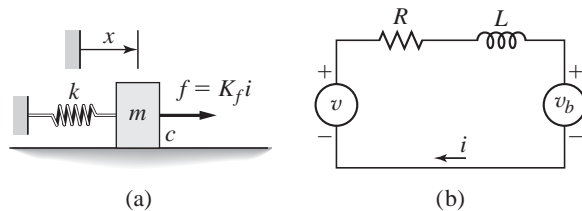
Figure 6.6.3 Models of the mechanical and electrical subsystems of a speaker.

6.7 MATLAB APPLICATIONS

In this section, we illustrate how to use the `lsim` and `step` functions with motor models in transfer function form and state-variable form. We also show how to use the `ode` solvers to obtain the response of a nonlinear system.

STEP RESPONSE FROM TRANSFER FUNCTIONS

The following transfer functions of an armature controlled motor were developed in Section 6.4.

$$\frac{I_a(s)}{V(s)} = \frac{Is + c}{L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T}$$

$$\frac{\Omega(s)}{V(s)} = \frac{K_T}{L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T}$$

where the input is the armature voltage $v(t)$.

The following program contains the motor parameters. When this is executed, the values of the parameters will be available in the MATLAB workspace, for use by our other programs. The use of such a program is good practice, because it enables you to develop modular programs that do not depend on a specific set of parameters. To investigate another motor having different parameter values, simply edit the following program and run it. We declare the parameters to be global so they can be accessed by the user-defined function `nlmotor`, which is used in our second example. The parameter values are those of Example 6.5.2. We use `Im` and `cm` to represent the inertia and damping of the motor, so that we can later distinguish them from the inertia and damping of the load.

```
% Program motor_par.m (Motor parameters in SI units)
global KT Kb La Ra Im cm
KT = 0.05;Kb = KT;
La = 2e-3;Ra = 0.5;
Im = 9e-5;cm = 1e-4;
```

The following program creates the LTI models based on the motor transfer functions.

```
% Program motor_tf.m (Transfer functions for voltage input)
I = Im;
c = cm;
% current:
current = tf([I,c],[La*I,Ra*I+c*La,c*Ra+Kb*KT]);
% speed:
speed = tf(KT,[La*I,Ra*I+c*La,c*Ra+Kb*KT]);
```

The next program computes and plots the step response for an input of 10 V. Note that since `step` gives the *unit* step response, we must multiply the response by 10. This is done in the `plot` functions.

```
% Program motor_step.m (Motor step response)
motor_par
motor_tf
[current, tc] = step(current);
[speed, ts] = step(speed);
subplot(2,1,1),plot(tc,10*current),...
   xlabel('t (s)'),ylabel('Current (A)')
subplot(2,1,2),plot(ts,10*speed),...
   xlabel('t (s)'),ylabel('Speed (rad/s)')
```

The result is shown in Figure 6.7.1.
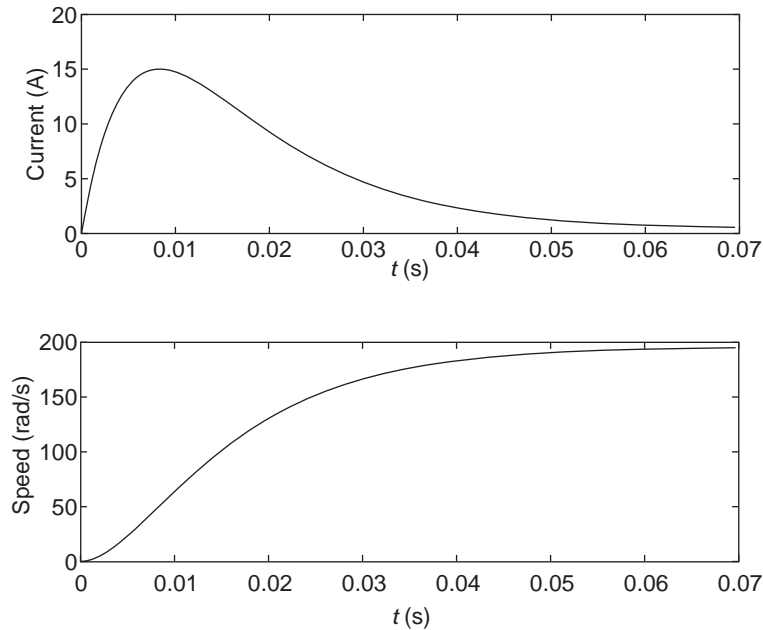
Figure 6.7.1 Motor step response.

MODIFIED STEP RESPONSE

In Example 6.5.2 we noted that the large current response is due partly to the step input acting on the numerator dynamics of the transfer function $I_a(s)/V(s)$. A more realistic model of a suddenly applied voltage input is $v(t) = 10(1 - e^{-t/\tau})$, where τ is no larger than the system's dominant time constant. The following program computes motor response to this modified step input for $\tau = 0.01$ s.

```
% Program motor_mod (Motor response with modified step)
motor_par
motor_tf
mod_step
```

where program mod_step is

```
% Program mod_step.m
% Motor simulation with modified step input
t = (0:0.0001:0.07);
v = 10*(1-exp(-t/0.01));
ia = lsim(current,v,t);
plot(t,ia,t,v)
```

The plot is shown in Figure 6.7.2. Note that the maximum current is now less than the 15 A that results from a pure step input.

STEP RESPONSE FROM STATE-VARIABLE MODEL

We now use the state variable model of the motor to plot the step response. The following model was developed in Section 6.4.

$$\frac{di_a}{dt} = \frac{v - i_a R_a - K_b \omega}{L_a} \tag{6.7.1}$$

$$\frac{d\omega}{dt} = \frac{K_T i_a - c\omega - T_d}{I} \tag{6.7.2}$$

Figure 6.7.2 Motor response
to a modified step input.
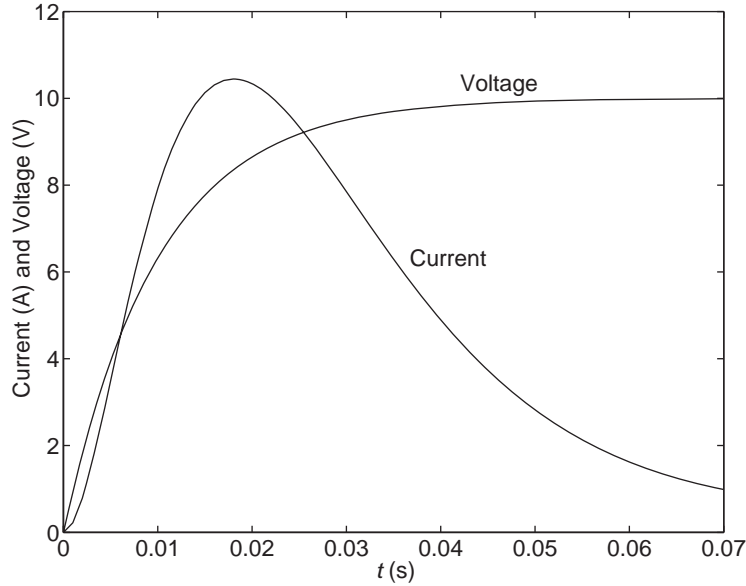
This model is in state-variable form, where the state variables are the armature current i_a and the speed ω. The inputs are the applied voltage v and the load torque T_L, which is reflected through the gear ratio N to produce $T_d = T_L/N$. In this case the appropriate state and input matrices are (see Section 5.1 for a discussion of the standard matrix-vector form of a state-variable model).

$$\mathbf{A} = \begin{bmatrix} -R_a/L_a & -K_b/L_a \\ K_T/I & -c/I \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1/L_a & 0 \\ 0 & -1/(NI) \end{bmatrix}$$

where the state vector and input vector are

$$\mathbf{x} = \begin{bmatrix} i_a \\ \omega \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} v \\ T_L \end{bmatrix}$$

Choosing the outputs to be i_a and ω, we use the following output matrices.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following program contains the parameter values for the load.

```
% load_par.m Load parameters.
IL = 0;
cL = 0;
N = 1;
TL = 0;
```

These particular values match those of Example 6.5.2, in which there is no reducer, no load torque, no load damping, and no load inertia. This program, however, is useful for solving the general problem where a reducer and a load are present.

The following program computes reflected inertia and damping, the matrices for the motor model, and the state space model `sysmotor`. When this is executed the matrices and state space model will be available in the MATLAB workspace. Note that the inertia I and damping c depend on both the motor and the load.

```
% motor_mat.m Motor state matrices.
I = Im +IL/N^2;
c = cm + cL/N^2;
A = [-Ra/La,-Kb/La;KT/I,-c/I;];
B = [1/La,0;0, -1/(N*I)];
C = [1,0;0,1];
D = [0,0;0,0];
sysmotor = ss(A,B,C,D);
```

The following program computes and plots the response due to a step voltage of magnitude 10 v, with the load torque T_L equal to zero.

```
% state_step.m (Motor step response with state model)
motor_par
load_par
motor_mat
[y, t] = step(sysmotor);
subplot(2,1,1),plot(t,10*y(:,1)),...
   xlabel('t (s)'),ylabel('Current (A)')
subplot(2,1,2),plot(t,10*y(:,2)),...
   xlabel('t (s)'),ylabel('Speed (rad/s)')
```

The resulting plot looks like Figure 6.7.1.

TRAPEZOIDAL RESPONSE

Suppose the applied voltage is the following trapezoidal function, which is shown in Figure 6.7.3 for the case where $v_{max} = 20$ V, $t_1 = 0.3$ s, $t_2 = 0.9$ s, $t_f = 1.2$ s,
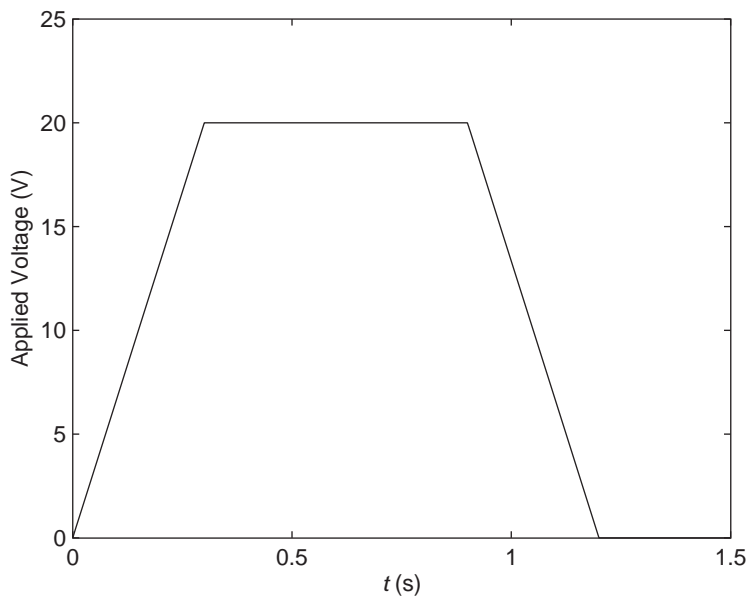
Figure 6.7.3 Trapezoidal voltage profile.

and $t_3 = 1.5$ s.

$$
v(t) = \begin{cases}
\dfrac{v_{\max}}{t_1} t & 0 \leq t \leq t_1 \\[2mm]
v_{\max} & t_1 < t < t_2 \\[2mm]
\dfrac{v_{\max}}{t_1}(t_f - t) & t_2 \leq t \leq t_f \\[2mm]
0 & t_f < t \leq t_3
\end{cases}
$$

The following program creates the voltage array v.

```
% Program trapezoid.m (Trapezoidal voltage profile)
t1 = 0.3; t2 = 0.9; tfinal = 1.2; t3 = 1.5;
v_max = 20;
dt = t3/1000;
t = (0:dt:t3);
for k = 1:1001
  if t(k) <= t1
     v(k) = (v_max/t1)*t(k);
  elseif t(k) <= t2
     v(k) = v_max;
  elseif t(k) <= tfinal
     v(k) = (v_max/t1)*(tfinal-t(k));
  else
     v(k) = 0;
  end
end
```

The next program computes the performance measures relating to energy consumption, maximum current, voltage, and torque required and rms current and torque. It uses the `trapz` function, which computes an integral with the trapezoidal rule. Note that this program does not use the formulas in Table 6.5.1 which represent a simplified case, but rather it computes the measures directly from the computed response.

```
% Program performance.m
% Computes motor performance measures.
ia = y(:,1);
speed = y(:,2);
E = trapz(t,Ra*ia.^2)+trapz(t,c*speed.^2)
i_max = max(ia)
i_rms = sqrt(trapz(t,ia.^2)/t3)
T_max = KT*i_max
T_rms = KT*i_rms
speed_max = max(speed);
v_max = Ra*i_max+Kb*speed_max
```

The next program uses `performance` and `trapezoid` to compute the response.

```
% Program trapresp.m (Motor trapezoidal response)
motor_par
load_par
```
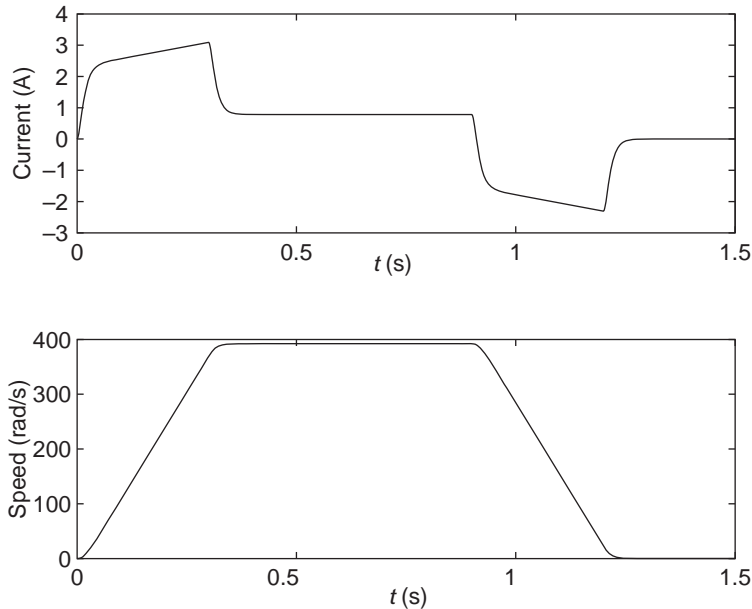
Figure 6.7.4 Motor response to a trapezoidal voltage profile.

```
motor_mat
trapezoid
u = [v', TL*ones(size(v'))];
y = lsim(sysmotor,u,t);
subplot(2,1,1),plot(t,y(:,1)),...
   xlabel('t (s)'),ylabel('Current (A)')
subplot(2,1,2),plot(t,y(:,2)),...
   xlabel('t (s)'),ylabel('Speed (rad/s)')
performance
```

The resulting plot is shown in Figure 6.7.4. The computed performance measures are

$$E = 14.12 \text{ J/cycle} \qquad i_{max} = 3.09 \text{ A} \qquad i_{rms} = 1.56 \text{ A}$$

$$T_{max} = 0.15 \text{ N} \cdot \text{m} \qquad T_{rms} = 0.08 \text{ N} \cdot \text{m} \qquad v_{max} = 21.15 \text{ V}$$

$$\omega_{max} = 392 \text{ rad/s}$$

Note that the computed value of v_{max} is different than that computed from the formula $v_{max} = Ri_{max} + K_b\omega_{max}$ because this formula neglects the electrical time constant L/R.

NONLINEAR DAMPING

In Chapter 4 we saw that the force or torque due to viscous friction is sometimes a nonlinear function of the speed. Let us investigate the effects of a nonlinear damping torque $c_N\omega^2$ on the speed and current, for a step input voltage. Suppose that $c_N = 5 \times 10^{-6} \text{ N} \cdot \text{m} \cdot \text{s}^2/\text{rad}^2$ and suppose that we are interested in the speed range from 0 to 2000 rpm. The plot of the nonlinear damping torque is shown in Figure 6.7.5. As discussed in Chapter 1, a linear approximation that gives an overestimate of the damping torque is the straight line shown on the plot. The coefficient for the linear model is $c = 10^{-3} \text{ N} \cdot \text{m} \cdot \text{s/rad}$.
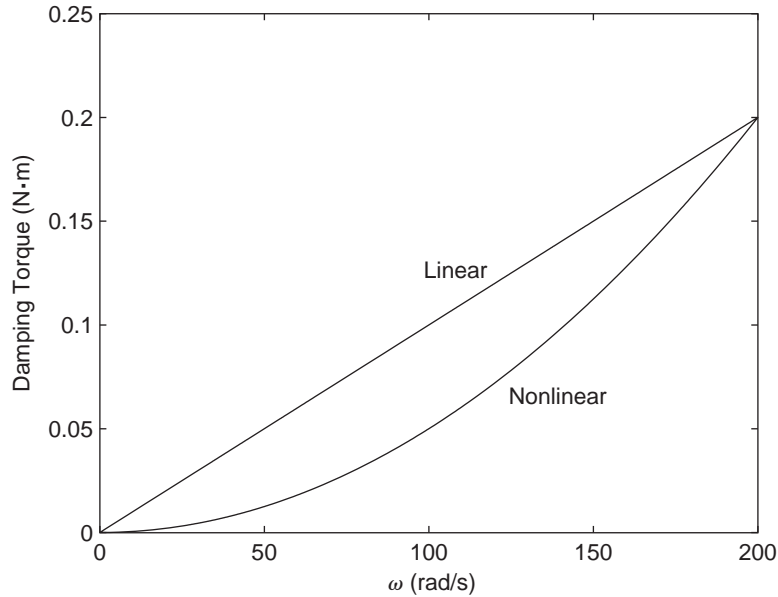
The current equation is given by (6.7.1). The nonlinear motor equation is a modification of (6.7.2) with $T_d = 0$:

$$\frac{d\omega}{dt} = \frac{K_T i_a - c_N \omega^2}{I} \tag{6.7.3}$$

To use one of the `ode` solvers we must first create a function file that computes the derivatives di_a/dt and $d\omega/dt$. This file is based on (6.7.1) and (6.7.3). The current is `x(1)` and the speed is `x(2)`.

```
function xdot = nlmotor(t,x)
% nonlinear damping in motor.
global KT Kb La Ra I cN v
xdot = [(-Ra*x(1)-Kb*x(2)+v)/La; (KT*x(1)-cN*x(2).^2)/I];
```

Note that the use of `global` in the program `motor_par` enables the parameters to be accessed by the function `nlmotor`.

The following program computes the step response of the linear and nonlinear models for an applied voltage of 10 V. Note that because we have changed the value of the motor damping c_m from that originally used in `motor_par`, we must type `cm = 1e-3` *after* running `motor_par` but *before* running `motor_mat`. Because `step` gives the *unit*-step response, we must multiply its result by $v = 10$ V. Note that the output order of the linear solvers `lsim` and `step` is `[y, t]`, whereas for the `ode` solvers the order is `[t, y]`.

```
Program motor.m (Computes response for nonlinear damping)
clear
global I cN v
motor_par
load_par
cm = 1e-3;
motor_mat
```
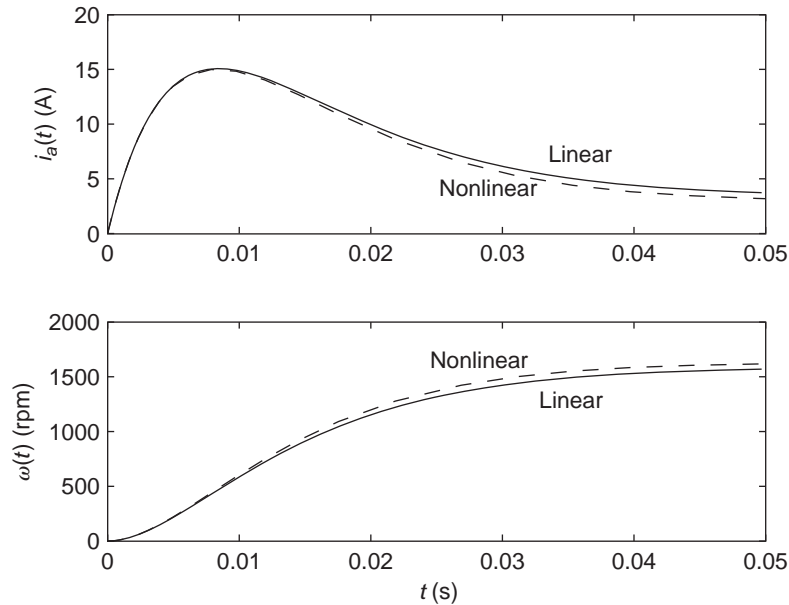
Figure 6.7.6 Motor response with linear and nonlinear damping.

```
cN = 5e-6; v = 10;
[y1, t1]= step(sysmotor);
y1 = v*y1;
speed = (0:2000)*(2*pi)/60;
[t2, y2] = ode23(@nlmotor, [0, 0.05], [0, 0]);
subplot(2,1,1), plot(t2,y2(:,1),'--',t1,y1(:,1)),...
   ylabel('i_a(t) (A)'),...
   axis([0 0.05 0 20]), gtext('Linear'),gtext('Nonlinear'),
subplot(2,1,2), ...
   plot(t2,y2(:,2)*60/(2*pi),'--',t1,y1(:,2)*60/(2*pi)),...
   xlabel('t (s)'),axis([0 0.05 0 2000]),...
   ylabel('\omega(t) (rpm)'),gtext('Linear'),...
   gtext('Nonlinear')
```

The resulting plots are shown in Figure 6.7.6. Note that the current of the linear model is greater than that of the nonlinear model but its speed is less. This is because the linear damping model used here overestimates the damping torque.

6.8 SIMULINK APPLICATIONS

Simulink is especially useful for obtaining the response of systems to input functions that are more complicated than step, impulse, ramp, or sine functions. Simulink is also helpful for computing the response of systems that contain nonlinear elements whose behavior is difficult to analyze by hand and tedious to program in MATLAB. In this section, we use several electrical systems to illustrate how to accomplish this.

SIMULATION WITH A PULSE INPUT

Example 6.2.9 derived an expression for the pulse response of a series *RC* circuit. To derive the response expression when the input is a series of pulses is, however, much

more difficult. We can use Simulink to find the response rather easily. We will introduce another block, the Pulse Generator block, to accomplish this. The Pulse Generator block is in the Sources library.

We will first use a single pulse of amplitude 12 V and duration 0.02 s as an input to check the results with the plot shown in Figure 6.2.11, using the values $R = 10^4$ Ω and $C = 10^{-6}$ F. Thus the transfer function for the circuit is

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{RCs + 1} = \frac{1}{0.01s + 1}$$

After placing the blocks as shown in Figure 6.8.1 and entering the coefficients of the transfer function, double-click on the Pulse Generator block and set the Amplitude to 12, the Period to 1, the Pulse Width to 2%, and the Phase Delay to 0. Use the default value, Time-based, for the Pulse type. Set the Save format for the Simout block to Array. Set the Stop time to 0.1 under the Simulation menu. The output of the program when plotted should look like Figure 6.8.2 after being enhanced with the Plot Editor. Compare the plot of the output voltage with Figure 6.2.12.

Figure 6.8.1 Simulink model using the Pulse Generator block.
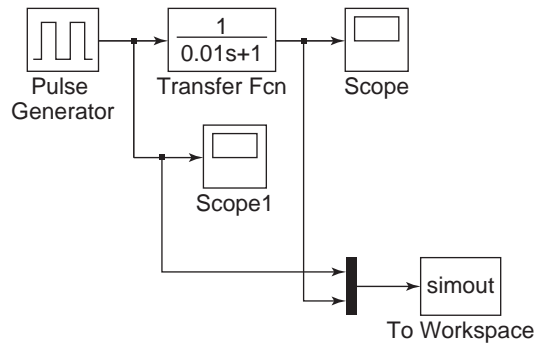
Figure 6.8.2 *RC* circuit response to a pulse input.
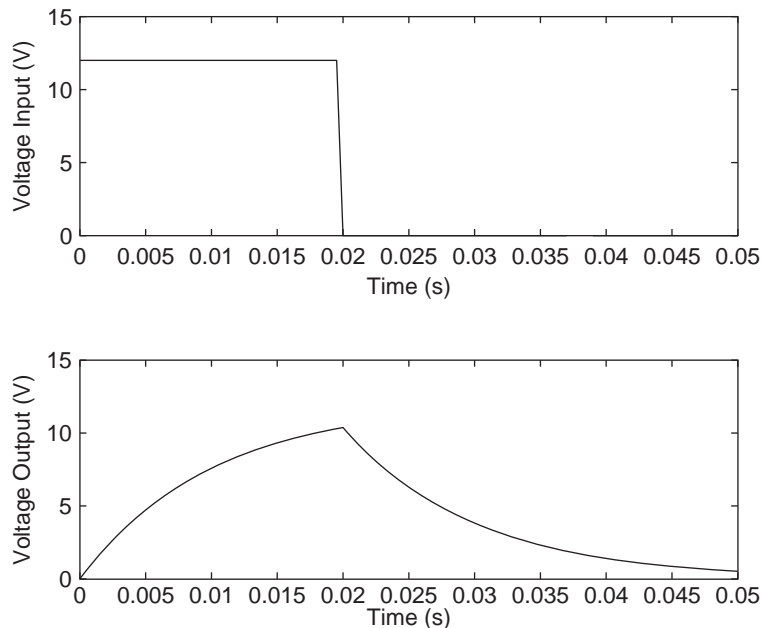
To obtain the response for a series of pulses of amplitude 12 V, of duration 0.005 s, and period 0.02 s, double-click on the Pulse Generator block and change the Period to 0.02, and the Pulse width to 25%. The output of the program when plotted should look like Figure 6.8.3 after being enhanced with the Plot Editor.

TORQUE LIMITATION IN MOTORS

In Section 6.5 we saw that motors have a maximum available torque. We can use Simulink to examine the effects of torque limits on the step response. The Simulink diagram in Figure 6.8.4 is based on the following equations.

$$\Omega(s) = \frac{1}{I_m s + c_m}[T(s) - T_d(s)]$$

$$T(s) = K_T I_a(s) = K_T \frac{1}{L_a s + R_a} V_a(s)$$

$$V_a(s) = V(s) - K_b \Omega(s)$$
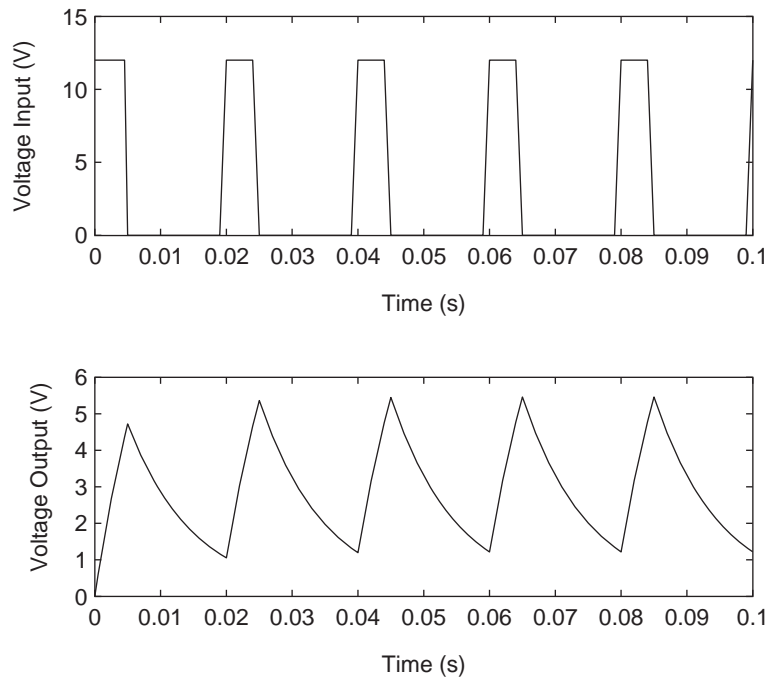
Figure 6.8.3 *RC* circuit response to a series of pulses.
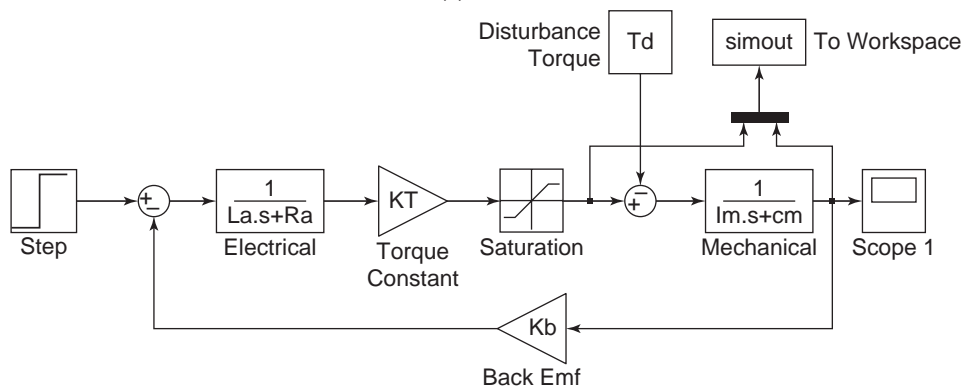
Figure 6.8.4 Simulink model of a torque-limited motor.

Figure 6.8.5 Response of a torque-limited motor.
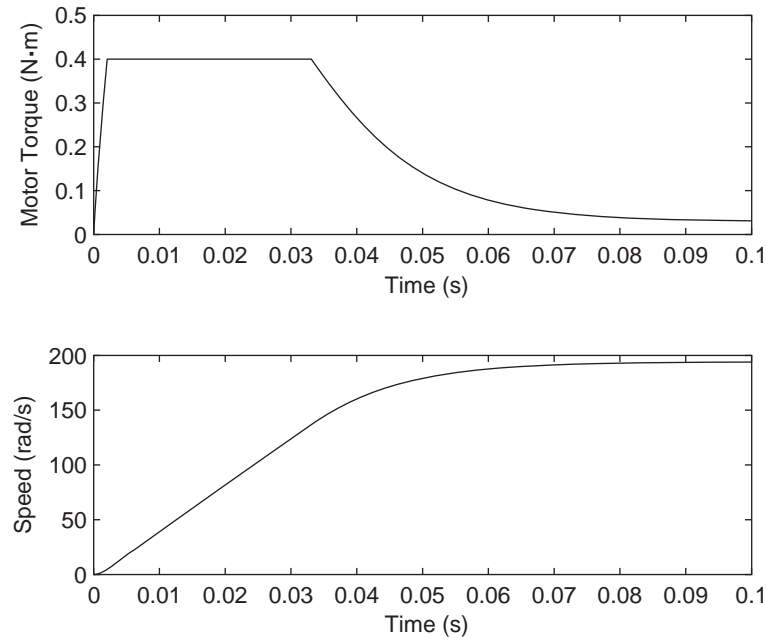
It uses two features we have not yet seen. These are the Constant block, labeled Disturbance Torque, and the use of variables as coefficients in a block. The Constant block gives a fixed constant input for all time, whereas the Step block can be set to switch on at a specified time.

The blocks labeled Electrical and Mechanical are Transfer Fcn blocks. The Torque Constant and Back Emf blocks are Gain blocks. After placing the blocks as shown, enter the denominator of the Electrical block as [La, Ra] and the denominator of the Mechanical block as [Im, cm]. Similarly, set the gains to KT and Kb, and set the constant in the Disturbance Torque block to Td. Note that these parameters do not yet have values.

Next set the lower and upper limits in the Saturation block to −0.4 and 0.4, respectively, to limit the motor torque to ±0.4 N·m. Set the Step Time of the Step block to 0 and the Final Value to 10, which corresponds to a 10 V input. Then, in the MATLAB Command window, run the program motor_par described in Section 6.7. This will set the values of all the parameters except for the torque T_d, which can be set to 0.01 N·m by typing Td = 0.01 in the Command window. Then run the Simulink model.

The results can be plotted as shown in Figure 6.8.5. Note that the motor torque is limited, and this limits the slope of the speed curve. You can check the effects of this limitation by setting larger limits (say ±1 N·m) in the Saturation block. If the torque were not limited, it would reach a peak value of about 0.8 N·m, and the speed would approach its steady-state value faster.

Note that it would be difficult to use the State Space block to simulate torque limitation. As we will see many more times, each model form—state variable and transfer function—have their own advantages.

6.9 CHAPTER REVIEW

This chapter introduced the basic physics, common elements, and terminology of electric circuits, and treated the two main physical laws needed to develop circuit models. These are Kirchhoff's current and voltage laws. Impedance, which is a generalization

of the electrical resistance concept, enables you to derive circuit models more easily, especially for more complex circuits, and is especially useful for obtaining models of circuits containing operational amplifiers. The principles of electromechanical devices, of which direct-current (dc) motors are the most important example, were extended to other electromechanical devices such as sensors and speakers.

Some of the systems treated in this chapter are more easily analyzed with computer simulation, and the chapter showed how to apply MATLAB and Simulink to analyze electromechanical systems.

Now that you have finished this chapter, you should be able to

■ Develop models of electrical circuits. This includes

Application of Kirchhoff's voltage and current laws,
Application of the voltage- and current-divider rules,
Use of loop analysis,
Selection of appropriate state variables, and
Use of impedance.

■ Apply impedance methods to obtain models of op-amp systems.
■ Apply Newton's laws, electrical circuit laws, and electromagnetic principles to develop models of electromechanical systems.
■ Analyze the performance of motors and amplifiers in motion control systems.
■ Apply MATLAB and Simulink to analyze models of circuits and electromechanical systems in state-variable and transfer function form.

PROBLEMS

Section 6.1 Electrical Elements

6.1 Determine the equivalent resistance R_e of the circuit shown in Figure P6.1, such that $v_s = R_e i$. All the resistors are identical and have the resistance R.

6.2 Determine the voltage v_1 in terms of the supply voltage v_s for the circuit shown in Figure P6.2.

6.3 The Wheatstone bridge, like that shown in Figure P6.3, is used for various measurements. For example, a strain gage sensor utilizes the fact that the resistance of wire changes when deformed. If the sensor is one resistance leg of the bridge, then the deformation can be determined from the voltage v_1. Determine the relation between the voltage v_1 and the supply voltage v_s.

Figure P6.1
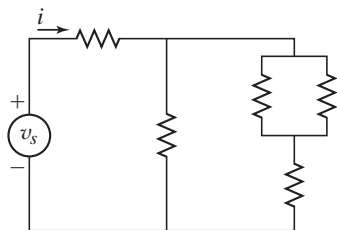
Figure P6.2
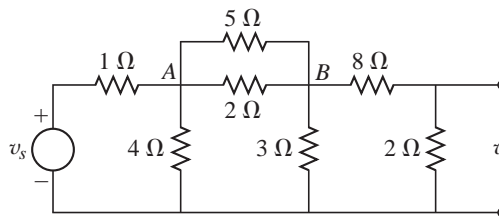
Figure P6.3
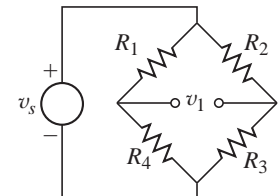
Section 6.2 Circuit Examples

6.4 The power supply of the circuit shown in Figure P6.4 supplies a voltage of 9 V. Compute the current i and the power P that must be supplied.

Figure P6.4
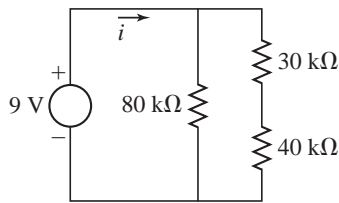
Figure P6.5
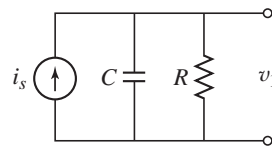
Figure P6.6
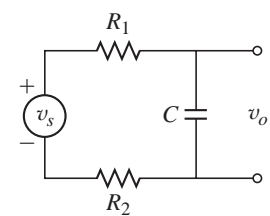
6.5 Obtain the model of the voltage v_1, given the current i_s, for the circuit shown in Figure P6.5.

6.6 (a) Obtain the model of the voltage v_o, given the supply voltage v_s, for the circuit shown in Figure P6.6. (b) Suppose $v_s(t) = V u_s(t)$. Obtain the expressions for the free and forced responses for $v_o(t)$.

6.7 (a) Obtain the model of the voltage v_o, given the supply voltage v_s, for the circuit shown in Figure P6.7. (b) Suppose $v_s(t) = V u_s(t)$. Obtain the expressions for the free and forced responses for $v_o(t)$.

6.8 (a) Obtain the model of the voltage v_o, given the supply voltage v_s, for the circuit shown in Figure P6.8. (b) Suppose $v_s(t) = V u_s(t)$. Obtain the expressions for the free and forced responses for $v_o(t)$.

6.9 (a) The circuit shown in Figure P6.9 is a model of a solenoid, such as that used to engage the gear of a car's starter motor to the engine's flywheel. The solenoid is constructed by winding wire around an iron core to make an electromagnet. The resistance R is that of the wire, and the inductance L is due to the electromagnetic effect. When the supply voltage v_s is turned on, the resulting current activates the magnet, which moves the starter gear. Obtain the model of the current i given the supply voltage v_s. (b) Suppose $v_s(t) = V u_s(t)$ and $i(0) = 0$. Obtain the expression for the response for $i(t)$.

Figure P6.7
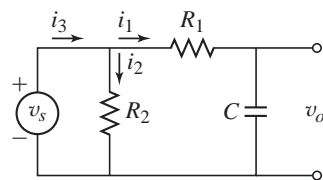
Figure P6.8
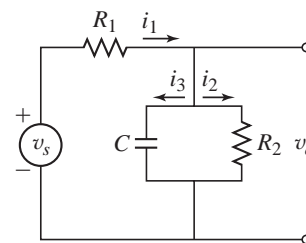
Figure P6.9
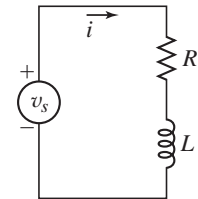
6.10 The resistance of a telegraph line is $R = 10\ \Omega$, and the solenoid inductance is $L = 5$ H. Assume that when sending a "dash," a voltage of 12 V is applied while the key is closed for 0.3 s. Obtain the expression for the current $i(t)$ passing through the solenoid. (See Figure 6.2.15)

6.11 Obtain the model of the voltage v_o, given the supply voltage v_s, for the circuit shown in Figure P6.11.

6.12 Obtain the model of the voltage v_o, given the supply voltage v_s, for the circuit shown in Figure P6.12.

6.13 Obtain the model of the current i, given the supply voltage v_s, for the circuit shown in Figure P6.13.

Figure P6.11
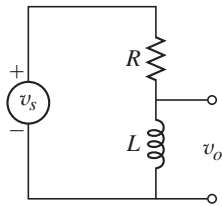
Figure P6.12
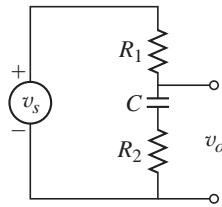
Figure P6.13
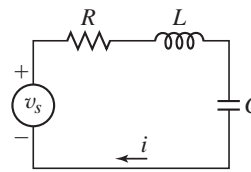
6.14 Obtain the model of the voltage v_o, given the supply current i_s, for the circuit shown in Figure P6.14.

6.15 Obtain the model of the currents i_1, i_2, and i_3, given the input voltages v_1 and v_2, for the circuit shown in Figure P6.15.

Figure P6.14
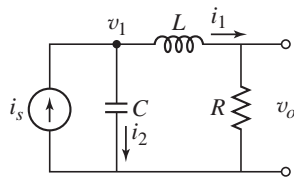
Figure P6.15
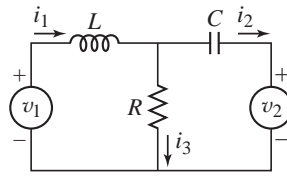
6.16 Obtain the model of the currents i_1, i_2, and the voltage v_3, given the input voltages v_1 and v_2, for the circuit shown in Figure P6.16.

Figure P6.16
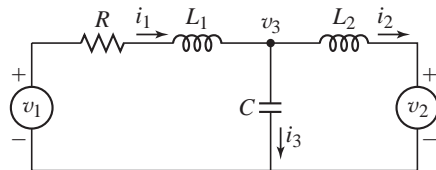
6.17 For the circuit shown in Figure P6.14, determine a suitable set of state variables, and obtain the state equations.

6.18 For the circuit shown in Figure P6.15, determine a suitable set of state variables, and obtain the state equations.

6.19 For the circuit shown in Figure P6.16, determine a suitable set of state variables, and obtain the state equations.

Section 6.3 Impedance and Amplifiers

6.20 Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the circuit shown in Figure P6.20.

6.21 Use the impedance method to obtain the transfer function $I(s)/V_s(s)$ for the circuit shown in Figure P6.21.

6.22 Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the circuit shown in Figure P6.22.

6.23 Use the impedance method to obtain the transfer function $V_o(s)/I_s(s)$ for the circuit shown in Figure P6.23.

Figure P6.20
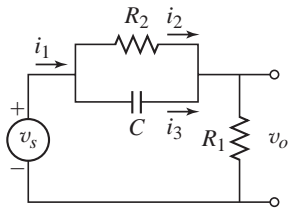
Figure P6.21
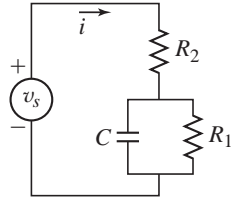
Figure P6.22
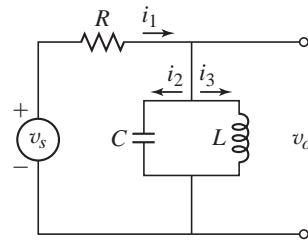
Figure P6.23
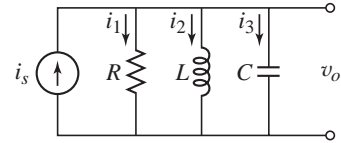
6.24   Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the circuit shown in Figure P6.24.

6.25   Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the circuit shown in Figure P6.25.

6.26   Use the impedance method to obtain the transfer function $V_o(s)/V_s(s)$ for the circuit shown in Figure P6.26.

Figure P6.24
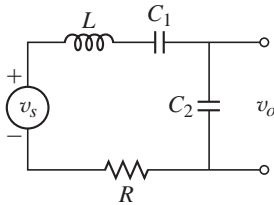
Figure P6.25
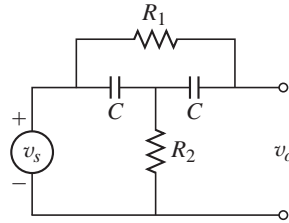
Figure P6.26
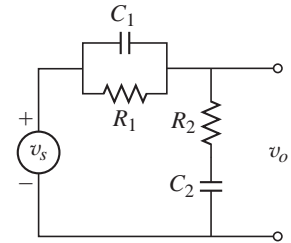
6.27   Obtain the transfer function $V_o(s)/V_i(s)$ for the op-amp system shown in Figure P6.27.

6.28   Obtain the transfer function $V_o(s)/V_i(s)$ for the op-amp system shown in Figure P6.28.

Figure P6.27
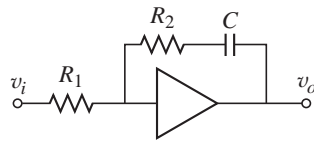
Figure P6.28
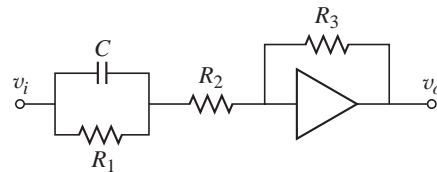
6.29   Obtain the transfer function $V_o(s)/V_i(s)$ for the op-amp system shown in Figure P6.29.

6.30   Obtain the transfer function $V_o(s)/V_i(s)$ for the op-amp system shown in Figure P6.30.

6.31   Obtain the transfer function $V_o(s)/V_i(s)$ for the op-amp system shown in Figure P6.31.

Figure P6.29
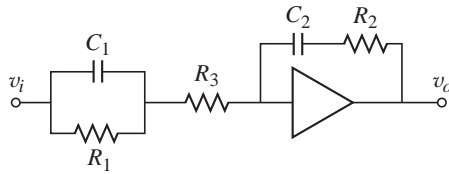
Figure P6.30
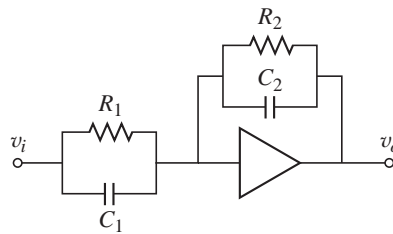
Figure P6.31
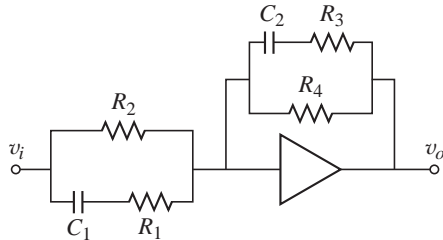
Section 6.4 Electric Motors

6.32 (a) Obtain the transfer function $\Theta(s)/V_i(s)$ for the D'Arsonval meter. (b) Use the final value theorem to obtain the expression for the steady-state value of the angle θ if the applied voltage v_i is a step function.

6.33 (a) Obtain the transfer function $\Omega(s)/T_L(s)$ for the field-controlled motor of Example 6.4.2. (b) Modify the field-controlled motor model in Example 6.4.2 so that the output is the angular displacement θ, rather than the speed ω, where $\omega = \dot{\theta}$. Obtain the transfer functions $\Theta(s)/V_f(s)$ and $\Theta(s)/T_L(s)$.

6.34 Modify the motor model given in Example 6.4.2 to account for a gear pair between the motor shaft and the load. The ratio of motor speed to load speed ω_L is N. The motor inertia is I_m and the motor damping is c_m. The load inertia is I_L and the load damping is c_L. The load torque T_L acts directly on the load inertia. Obtain the transfer functions $\Omega_L(s)/V_f(s)$ and $\Omega_L(s)/T_L(s)$.

6.35 The derivation of the field-controlled motor model in Section 6.4 neglected the elasticity of the motor-load shaft. Figure P6.35 shows a model that includes this elasticity, denoted by its equivalent torsional spring constant k_T. The motor inertia is I_1, and the load inertia is I_2. Derive the differential equation model with θ_2 as output and v_f as input.
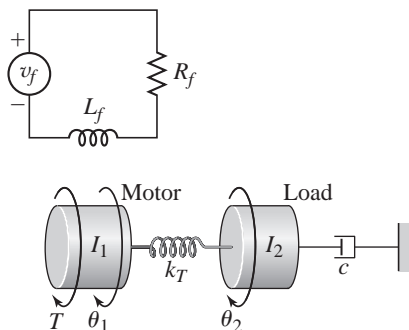
Figure P6.35
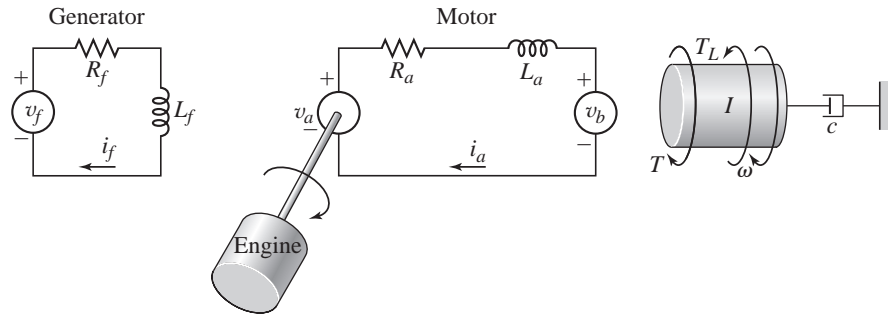
6.36   Figure P6.36 is the circuit diagram of a speed-control system in which the dc
motor voltage v_a is supplied by a generator driven by an engine. This system
has been used on locomotives whose diesel engine operates most efficiently at
one speed. The efficiency of the electric motor is not as sensitive to speed and
thus can be used to drive the locomotive at various speeds. The motor voltage
v_a is varied by changing the generator input voltage v_f. The voltage v_a is
related to the generator field current i_f by $v_a = K_f i_f$.

   Derive the system model relating the output speed ω to the voltage v_f, and
obtain the transfer function $\Omega(s)/V_f(s)$.

Section 6.5 Analysis of Motor Performance

6.37   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.2 \, \text{N} \cdot \text{m/A}$$

$$c = 5 \times 10^{-4} \, \text{N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.8 \, \Omega$$

The manufacturer's data states that the motor's maximum speed is 3500 rpm, and
the maximum armature current it can withstand without demagnetizing is 40 A.

   Compute the no-load speed, the no-load current, and the stall torque.
Determine whether the motor can be used with an applied voltage of $v_a = 15$ V.

6.38   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.05 \, \text{N} \cdot \text{m/A}$$

$$R_a = 0.8 \, \Omega$$

$$L_a = 3 \times 10^{-3} \, \text{H} \qquad I = 8 \times 10^{-5} \, \text{kg} \cdot \text{m}^2$$

where I includes the inertia of the armature and that of the load. Investigate the
effect of the damping constant c on the motor's characteristic roots and on its
response to a step voltage input. Use the following values of c (in $\text{N} \cdot \text{m} \cdot \text{s/rad}$):
$c = 0$, $c = 0.01$, and $c = 0.1$. For each case, estimate how long the motor's
speed will take to become constant, and discuss whether or not the speed will
oscillate before it becomes constant.

6.39   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.2 \, \text{N} \cdot \text{m/A}$$

$$c = 5 \times 10^{-4} \, \text{N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.8 \, \Omega$$

$$L_a = 4 \times 10^{-3} \, \text{H} \qquad I = 5 \times 10^{-4} \, \text{kg} \cdot \text{m}^2$$

where c and I include the effect of the load.

a. Obtain the step response of $i_a(t)$ and $\omega(t)$ if the applied voltage is $v_a = 10$ V.

b. Obtain the step response of $i_a(t)$ and $\omega(t)$ if the load torque is $T_L = 0.2$ N · m.

6.40 The following measurements were performed on a permanent magnet motor when the applied voltage was $v_a = 20$ V. The measured stall current was 25 A. The no-load speed was 2400 rpm and the no-load current was 0.6 A. Estimate the values of K_b, K_T, R_a, and c.

6.41 A single link of a robot arm is shown in Figure P6.41. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through spur gears. Suppose that the equivalent inertia felt at the motor shaft is 0.215 kg · m². As the arm rotates, the effect of the arm weight generates an opposing torque that depends on the arm angle, and is therefore nonlinear. For this problem, however, assume that the effect of the opposing torque is a constant 4.2 N · m at the motor shaft. Neglect damping in the system. It is desired to have the motor shaft rotate through $3\pi/4$ rad in a total time of 2 s, using a trapezoidal speed profile with $t_1 = 0.3$ s and $t_2 = 1.7$ s.

The given motor parameters are $R_a = 4$ Ω, $L_a = 3 \times 10^{-3}$ H, and $K_T = 0.3$ N · m/A. Compute the energy consumption per cycle; the maximum required torque, current, and voltage; the rms torque; and the rms current.
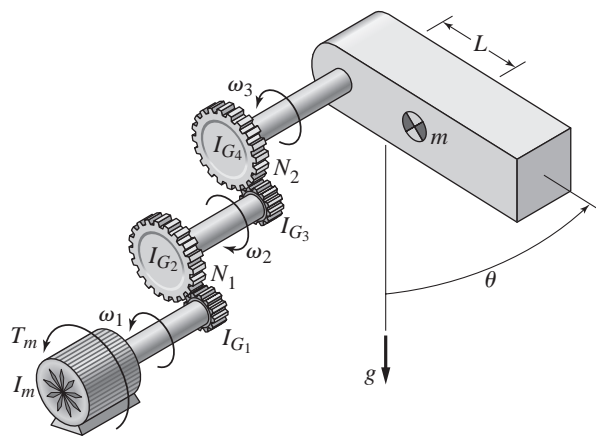
Figure P6.41   **Figure P6.42**
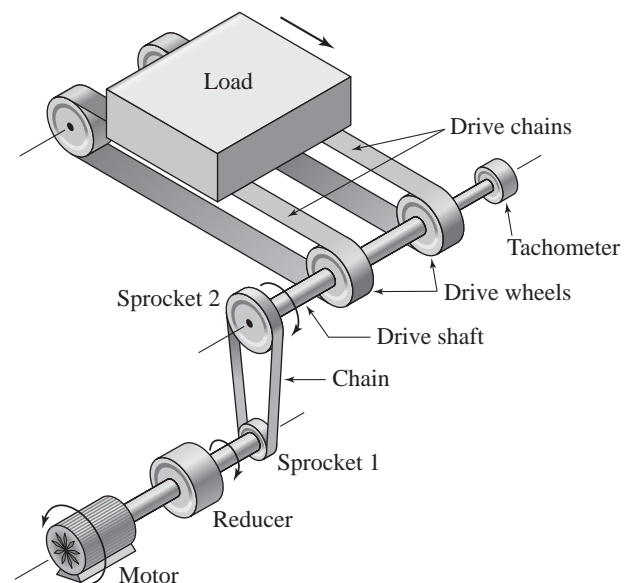
6.42 A conveyor drive system to produce translation of the load is shown in Figure P6.42. Suppose that the equivalent inertia felt at the motor shaft is 0.05 kg · m², and that the effect of Coulomb friction in the system produces an opposing torque of 3.6 N · m at the motor shaft. Neglect damping in the system. It is desired to have the motor shaft rotate through 11 revolutions in a total time of 3 s, using a trapezoidal speed profile with $t_1 = 0.5$ s and $t_2 = 2.5$ s.

The given motor parameters are $R_a = 3\ \Omega$, $L_a = 4 \times 10^{-3}$ H, and $K_T = 0.4 \text{N} \cdot \text{m/A}$. Compute the energy consumption per cycle; the maximum required torque, current, and voltage; the rms torque; and the rms current.

Section 6.6 Sensors and Electroacoustic Devices

6.43   Consider the accelerometer model in Section 6.6. Its transfer function can be expressed as

$$\frac{Y(s)}{Z(s)} = -\frac{s^2}{s^2 + (c/m)s + k/m}$$

Suppose that the input displacement is $z(t) = 10 \sin 120t$ mm. Consider two cases, in SI units: (a) $k/m = 100$ and $c/m = 18$ and (b) $k/m = 10^6$ and $c/m = 1800$. Obtain the steady-state response $y(t)$ for each case. By comparing the amplitude of $y(t)$ with the amplitudes of $z(t)$ and $\ddot{z}(t)$, determine which case can be used as a vibrometer (to measure displacement) and which can be used as an accelerometer (to measure acceleration).

6.44   An electromagnetic microphone has a construction similar to that of the speaker shown in Figure 6.6.2, except that there is no applied voltage and the sound waves are incoming rather than outgoing. They exert a force f_s on the diaphragm whose mass is m, damping is c, and stiffness is k. Develop a model of the microphone, whose input is f_s and output is the current i in the coil.

6.45   Consider the speaker model developed in Example 6.6.1. The model, whose transfer function is given by equation (3) in that example, is third order and therefore we cannot obtain a useful expression for the characteristic roots. Sometimes inductance L and damping c are small enough to be ignored. If $L = 0$, the model becomes second order. (a) Obtain the transfer function $X(s)/V(s)$ for the case where $L = c = 0$, and obtain the expressions for the two roots. (b) Compare the results with the third-order case where

$$m = 0.002 \text{ kg} \qquad k = 4 \times 10^5 \text{ N/m}$$
$$K_f = 16 \text{ N/A} \qquad K_b = 13 \text{ V} \cdot \text{s/m}$$
$$R = 12\ \Omega \qquad L = 10^{-3} \text{ H}$$
$$c = 0$$

Section 6.7 MATLAB Applications

6.46   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.2 \text{ N} \cdot \text{m/A}$$
$$c = 5 \times 10^{-4} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.8\ \Omega$$
$$L_a = 4 \times 10^{-3} \text{ H} \qquad I = 5 \times 10^{-4} \text{ kg} \cdot \text{m}^2$$

where c and I include the effect of the load. The load torque is zero.
    Use MATLAB to obtain a plot of the step response of $i_a(t)$ and $\omega(t)$ if the applied voltage is $v_a = 10$ V. Determine the peak value of $i_a(t)$.

6.47   Consider the motor whose parameters are given in Problem 6.46. Use MATLAB to obtain a plot of the response of $i_a(t)$ and $\omega(t)$ if the applied voltage is the modified step $v_a(t) = 10(1 - e^{-100t})$ V. Determine the peak value of $i_a(t)$.

6.48 Consider the circuit shown in Figure P6.48. The parameter values are $R = 10^3 \ \Omega$, $C = 2 \times 10^{-6}$ F, and $L = 2 \times 10^{-3}$ H. The voltage v_1 is a step input of magnitude 5 V, and the voltage v_2 is sinusoidal with frequency of 60 Hz and an amplitude of 4 V. The initial conditions are zero. Use MATLAB to obtain a plot of the current response $i_3(t)$.

Figure P6.48

6.49 The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.2 \ \text{N} \cdot \text{m/A}$$
$$c = 3 \times 10^{-4} \ \text{N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.8 \ \Omega$$
$$L_a = 4 \times 10^{-3} \ \text{H} \qquad I = 4 \times 10^{-4} \ \text{kg} \cdot \text{m}^2$$

The system uses a gear reducer with a reduction ratio of 3:1. The load inertia is $10^{-3} \ \text{kg} \cdot \text{m}^2$, the load torque is 0.04 N · m, and the load damping constant is $1.8 \times 10^{-3} \ \text{N} \cdot \text{m} \cdot \text{s/rad}$.

Use MATLAB to obtain a plot of the step response of $i_a(t)$ and $\omega(t)$ if the applied voltage is $v_a = 20$ V. Determine the peak value of $i_a(t)$.

6.50 The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.05 \ \text{N} \cdot \text{m/A}$$
$$c = 0 \qquad R_a = 0.8 \ \Omega$$
$$L_a = 3 \times 10^{-3} \ \text{H} \qquad I = 8 \times 10^{-5} \ \text{kg} \cdot \text{m}^2$$

where I includes the inertia of the armature and that of the load. The load torque is zero. The applied voltage is a trapezoidal function defined as follows.

$$v(t) = \begin{cases} 60t & 0 \leq t \leq 0.5 \\ 30 & 0.5 < t < 2 \\ 60(2.5 - t) & 2 \leq t \leq 2.5 \\ 0 & 2.5 < t \leq 4 \end{cases}$$

a. Use MATLAB to obtain of plot of the response of $i_a(t)$ and $\omega(t)$.

b. Compute the energy consumption per cycle; the maximum required torque, current, and voltage; the rms torque; and the rms current.

6.51 A single link of a robot arm is shown in Figure P6.41. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through spur gears. Suppose that the equivalent inertia felt at the motor shaft is $0.215 \ \text{kg} \cdot \text{m}^2$. As the arm rotates, the effect of the arm weight generates an opposing torque that depends on the arm angle, and is therefore nonlinear. The effect of the opposing torque at the motor shaft is $4.2 \sin \theta \ \text{N} \cdot \text{m}$. Neglect damping in the system. It is desired to have the motor shaft rotate through $3\pi/4$ rad in a total time of 2 s, using a trapezoidal speed profile with $t_1 = 0.3$ s and $t_2 = 1.7$ s.

The given motor parameters are $R_a = 4 \ \Omega$, $L_a = 3 \times 10^{-3}$ H, and $K_T = 0.3 \ \text{N} \cdot \text{m/A}$. Use MATLAB to obtain of plot of the response of the motor current and the motor speed.

Section 6.8 Simulink Applications

6.52 Consider the circuit shown in Figure P6.48. The parameter values are $R = 10^4 \ \Omega$, $C = 2 \times 10^{-6}$ F, and $L = 2 \times 10^{-3}$ H. The voltage v_1 is a single pulse of magnitude 5 V and duration 0.05 s, and the voltage v_2 is sinusoidal
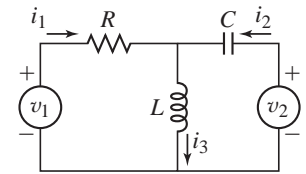
with frequency of 60 Hz and an amplitude of 4 V. The initial conditions are zero. Use Simulink to obtain a plot of the current response $i_3(t)$.

6.53   Consider the circuit shown in Figure P6.53. The parameter values are $R = 2 \times 10^4$ Ω and $C = 3 \times 10^{-6}$ F. The voltage v_s is $v_s(t) = 12u_s(t) + 3\sin 120\pi t$ V. The initial conditions are zero. Use Simulink to obtain a plot of the responses $v_o(t)$ and $v_1(t)$.

Figure P6.53
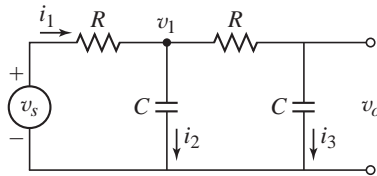
6.54   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.2 \text{ N} \cdot \text{m/A}$$
$$c = 5 \times 10^{-4} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad R_a = 0.8 \text{ Ω}$$
$$L_a = 4 \times 10^{-3} \text{ H} \qquad I = 5 \times 10^{-4} \text{ kg} \cdot \text{m}^2$$

where c and I include the effect of the load. The load torque is zero.

a.   Use Simulink to obtain a plot of the step response of the motor torque and speed if the applied voltage is $v_a = 10$ V. Determine the peak value of the motor torque.

b.   Now suppose that the motor torque is limited to one-half the peak value found in part (a). Use Simulink to obtain a plot of the step response of the motor torque and speed if the applied voltage is $v_a = 10$ V.

6.55   The parameter values for a certain armature-controlled motor are

$$K_T = K_b = 0.05 \text{ N} \cdot \text{m/A}$$
$$c = 0 \qquad R_a = 0.8 \text{ Ω}$$
$$L_a = 3 \times 10^{-3} \text{ H} \qquad I = 8 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

where I includes the inertia of the armature and that of the load. The load torque is zero. The applied voltage is a trapezoidal function defined as follows.

$$v(t) = \begin{cases} 60t & 0 \le t \le 0.5 \\ 30 & 0.5 < t < 2 \\ 60(2.5 - t) & 2 \le t \le 2.5 \\ 0 & 2.5 < t \le 4 \end{cases}$$

A trapezoidal profile can be created by adding and subtracting ramp functions starting at different times. Use several Ramp source blocks and Sum blocks in Simulink to create the trapezoidal input. Obtain a plot of the response of $i_a(t)$ and $\omega(t)$.

Fluid and Thermal Systems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1.   Apply the conservation of mass principle to model simple hydraulic and pneumatic systems.

2.   Determine the appropriate resistance relation to use for laminar, turbulent, and orifice flow.

3.   Develop a dynamic model of hydraulic and pneumatic systems containing one or more fluid containers.

4.   Determine the appropriate thermal resistance relation to use for conduction, convection, and radiation heat transfer.

5.   Develop a model of a thermal process having one or more thermal storage compartments.

6.   Apply MATLAB and Simulink to solve fluid and thermal system models.

A fluid system uses one or more fluids to achieve its purpose. The dampers, shock absorbers, and door closer we saw in Chapter 4 are examples of fluid systems because they depend on the viscous nature of a fluid to provide damping. A fluid might be either a liquid or a gas. Part I of this chapter concerns the study of fluid systems, which can be divided into *hydraulics* and *pneumatics*. Hydraulics is the study of systems in which the fluid is *incompressible,* that is, its density stays approximately constant over a range of pressures. Pneumatics is the study of systems in which the fluid is *compressible*. Hydraulics and pneumatics share a common modeling

principle: conservation of mass. It will form the basis of all our models of such systems. Modeling pneumatic systems also requires application of thermodynamics, because the temperature and density of a gas can change when its pressure changes.

Thus pneumatics provides a bridge to the treatment of *thermal* systems, which is the subject of Part II of the chapter. Thermal systems are systems that operate due to temperature differences. They thus involve the flow and storage of thermal energy, or *heat,* and conservation of heat energy forms the basis of our thermal models.

Part III illustrates applications of MATLAB and Simulink to fluid and thermal systems.

Fluid and thermal systems are more complicated than most electrical and mechanical systems. While, for example, there are formulas available to compute the spring constant of typical elastic elements, few formulas are available for the coefficients that will appear in our fluid and thermal models, and the coefficients' values often must be determined experimentally. For this reason, the methods for developing models from data, covered in Chapter 1, are most important for modeling fluid and thermal systems. ■

PART I. FLUID SYSTEMS

In addition to providing damping, other applications of fluid systems include actuators and processes that involve mixing, heating, and cooling of fluids. Active vehicle suspensions use hydraulic and pneumatic actuators to provide forces that supplement the passive spring and damping elements. Water supply, waste treatment, and other chemical processing applications are examples of a general category of fluid systems called *liquid-level systems,* because they involve regulating the volumes, and therefore the levels of liquids in containers such as tanks.

Because all real fluids are compressible to some extent, incompressibility is an approximation. But this approximation is usually sufficiently accurate for most liquids under typical conditions, and it results in a simpler model of the system. For this reason we will begin our study of fluid systems with hydraulics.

7.1 CONSERVATION OF MASS

We will avoid complex system models by describing only the gross system behavior instead of the details of the fluid motion patterns. The study of such motion belongs to the specialized subject of fluid mechanics and will not be treated here.

For incompressible fluids, conservation of mass is equivalent to conservation of volume, because the fluid density is constant. If we know the mass density ρ and the volume flow rate, we can compute the mass flow rate. That is, $q_m = \rho q_v$, where q_m and q_v are the mass and volume flow rates. The FPS and SI units for mass flow rate are slug/sec and kg/s, respectively. The units for volume rates are ft^3/sec and m^3/s, respectively. Other common units for volume are the *U.S. gallon,* which is 0.13368 ft^3, and the *liter,* which is 0.001 m^3.

DENSITY AND PRESSURE

The units for mass density are slug/ft^3 and kg/m^3. Sometimes one encounters *weight density,* whose common symbol is γ. Its units are lb/ft^3 or N/m^3, and it is related to the mass density as $\gamma = \rho g$, where g is the acceleration due to gravity. The mass density of fresh water near room temperature is 1.94 slug/ft^3, or 1000 kg/m^3. The mass

density of air at sea level and near room temperature is approximately 0.0023 slug/ft^3 or 1.185 kg/m^3.

Pressure is the force per unit area that is exerted by the fluid. The FPS and SI units of pressure are lb/ft^2 and the Pascal (1 Pa $= 1$ N/m^2), respectively. Another common unit is psi (lb/in.^{2}). At sea level near room temperature, atmospheric pressure, usually abbreviated p_a, is 14.7 psi (2117 lb/ft^2) or 1.0133×10^5 Pa. *Gage pressure* is the pressure difference between the *absolute pressure* and atmospheric pressure, and is often abbreviated as psig. For example, 3 psig is 17.7 psi absolute (which is abbreviated as psia).

Hydrostatic pressure is the pressure that exists in a fluid at rest. It is caused by the weight of the fluid. For example, the hydrostatic pressure at the bottom of a column of fluid of height h is ρgh. If the atmospheric pressure above the column of liquid is p_a, then the total pressure at the bottom of the column is $\rho gh + p_a$.

A Hydraulic Brake System | **EXAMPLE 7.1.1**

■ Problem

Figure 7.1.1 is a representation of a hydraulic brake system. The piston in the master cylinder moves in response to the foot pedal. The resulting motion of the piston in the slave cylinder causes the brake pad to be pressed against the brake drum with a force f_3. Obtain the expression for the force f_3 with the force f_1 as the input. The force f_1 depends on the force f_4 applied by the driver's foot. The precise relation between f_1 and f_4 depends on the geometry of the pedal arm.

■ Solution

The forces are related to the pressures and the piston areas as follows: $f_1 = p_1 A_1$ and $f_2 = p_2 A_2$. Assuming the system is in static equilibrium after the brake pedal has been pushed, we see that $p_1 = p_2 + \rho gh$, where h is the height between points 1 and 2. Thus, if h is small, that is, if the pressure ρgh is negligible compared to p_2, then $p_1 = f_1/A_1 = p_2 = f_2/A_2$. The forces are therefore related as $f_2 = f_1 A_2/A_1$, and if the area A_2 of the slave piston is greater than the area A_1 of the master piston, the force f_2 will be greater than the force f_1. So we see that this system serves to amplify the pedal force.

The force f_3 can be obtained from the lever relation $f_3 = f_2 L_1/L_2$, assuming static equilibrium or negligible lever inertia.
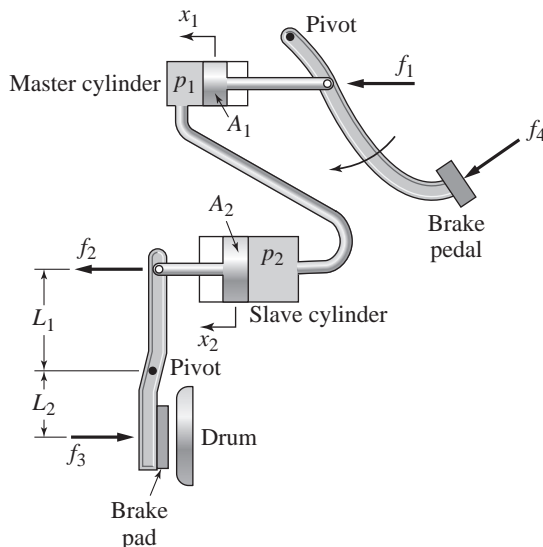
Figure 7.1.1 A hydraulic brake system.

The tradeoff for force amplification is that the master piston must move a distance greater than that of the slave piston. We may see this effect by equating the fluid volume displaced by each piston.

$$\text{volume} = A_1 x_1 = A_2 x_2$$

Thus $x_2 = x_1 A_1 / A_2$, and so $x_2 < x_1$ if $A_1 < A_2$.

Conservation of mass can be stated as follows. For a container holding a mass of fluid m, the time rate of change $\dot{m}$ of mass in the container must equal the total mass inflow rate minus the total mass outflow rate. That is,

$$\dot{m} = q_{mi} - q_{mo} \tag{7.1.1}$$

where q_{mi} is the mass inflow rate and q_{mo} is the mass outflow rate.

The fluid mass m is related to the container volume V by $m = \rho V$. For an incompressible fluid, ρ is constant, and thus $\dot{m} = \rho \dot{V}$. Let q_{vi} and q_{vo} be the total *volume* inflow and outflow rates. Thus, $q_{mi} = \rho q_{vi}$, and $q_{mo} = \rho q_{vo}$. Substituting these relationships into (7.1.1) gives

$$\rho \dot{V} = \rho q_{vi} - \rho q_{vo}$$

Cancel ρ to obtain

$$\dot{V} = q_{vi} - q_{vo} \tag{7.1.2}$$

This is a statement of conservation of *volume* for the fluid, and it is equivalent to conservation of mass, equation (7.1.1), when the fluid is incompressible.

EXAMPLE 7.1.2 | A Water Supply Tank

■ **Problem**

Water is pumped as needed at the mass flow rate $q_{mo}(t)$ from the tank shown in Figure 7.1.2a. Replacement water is pumped from a well at the mass flow rate $q_{mi}(t)$. Determine the water height $h(t)$, assuming that the tank is cylindrical with a cross section A.

Figure 7.1.2 A water supply tank.

(a)   (b)
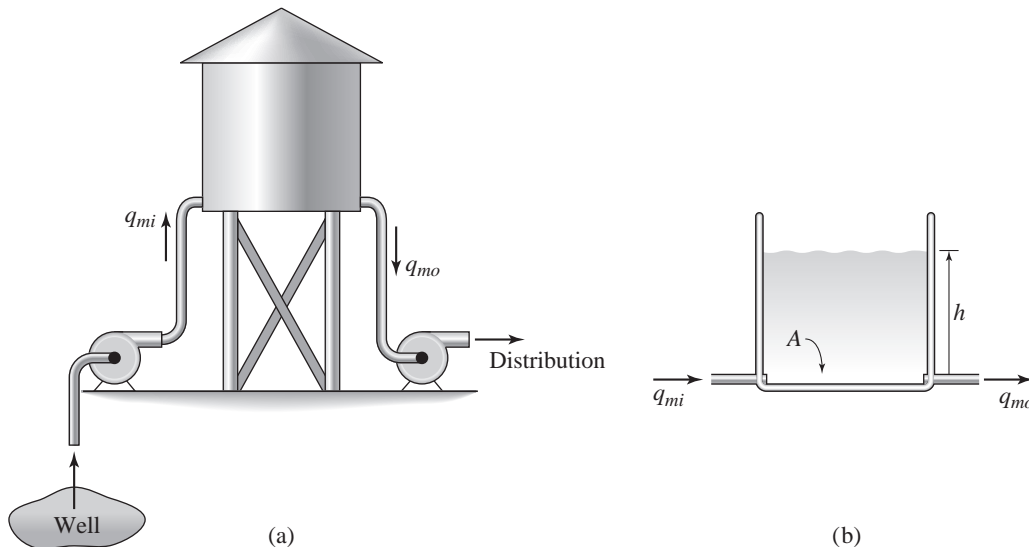
■ Solution

We model the system as shown in part (b) of the figure. The volume of water in the tank is Ah, and therefore the mass of water in the tank is ρAh, where ρ is the mass density of water. From conservation of mass, we have

$$\frac{d}{dt}(\rho Ah) = q_{mi}(t) - q_{mo}(t)$$

Since ρ and A are constant, we have

$$\rho A \frac{dh}{dt} = q_{mi}(t) - q_{mo}(t)$$

which can be integrated as follows:

$$h(t) = h(0) + \frac{1}{\rho A} \int_0^t [q_{mi}(u) - q_{mo}(u)]\, du$$

Once we know the flow rates, we can evaluate the integral.

A common hydraulic actuator is the piston-and-cylinder actuator used on many types of heavy equipment, such as the backhoe shown in Figure 7.1.3. When the operator moves a handle, hydraulic fluid under high pressure is sent through the line to the cylinder. The fluid acts on the piston within the cylinder and produces a force that is equal to the pressure times the piston area. This large force moves the linkage. Example 7.1.3 develops a simple model of such a device.
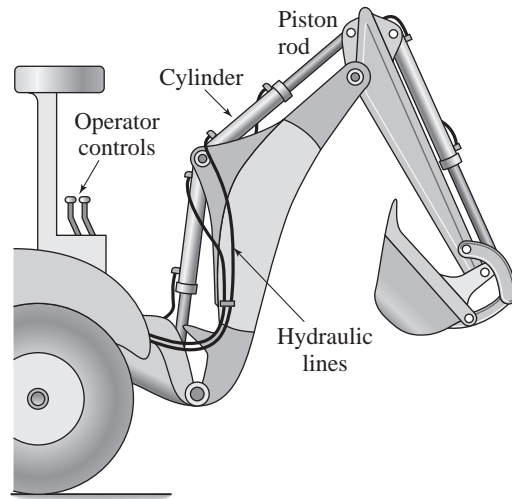
Figure 7.1.3 A backhoe.

| A Hydraulic Cylinder | **EXAMPLE 7.1.3** |

■ Problem

Figure 7.1.4a shows a cylinder and piston connected to a load mass m, which slides on a frictionless surface. Part (b) of the figure shows the piston rod connected to a rack-and-pinion gear. The pressures p_1 and p_2 are applied to each side of the piston by two pumps. Assume the piston rod diameter is small compared to the piston area, so the effective piston area A is the same on both sides of the piston. Assume also that the piston and rod mass have been lumped into m and that any friction is negligible. (a) Develop a model of the motion of the displacement

Figure 7.1.4 A hydraulic
cylinder for (a) translating a
mass and for (b) rotating
a pinion gear.
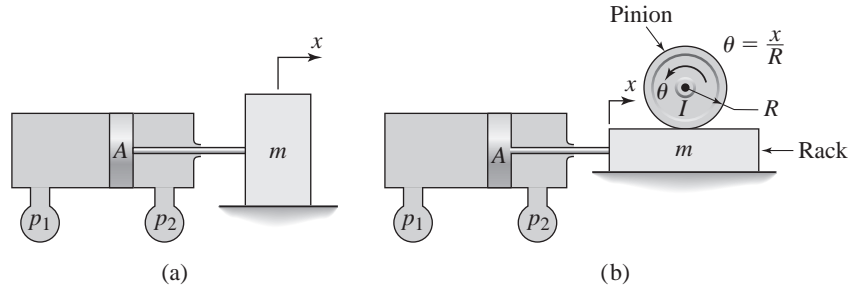
x of the mass in part (a) of the figure, assuming that p_1 and p_2 are given functions of time.
Also, obtain the expression for the mass flow rate that must be delivered or absorbed by the two
pumps. (b) Develop a model of the displacement x in part (b) of the figure. The inertia of the
pinion and the load connected to the pinion is I.

■ **Solution**

a.   Assuming that $p_1 > p_2$, the net force acting on the piston and mass m is $(p_1 - p_2)A$, and
     thus from Newton's law,

$$m\ddot{x} = (p_1 - p_2)A$$

Because p_1 and p_2 are given functions of time, we can integrate this equation once to
obtain the velocity:

$$\dot{x}(t) = \dot{x}(0) + \frac{A}{m} \int_0^t [p_1(u) - p_2(u)]\, du$$

The rate at which fluid volume is swept out by the piston is $A\dot{x}$, and thus if $\dot{x} > 0$, the
pump providing pressure p_1 must supply fluid at the mass rate $\rho A\dot{x}$, and the pump
providing pressure p_2 must absorb fluid at the same mass rate.

b.   Because we want an expression for the displacement x, we obtain an expression for the
     equivalent mass of the rack, pinion, and load. The kinetic energy of the system is

$$\text{KE} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2$$

because $R\dot{\theta} = \dot{x}$.

Thus the equivalent mass is

$$m_e = m + \frac{I}{R^2}$$

The required model can now be obtained by replacing m with m_e in the model developed
in part (a).

EXAMPLE 7.1.4 | A Mixing Process

■ **Problem**

A mixing tank is shown in Figure 7.1.5. Pure water flows into the tank of volume $V = 600\text{ m}^3$
at the constant volume rate of $5\text{ m}^3/\text{s}$. A solution with a salt concentration of s_i kg/m^3 flows
into the tank at a constant volume rate of $2\text{ m}^3/\text{s}$. Assume that the solution in the tank is well
mixed so that the salt concentration in the tank is uniform. Assume also that the salt dissolves
completely so that the volume of the mixture remains the same. The salt concentration s_o kg/m^3
in the outflow is the same as the concentration in the tank. The input is the concentration $s_i(t)$,
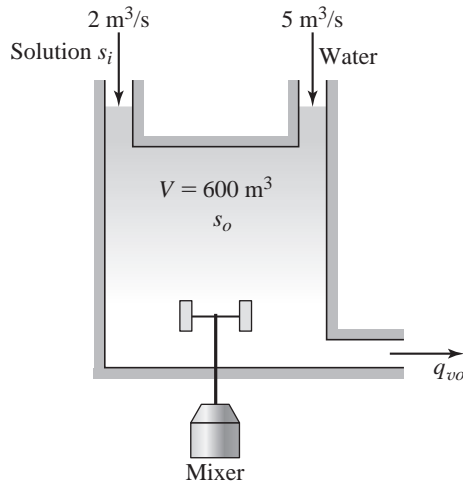
Figure 7.1.5 A mixing process.

whose value may change during the process, thus changing the value of s_o. Obtain a dynamic model of the concentration s_o.

■ Solution

Two mass species are conserved here: water mass and salt mass. The tank is always full, so the mass of water m_w in the tank is constant, and thus conservation of water mass gives

$$\frac{dm_w}{dt} = 5\rho_w + 2\rho_w - \rho_w q_{vo} = 0$$

where ρ_w is the mass density of fresh water, and q_{vo} is the volume outflow rate of the mixed solution. This equation gives $q_{vo} = 5 + 2 = 7$ m³/s.

The salt mass in the tank is $s_o V$, and conservation of salt mass gives

$$\frac{d}{dt}(s_o V) = 0(5) + 2s_i - s_o q_{vo} = 2s_i - 7s_o$$

or, with $V = 600$,

$$600\frac{ds_o}{dt} = 2s_i - 7s_o \tag{1}$$

This is the model. The time constant for the mixing process is $600/7 = 85.7$ s. Thus, if s_i is initially zero and then becomes a nonzero constant value S, the salt concentration in the outflow will eventually become constant at the value $2S/7$ after approximately $4(85.7) = 343$ s.

7.2 FLUID CAPACITANCE

Sometimes it is very useful to think of fluid systems in terms of electrical circuits. Table 7.2.1 gives the fluid quantity, its common nomenclature, its linear relation, and its analogous electrical property. *Fluid resistance* is the relation between pressure and mass flow rate. *Fluid capacitance* is the relation between pressure and stored mass.

We will concentrate on fluid resistance and capacitance. Fluid resistance relates to energy dissipation while fluid capacitance relates to potential energy. *Fluid inertance* relates to fluid acceleration and kinetic energy.

Fluid systems obey two laws that are analogous to Kirchhoff's current and voltage laws; these laws are the *continuity* and the *compatibility* laws. The continuity law is simply a statement of conservation of fluid mass. This says that the total mass flow into

Table 7.2.1 Analogous fluid and electrical quantities.

Fluid quantity	Electrical quantity
Fluid mass, m	Charge, Q
Mass flow rate, q_m	Current, i
Pressure, p	Voltage, v
Fluid linear resistance, R	Electrical resistance, R
$\quad R = p/q_m$	$\quad R = v/i$
Fluid capacitance, C	Electrical capacitance, C
$\quad C = m/p$	$\quad C = Q/v$
Fluid inertance, I	Electrical inductance, L
$\quad I = p/(dq_m/dt)$	$\quad L = v/(di/dt)$

a junction must equal the total flow out of the junction. This is analogous to Kirchhoff's current law. Flow through two rigid pipes joined together to make one pipe is an example where this applies. If, however, the flow is through *flexible* tubes that can expand and contract under pressure, then the outflow rate is not the sum of the inflow rates. This is an example where fluid mass can accumulate within the system and is analogous to having a capacitor in an electrical circuit.

The compatibility law is analogous to Kirchhoff's voltage law, which states that the sum of signed voltage differences around a closed loop must be zero. It is an expression of conservation of energy. The compatibility law states that the sum of signed pressure differences around a closed loop must be zero.

FLUID SYMBOLS AND SOURCES

Figure 7.2.1 shows the commonly used symbols for fluid system elements. The resistance symbol is used to represent both linear and nonlinear fixed resistances, for example, due to pipe flow, orifice flow, or a restriction. A valve that can be manually adjusted, such as a faucet, is a variable resistance and has a slightly different symbol. An actuated valve, driven, for example, by an electric motor or a pneumatic device, has a different symbol. Such valves are usually operated under computer control.

Just as there are ideal voltage and current sources in electrical systems, so we use ideal pressure and flow sources in our fluid system models. An ideal pressure source is capable of supplying the specified pressure at any flow rate. An ideal flow source is capable of supplying the specified flow.

These ideal sources are approximations to real devices such as pumps. For example, Figure 7.2.2 shows the steady-state flow-pressure relation for a centrifugal pump, where q_m is the mass flow rate produced by the pump when the pressure difference across the pump is Δp. When the outlet pressure is greater than the inlet pressure, $\Delta p > 0$. Such

Figure 7.2.1 Fluid system symbols.

Resistance   Manually adjusted valve   Actuated valve

p_1 p_s p_2

$p_s = p_2 - p_1$

Ideal pressure source q_s Ideal flow source   Pump
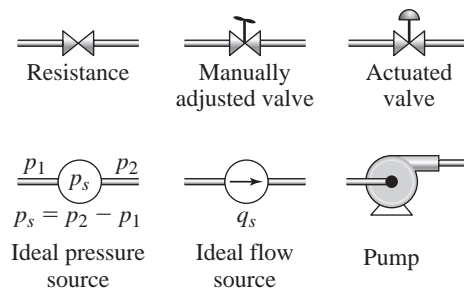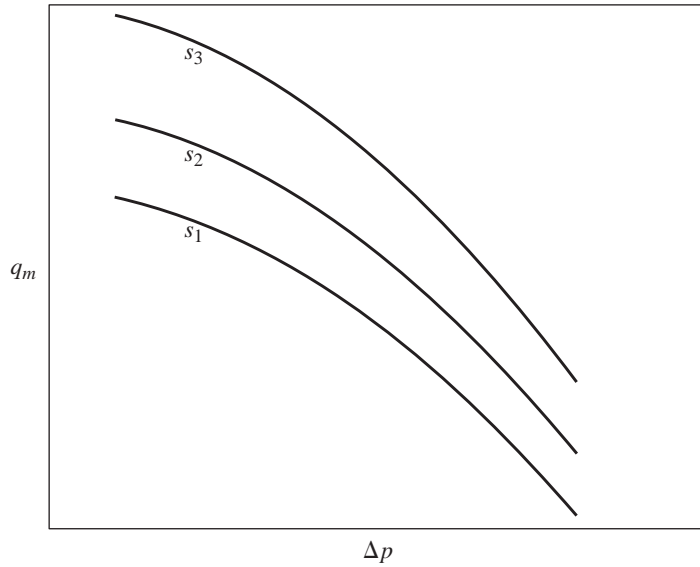
Figure 7.2.2 Steady-state flow-pressure relation for a centrifugal pump.
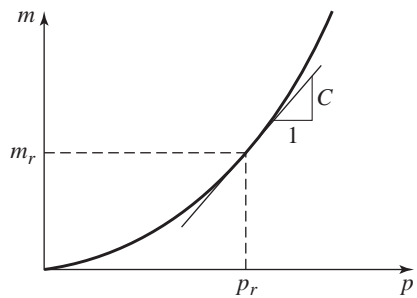
Figure 7.2.3 General fluid capacitance relation and its linear approximation.

curves depend on the pump speed, labeled s_1, s_2, and so on in the figure. To determine the operating condition of the pump for a given speed, we need another relation between q_m and Δp. This relation depends on the load connected to the pump outlet. We will see how such a relation is obtained in Example 7.4.8 in Section 7.4.

CAPACITANCE RELATIONS

Fluid capacitance is the relation between stored fluid mass and the resulting pressure caused by the stored mass. Figure 7.2.3 illustrates this relation, which holds for both pneumatic and hydraulic systems. At a particular reference point (p_r, m_r) the slope is C, where

$$C = \left. \frac{dm}{dp} \right|_{p=p_r} \tag{7.2.1}$$

Thus, fluid capacitance C is the ratio of the change in stored mass to the change in pressure.

EXAMPLE 7.2.1   Capacitance of a Storage Tank

■ Problem

Consider the tank shown in Figure 7.2.4. Assume that the sides are vertical so that the cross-sectional area A is constant. This is the case, for example, with a cylindrical tank whose horizontal cross section is circular, or with a tank having vertical sides and a rectangular horizontal cross section. Derive the expression for the tank's capacitance.

■ Solution

Because the tank's sides are vertical, the liquid height h is related to m, the liquid mass in the tank, by $m = \rho A h$. The total pressure at the bottom of the tank is $\rho g h + p_a$, but the pressure due only to the stored fluid mass is $p = \rho g h$. We can therefore express the pressure as a function of the mass m stored in the tank as $p = mg/A$.

Thus,

$$m = \frac{pA}{g}$$

and the capacitance of the tank is given by

$$C = \frac{dm}{dp} = \frac{A}{g}$$

Figure 7.2.4 Capacitance of a storage tank.

(a)   (b)
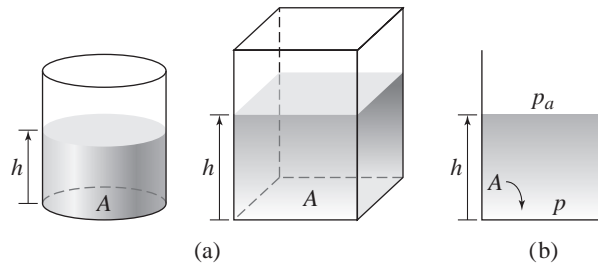
When the container does not have vertical sides, such as the one shown in Figure 7.2.5, the cross-sectional area A is a function of the liquid height h, and the relations between m and h and between p and m are nonlinear. In such cases, there is no single value for the container's capacitance. The fluid mass stored in the container is

$$m = \rho V = \rho \int_0^h A(x)dx,$$

which gives

$$\frac{dm}{dh} = \rho A$$

Figure 7.2.5 A storage tank of arbitrary shape.
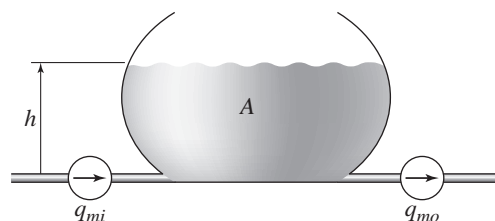
For such a container, conservation of mass gives

$$\frac{dm}{dt} = q_{mi} - q_{mo} \tag{7.2.2}$$

but

$$\frac{dm}{dt} = \frac{dm}{dp}\frac{dp}{dt} = C\frac{dp}{dt}$$

Thus

$$C\frac{dp}{dt} = \frac{dm}{dt} = q_{mi} - q_{mo} \tag{7.2.3}$$

Also

$$\frac{dm}{dt} = \frac{dm}{dh}\frac{dh}{dt} = \rho A\frac{dh}{dt}$$

So

$$\rho A\frac{dh}{dt} = q_{mi} - q_{mo} \tag{7.2.4}$$

Equations (7.2.2), (7.2.3), and (7.2.4) are alternative, but equivalent, hydraulic models of a container of fluid. They suggest that either pressure p, mass m, or height h can be chosen as the model's variable. These variables are all indicators of the system's potential energy, and as such any one can be chosen as a state variable. If the container's cross-sectional area is constant, then $V = Ah$ and thus the liquid volume V can also be used as the model variable.

| Capacitance of a V-Shaped Trough | **EXAMPLE 7.2.2** |

■ Problem
(a) Derive the capacitance of the V-shaped trough shown in Figure 7.2.6a. (b) Use the capacitance to derive the dynamic models for the bottom pressure p and the height h. The mass inflow rate is $q_{mi}(t)$, and there is no outflow.

■ Solution
a. From part (b) of the figure, $D = 2h \tan\theta$, and the vertical cross-sectional area of the liquid is $hD/2$. Thus the fluid mass is given by

$$m = \rho V = \rho\left(\frac{1}{2}hD\right)L = (\rho L \tan\theta)h^2$$
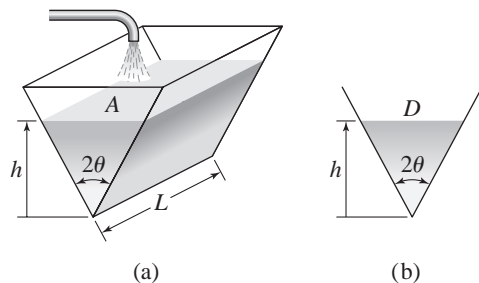
Figure 7.2.6 A V-shaped trough.

(a)   (b)

But $p = \rho g h$ and thus,

$$m = (\rho L \tan\theta)\left(\frac{p}{\rho g}\right)^2 = \left(\frac{L \tan\theta}{\rho g^2}\right) p^2$$

From the definition of capacitance,

$$C = \frac{dm}{dp} = \left(\frac{2L \tan\theta}{\rho g^2}\right) p$$

b.   From (7.2.3) with $q_{mo} = 0$, $C\dot{p} = q_{mi}$, or

$$\left(\frac{2L \tan\theta}{\rho g^2}\right) p\frac{dp}{dt} = q_{mi}$$

which is a nonlinear equation because of the product $p\dot{p}$. We can obtain the model for the height by substituting $h = p/\rho g$. The result is

$$(2\rho L \tan\theta)h\frac{dh}{dt} = q_{mi}$$

7.3 FLUID RESISTANCE

Fluid meets resistance when flowing through a conduit such as a pipe, through a component such as a valve, or even through a simple opening or *orifice,* such as a hole. We now consider appropriate models for each type of resistance.

The mass flow rate q_m through a resistance is related to the pressure difference p across the resistance. This relation, $p = f(q_m)$, is illustrated in general by Figure 7.3.1. We define the *fluid resistance R* as

$$R = \frac{dp}{dq_m}\bigg|_{q=q_{mr}} \tag{7.3.1}$$

Thus the resistance is the slope of the p versus q_m curve at some reference flow rate q_{mr} and reference pressure p_r. The values of q_{mr} and p_r depend on the particular application.

In a limited number of cases, such as pipe flow under certain conditions, the relation of p versus q_m is linear so that $p = Rq_m$, or

$$q_m = \frac{p}{R} \tag{7.3.2}$$

In some cases a formula is available to compute the numerical value of R.

Figure 7.3.1 General fluid resistance relation and its linear approximations.
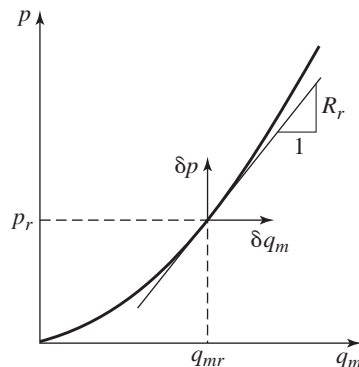
In some other applications the relation is a square-root relation.

$$q_m = \sqrt{\frac{p}{R_1}} \qquad (7.3.3)$$

We usually must determine the value of R_1 empirically.

If we need to obtain an approximate linear model, we can linearize the expression $p = f(q_m)$ near a reference operating point (q_{mr}, p_r) as follows:

$$p = p_r + \left(\frac{dp}{dq_m}\right)_r (q_m - q_{mr}) = p_r + R_r(q_m - q_{mr}) \qquad (7.3.4)$$

where R_r is the linearized resistance at the reference condition (q_{mr}, p_r).

Note that the symbol dp represents an *infinitesimal* change in the variable p. We will use the symbol δp to represent a small but finite change in p. The symbol δp is called a *deviation variable* because it often represents a small deviation in p from some reference value p_r. Thus, $\delta p = p - p_r$.

In terms of the deviation variables $\delta q_m = q_m - q_{mr}$ and $\delta p = p - p_r$, we can rewrite (7.3.4) as

$$\delta p = \left(\frac{dp}{dq_m}\right)_r \delta q_m = R_r \delta q_m \qquad (7.3.5)$$

or

$$\delta q_m = \frac{1}{R_r}\delta p$$

For the relation (7.3.3), $p = R_1 q_m^2$ and

$$R_r = \left(\frac{dp}{dq_m}\right)_r = 2R_1 q_{mr}$$

Thus, since $q_{mr} = \sqrt{p_r/R_1}$,

$$\delta p = R_r \delta q_m = 2R_1 \sqrt{\frac{p_r}{R_1}} \delta q_m = 2\sqrt{R_1 p_r} \delta q_m$$

or

$$\delta q_m = \frac{1}{2\sqrt{R_1 p_r}}\delta p \qquad (7.3.6)$$

When only the curve of p versus q_m is available, we can obtain a linearized model by graphically computing the slope S of the tangent line that passes through the reference point (q_{mr}, p_r). The equivalent, linearized resistance R_r is the slope S.

The resistance symbol shown in Figure 7.3.2 represents all types of fluid resistance, whether linear or not. Although the symbol looks like a valve, it can represent fluid resistance due to other causes, such as pipe wall friction and orifices.

As with electrical resistances, linear fluid resistance elements obey the series and parallel combination rules. These are illustrated in Figure 7.3.2. Series fluid resistances carry the same flow rate; parallel fluid resistances have the same pressure difference across them.

LAMINAR PIPE RESISTANCE

Fluid motion is generally divided into two types: *laminar flow* and *turbulent flow*. Laminar flow can be described as "smooth" in the sense that the average fluid particle velocity is the same as the actual particle velocity. If the flow is "rough," the *average*

Figure 7.3.2 Combination of
(a) series resistances and
(b) parallel resistances.

$$R = R_1 + R_2$$

(a)

$$q_m = q_{m_1} + q_{m_2}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

(b)
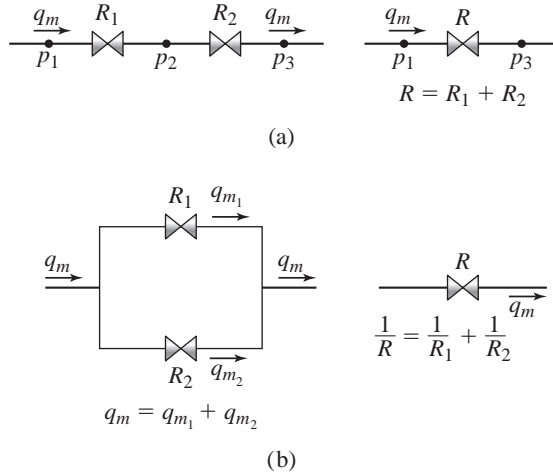
particle velocity will be less than the *actual* particle velocity, because the fluid particles meander while moving downstream. This is turbulent flow. You can see the difference between laminar and turbulent flow by slightly opening a faucet; the flow will be smooth. As you open the faucet more, eventually the flow becomes rough.

If the pipe flow is laminar, the linear relation (7.3.2) applies. The laminar resistance for a level pipe of diameter D and length L is given by the *Hagen-Poiseuille* formula

$$R = \frac{128\mu L}{\pi\rho D^4} \tag{7.3.7}$$

where μ is the fluid viscosity. The viscosity is a measure of the "stickiness" of the fluid. Thus molasses has a higher value of μ than that of water.

Not all pipe flow is laminar. A useful criterion for predicting the existence of laminar flow is the *Reynolds number* N_e, the ratio of the fluid's inertial forces to the viscosity forces. For a circular pipe,

$$N_e = \frac{\rho\overline{v}D}{\mu} \tag{7.3.8}$$

where $\overline{v} = q_v/(\pi D^2/4)$, the average fluid velocity. For $N_e > 2300$ the flow is often turbulent, while for $N_e < 2300$ laminar flow usually exists. The precise value of N_e above which the flow becomes turbulent depends on, for example, the flow conditions at the pipe inlet. However, the criterion is useful as a rule of thumb.

The resistance formula (7.3.7) applies only if the so-called "entrance length" L_e, which is the distance from the pipe entrance beyond which the velocity profile no longer changes with increasing distance, is much less than $0.06DN_e$. Because laminar flow can be expected only if $N_e < 2300$, L_e might be as long as 138 pipe diameters. Of course, for small Reynolds numbers, L_e is shorter. The smaller L_e is relative to the pipe length, the more reliable will be our resistance calculations.

SYSTEM MODELS

In liquid-level systems such as shown in Figure 7.3.3, energy is stored in two ways: as potential energy in the mass of liquid in the tank, and as kinetic energy in the mass of liquid flowing in the pipe. In many systems, the mass of the liquid in the pipes is small compared to the liquid mass in the tanks. If the mass of liquid in a pipe is small

enough or is flowing at a small enough velocity, the kinetic energy contained in it will be negligible compared to the potential energy stored in the liquid in the tank. If the kinetic energy of the liquid is significant, more advanced fluid-flow theory is required. This is usually not the case for the scope of applications considered here.
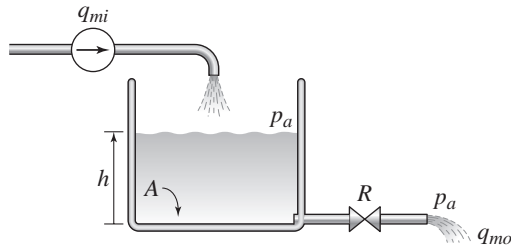
| Liquid-Level System with a Flow Source | **EXAMPLE 7.3.1** |

■ Problem

The cylindrical tank shown in Figure 7.3.3 has a bottom area A. The mass inflow rate from the flow source is $q_{mi}(t)$, a given function of time. The outlet resistance is linear and the outlet discharges to atmospheric pressure p_a. Develop a model of the liquid height h.

■ Solution

The total mass in the tank is $m = \rho A h$, and from conservation of mass

$$\frac{dm}{dt} = \rho A \frac{dh}{dt} = q_{mi} - q_{mo} \qquad (1)$$

since ρ and A are constants. Because the outlet resistance is linear,

$$q_{mo} = \frac{1}{R}\left[(\rho g h + p_a) - p_a\right] = \frac{1}{R}\rho g h$$

Substituting this into equation (1) gives the desired model:

$$\rho A \frac{dh}{dt} = q_{mi} - \frac{\rho g}{R}h \qquad (2)$$

The time constant is $\tau = RA/g$.

Some engineers are helped by thinking of a fluid system in terms of an analogous electric circuit, in which pressure difference plays the role of voltage difference, and mass flow rate is analogous to current. A fluid resistance resists flow just as an electrical resistor resists current. A fluid capacitance stores fluid mass just as an electrical capacitor stores charge. Figure 7.3.4 shows an electric circuit that is analogous to the tank system of Figure 7.3.3. The circuit model is

$$C\frac{dv}{dt} = i_s - \frac{1}{R}v$$

The input current i_s is analogous to the inflow rate q_{mi}, the voltage v across the capacitor is analogous to the fluid pressure $\rho g h$, and the electrical capacitance C is analogous to the fluid capacitance A/g. It is a matter of personal opinion as to whether such analogies help to understand the dynamics of fluid systems, and you should decide for yourself. Always keep in mind, however, that we should not get too dependent on analogies for
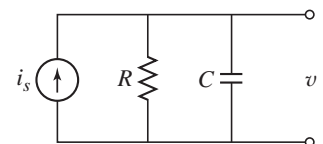
Figure 7.3.5 Derivation of
Torricelli's principle.
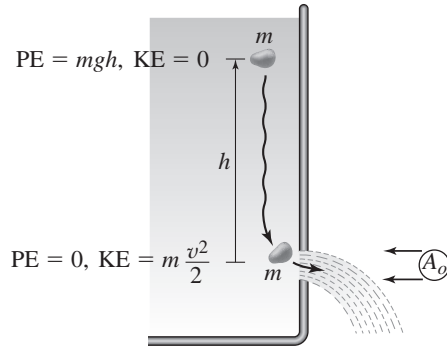
developing models, because they might not always properly represent the underlying physics of the original system.

TORRICELLI'S PRINCIPLE

An orifice can simply be a hole in the side of a tank or it can be a passage in a valve. We saw an example of orifice flow in Example 1.5.2 in Chapter 1, in which we analyzed the flow rate of water through a small hole in the side of a plastic milk bottle. We found that the fitted function is $f = 9.4h^{0.558}$, where f is the outflow rate in ml/s and the water height h is in centimeters. It turns out that the empirically determined exponent 0.558 is close to its theoretical value of 0.5, as we will now demonstrate.

Around 1640 Torricelli discovered that the flow rate through an orifice is proportional to the square root of the pressure difference. This observation can be simply derived by considering a mass m of fluid a height h above the orifice (see Figure 7.3.5). The potential energy of the mass is mgh. As the mass falls toward the orifice its potential energy is converted to kinetic energy $mv^2/2$. If all the potential energy is converted to kinetic energy at the orifice, then $mgh = mv^2/2$, and the maximum speed the fluid mass can attain through the orifice is $v = \sqrt{2gh}$. Because the pressure drop across the orifice is $p = \rho gh$, we can express the maximum speed as $v = \sqrt{2p/\rho}$. Thus the mass flow rate q_m through the orifice of area A_o can be no greater than $A_o \rho v = A_o \rho \sqrt{2p/\rho} = A_o \sqrt{2p\rho}$. The actual flow rate will be less than this value because of friction effects. To account for these frictional effects, we introduce a factor C_d as follows:

$$q_m = C_d A_o \sqrt{2p\rho} \qquad (7.3.9)$$

The factor C_d is the *discharge coefficient,* which must lie in the range $0 < C_d \leq 1$. A typical value for water is 0.6.

Because $p = \rho gh$, (7.3.9) can be expressed in terms of the volume flow rate q_v and the height h as follows:

$$q_v = \left(C_d A_o \sqrt{2g} \right) h^{0.5}$$

Thus the theoretical value of the exponent (0.5) is close to the value obtained in the bottle experiment.

Equation (7.3.9) depends on the orifice area being small enough so that the pressure variation over the orifice area is negligible compared to the average pressure at the orifice. For a liquid-level system with a circular orifice, this implies that the liquid height above the orifice must be large compared to the orifice diameter.

The orifice relation (7.3.9) can be rearranged in the form of (7.3.3).

$$q_m = C_d A_o \sqrt{2\rho} \sqrt{p} = \sqrt{\frac{p}{R_o}} \qquad (7.3.10)$$

where the *orifice resistance* is defined as

$$R_o = \frac{1}{2\rho C_d^2 A_o^2} \qquad (7.3.11)$$

| Liquid-Level System with an Orifice | **EXAMPLE 7.3.2** |

■ Problem

The cylindrical tank shown in Figure 7.3.6 has a circular bottom area A. The volume inflow rate from the flow source is $q_{vi}(t)$, a given function of time. The orifice in the side wall has an area A_o and discharges to atmospheric pressure p_a. Develop a model of the liquid height h, assuming that $h_1 > L$.

■ Solution

From conservation of mass and the orifice flow relation (7.3.9), we obtain

$$\rho A \frac{dh}{dt} = \rho q_{vi} - C_d A_o \sqrt{2p\rho}$$

where $p = \rho g h$. Thus the model becomes

$$A \frac{dh}{dt} = q_{vi} - C_d A_o \sqrt{2gh}$$

Note that the height L does not enter the model. It serves only to relate h_1 to h.
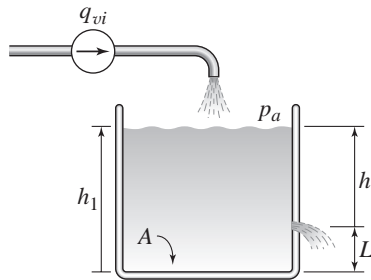
Figure 7.3.6 A liquid-level system with an orifice.

TURBULENT AND COMPONENT RESISTANCE

For us, the practical importance of the difference between laminar and turbulent flow lies in the fact that laminar flow can be described by the *linear* relation (7.3.2), while turbulent flow is described by the *nonlinear* relation (7.3.3). Components, such as valves, elbow bends, couplings, porous plugs, and changes in flow area resist flow and usually induce turbulent flow at typical pressures, and (7.3.3) is often used to model them. Experimentally determined values of R are available for common types of components.

7.4 DYNAMIC MODELS OF HYDRAULIC SYSTEMS

In this section we consider a number of hydraulic system examples dealing with liquid-level systems, dampers, actuators, pumps, and nonlinear systems.

EXAMPLE 7.4.1

Liquid-Level System with a Pressure Source

■ Problem

The tank shown in cross section in Figure 7.4.1 has a bottom area A. A pressure source p_s is connected through a linear resistance to the bottom of the tank, where $p_s(t)$ is a given function of time. The outlet resistance is linear and the outlet discharges to atmospheric pressure p_a. Develop a model of the liquid height h.

■ Solution

The total mass in the tank is $m = \rho A h$, and from conservation of mass

$$\frac{dm}{dt} = \rho A \frac{dh}{dt} = q_{mi} - q_{mo} \qquad (1)$$

since ρ and A are constants. Because the outlet resistance is linear,

$$q_{mo} = \frac{1}{R_2}[(\rho g h + p_a) - p_a] = \frac{\rho g h}{R_2}$$

The mass inflow rate is

$$q_{mi} = \frac{1}{R_1}[(p_s + p_a) - (\rho g h + p_a)] = \frac{1}{R_1}(p_s - \rho g h)$$

Substituting these expressions into equation (1) gives the desired model:

$$\rho A \frac{dh}{dt} = \frac{1}{R_1}(p_s - \rho g h) - \frac{\rho g}{R_2}h$$

which can be rearranged as

$$\rho A \frac{dh}{dt} = \frac{1}{R_1}p_s - \rho g\left(\frac{1}{R_1} + \frac{1}{R_2}\right)h = \frac{1}{R_1}p_s - \rho g\frac{R_1 + R_2}{R_1 R_2}h$$

The time constant is $\tau = R_1 R_2 A/g(R_1 + R_2)$.

Figure 7.4.1 A liquid-level system with a pressure source.
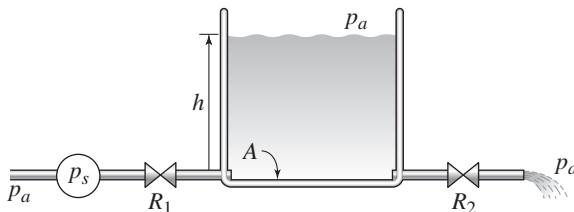
EXAMPLE 7.4.2

Water Tank Model

■ Problem

In the water tower model developed in Example 7.1.2, the input was the specified flow rate q_{mi}. We now treat the more realistic case in which the input is a pressure source such as a pump. Figure 7.4.2 is a representation of the situation. The linear resistance R represents the pipe resistance lumped at the outlet of the pressure source. The bottom of the water tank is a height L above the pressure source. Develop a model of the water height h with the supply pressure p_s and the flow rate $q_{mo}(t)$ as the inputs.

■ Solution

The mass flow rate into the bottom of the tank is

$$q_{mi} = \frac{1}{R}\{(p_s + p_a) - [\rho g(h + L) + p_a]\} = \frac{1}{R}[p_s - \rho g(h + L)]$$

From conservation of mass,

$$\frac{d}{dt}(\rho Ah) = q_{mi} - q_{mo}(t) = \frac{1}{R}[p_s - \rho g(h + L)] - q_{mo}(t)$$

Because ρ and A are constants, the model can be written as

$$\rho A \frac{dh}{dt} = \frac{1}{R}[p_s - \rho g(h + L)] - q_{mo}(t)$$
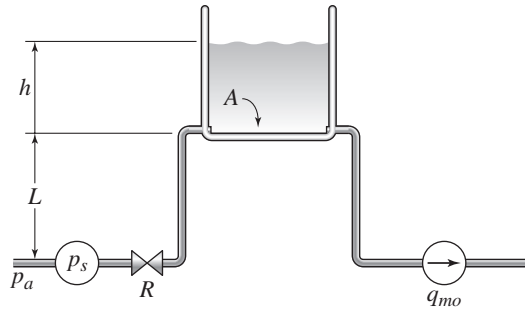
Figure 7.4.2 A water tank model.

When a fluid system contains more than one capacitance, you should apply the conservation of mass principle to each capacitance, and then use the appropriate resistance relations to couple the resulting equations. To do this you must assume that some pressures or liquid heights are greater than others and assign the positive-flow directions accordingly. If you are consistent, the mathematics will handle the reversals of flow direction automatically.

Two Connected Tanks | **EXAMPLE 7.4.3**

■ Problem

The cylindrical tanks shown in Figure 7.4.3a have bottom areas A_1 and A_2. The mass inflow rate $q_{mi}(t)$ from the flow source is a given function of time. The resistances are linear and the outlet discharges to atmospheric pressure p_a. (a) Develop a model of the liquid heights h_1 and
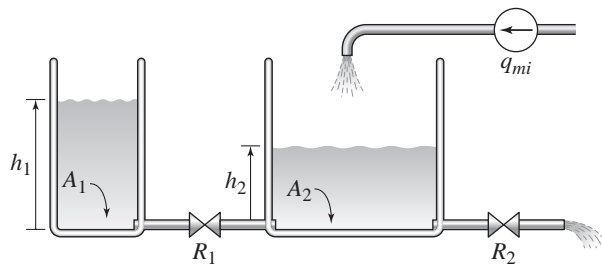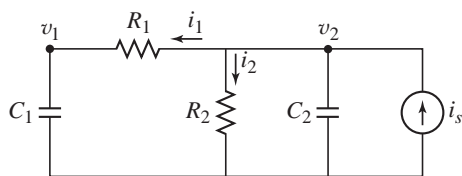
Figure 7.4.3 (a) Two connected tanks. (b) Analogous electric circuit.

(a)

(b)

h_2. (b) Suppose the resistances are equal: $R_1 = R_2 = R$, and the areas are $A_1 = A$ and $A_2 = 3A$. Obtain the transfer function $H_1(s)/Q_{mi}(s)$. (c) Use the transfer function to solve for the steady-state response for h_1 if the inflow rate q_{mi} is a unit-step function, and estimate how long it will take to reach steady state. Is it possible for liquid heights to oscillate in the step response?

■ Solution

a.   Assume that $h_1 > h_2$ so that the mass flow rate q_{m_1} is positive if flowing from tank 1 to tank 2. Conservation of mass applied to tank 1 gives

$$\rho A_1 \dot{h}_1 = -q_{m_1} = -\frac{\rho g}{R_1}(h_1 - h_2)$$

For tank 2,

$$\rho A_2 \dot{h}_2 = q_{mi} + q_{m_1} - q_{mo} = q_{mi} + q_{m_1} - \frac{\rho g}{R_2} h_2$$

Canceling ρ where possible, we obtain the desired model.

$$A_1 \dot{h}_1 = -\frac{g}{R_1}(h_1 - h_2) \tag{1}$$

$$\rho A_2 \dot{h}_2 = q_{mi} + \frac{\rho g}{R_1}(h_1 - h_2) - \frac{\rho g}{R_2} h_2 \tag{2}$$

b.   Substituting $R_1 = R_2 = R$, $A_1 = A$, and $A_2 = 3A$ into the differential equations and dividing by A, and letting $B = g/RA$ we obtain

$$\dot{h}_1 = -B(h_1 - h_2)$$

$$3\dot{h}_2 = \frac{q_{mi}}{\rho A} + B(h_1 - h_2) - Bh_2 = \frac{q_{mi}}{\rho A} + Bh_1 - 2Bh_2$$

Apply the Laplace transform of each equation, assuming zero initial conditions, and collect terms to obtain

$$(s + B)H_1(s) - BH_2(s) = 0 \tag{3}$$

$$-BH_1(s) + (3s + 2B)H_2(s) = \frac{1}{\rho A} Q_{mi}(s) \tag{4}$$

Solve equation (3) for $H_2(s)$, substitute the expression into equation (4), and solve for $H_1(s)$ to obtain

$$\frac{H_1(s)}{Q_{mi}(s)} = \frac{RB^2/\rho g}{3s^2 + 5Bs + B^2} \tag{5}$$

c.   The characteristic equation is $3s^2 + 5Bs + B^2 = 0$ and has the two real roots

$$s = \frac{-5 \pm \sqrt{13}}{6} B = -1.43B, -0.232B$$

Thus the system is stable, and there will be a constant steady-state response to a step input. The step response cannot oscillate because both roots are real. The steady-state height can be obtained by applying the final value theorem to equation (5) with $Q_{mi}(s) = 1/s$.

$$h_{1ss} = \lim_{s \to 0} s H_1(s) = \lim_{s \to 0} s \frac{RB^2/\rho g}{3s^2 + 5Bs + B^2} \frac{1}{s} = \frac{R}{\rho g}$$

The time constants are

$$\tau_1 = \frac{1}{1.43B} = \frac{0.699}{B} \qquad \tau_2 = \frac{1}{0.232B} = \frac{4.32}{B}$$

The largest time constant is τ_2 and thus it will take a time equal to approximately $4\tau_2 = 17.2/B$ to reach steady state.

Figure 7.4.3b shows an electrical circuit that is analogous to the hydraulic system shown in part (a) of the figure. The currents i_s, i_1, and i_2 are analogous to the mass flow rates q_{mi}, q_{m_1}, and q_{mo}. The voltages v_1 and v_2 are analogous to the pressures $\rho g h_1$ and $\rho g h_2$, and the capacitances C_1 and C_2 are analogous to the fluid capacitances A_1/g and A_2/g.

HYDRAULIC DAMPERS

Dampers oppose a velocity difference across them, and thus they are used to limit velocities. The most common application of dampers is in vehicle shock absorbers.

Linear Damper | **EXAMPLE 7.4.4**

■ Problem
A damper exerts a force as a result of a velocity difference across it. Figure 7.4.4 shows the principle used in automotive shock absorbers. A piston of diameter W and thickness L has a cylindrical hole of diameter D. The piston rod extends out of the housing, which is sealed and filled with a viscous incompressible fluid. Assuming that the flow through the hole is laminar and that the entrance length L_e is small compared to L, develop a model of the relation between the applied force f and $\dot{x}$, the relative velocity between the piston and the cylinder.

■ Solution
Assume that the rod's cross-sectional area and the hole area $\pi(D/2)^2$ are small compared to the piston area A. Let m be the combined mass of the piston and rod. Then the force f acting on the piston rod creates a pressure difference $(p_1 - p_2)$ across the piston such that

$$m\ddot{y} = f - A(p_1 - p_2) \tag{1}$$

If the mass m or the acceleration $\ddot{y}$ is small, then $m\ddot{y} \approx 0$, and we obtain

$$f = A(p_1 - p_2) \tag{2}$$

For laminar flow through the hole,

$$q_v = \frac{1}{\rho}q_m = \frac{1}{\rho R}(p_1 - p_2) \tag{3}$$

The volume flow rate q_v is the rate at which the piston sweeps out volume as it moves, and can be expressed as

$$q_v = A(\dot{y} - \dot{z}) = A\dot{x} \tag{4}$$
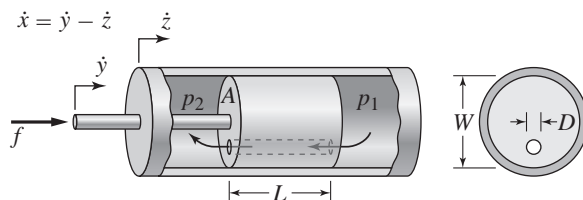
Figure 7.4.4 A damper.

because the fluid is incompressible. Combining equations (2), (3), and (4), we obtain

$$f = A(\rho R A \dot{x}) = \rho R A^2 \dot{x} = c \dot{x}$$

where the damping coefficient c is given by $c = \rho R A^2$. From the Hagen-Poiseuille formula (7.3.7) for a cylindrical conduit,

$$R = \frac{128 \mu L}{\pi \rho D^4}$$

and thus the damping coefficient can be expressed as

$$c = \frac{128 \mu L A^2}{\pi D^4}$$

The approximation $m \ddot{y} \approx 0$ is commonly used for hydraulic systems to simplify the resulting model. To see the effect of this approximation, use instead equation (1) with equations (3) and (4) to obtain

$$q_v = \frac{1}{\rho R}(p_1 - p_2) = \frac{1}{\rho R A}(f - m \ddot{y}) = A \dot{x}$$

Thus,

$$f = m \ddot{y} + \rho R A^2 \dot{x}$$

Therefore, if $m \ddot{y}$ cannot be neglected, the damper force is a function of the absolute acceleration as well as the relative velocity.

HYDRAULIC ACTUATORS

Hydraulic actuators are widely used with high pressures to obtain high forces for moving large loads or achieving high accelerations. The working fluid may be liquid, as is commonly found with construction machinery, or it may be air, as with the air cylinder-piston units frequently used in manufacturing and parts-handling equipment.

EXAMPLE 7.4.5

Hydraulic Piston and Load

■ Problem
Figure 7.4.5 shows a double-acting piston and cylinder. The device moves the load mass m in response to the pressure sources p_1 and p_2. Assume the fluid is incompressible, the resistances are linear, and the piston mass is included in m. Derive the equation of motion for m.

Figure 7.4.5 A double-acting piston and cylinder.
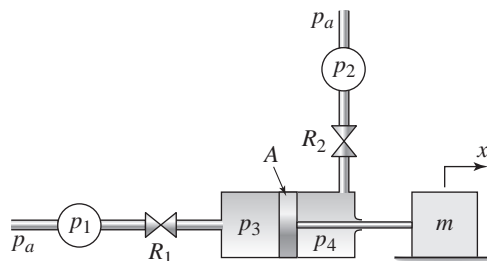
■ **Solution**

Define the pressures p_3 and p_4 to be the pressures on the left- and right-hand sides of the piston. The mass flow rates through the resistances are

$$q_{m_1} = \frac{1}{R_1}(p_1 + p_a - p_3) \tag{1}$$

$$q_{m_2} = \frac{1}{R_2}(p_4 - p_2 - p_a) \tag{2}$$

From conservation of mass, $q_{m_1} = q_{m_2}$ and $q_{m_1} = \rho A \dot{x}$. Combining these four equations we obtain

$$p_1 + p_a - p_3 = R_1 \rho A \dot{x} \tag{3}$$

$$p_4 - p_2 - p_a = R_2 \rho A \dot{x} \tag{4}$$

Adding equations (3) and (4) gives

$$p_4 - p_3 = p_2 - p_1 + (R_1 + R_2)\rho A \dot{x} \tag{5}$$

From Newton's law,

$$m\ddot{x} = A(p_3 - p_4) \tag{6}$$

Substitute equation (5) into (6) to obtain the desired model:

$$m\ddot{x} + (R_1 + R_2)\rho A^2 \dot{x} = A(p_1 - p_2) \tag{7}$$

Note that if the resistances are zero, the $\dot{x}$ term disappears, and we obtain

$$m\ddot{x} = A(p_1 - p_2)$$

which is identical to the model derived in part (a) of Example 7.1.3.

Hydraulic Piston with Negligible Load | **EXAMPLE 7.4.6**

■ **Problem**

Develop a model for the motion of the load mass m in Figure 7.4.5, assuming that the product of the load mass m and the load acceleration $\ddot{x}$ is very small.

■ **Solution**

If $m\ddot{x}$ is very small, from equation (7) of Example 7.4.5, we obtain the model

$$(R_1 + R_2)\rho A^2 \dot{x} = A(p_1 - p_2)$$

which can be expressed as

$$\dot{x} = \frac{p_1 - p_2}{(R_1 + R_2)\rho A} \tag{1}$$

From this we see that if $p_1 - p_2$ is constant, the mass velocity $\dot{x}$ will also be constant.

The implications of the approximation $m\ddot{x} = 0$ can be seen from Newton's law:

$$m\ddot{x} = A(p_3 - p_4) \tag{2}$$

If $m\ddot{x} = 0$, equation (2) implies that $p_3 = p_4$; that is, the pressure is the same on both sides of the piston. From this we can see that the pressure difference across the piston is produced by a large load mass or a large load acceleration. The modeling implication of this fact is that if we neglect the load mass or the load acceleration, we can develop a simpler model of a hydraulic system—a model based only on conservation of mass and not on Newton's law. The resulting model will be first order rather than second order.

EXAMPLE 7.4.7

Hydraulic Motor

■ **Problem**

A *hydraulic motor* is shown in Figure 7.4.6. The pilot valve controls the flow rate of the hydraulic fluid from the supply to the cylinder. When the pilot valve is moved to the right of its neutral position, the fluid enters the right-hand piston chamber and pushes the piston to the left. The fluid displaced by this motion exits through the left-hand drain port. The action is reversed for a pilot valve displacement to the left. Both return lines are connected to a sump from which a pump draws fluid to deliver to the supply line. Derive a model of the system assuming that $m\ddot{x}$ is negligible.

■ **Solution**

Let y denote the displacement of the pilot valve from its neutral position, and x the displacement of the load from its last position before the start of the motion. Note that a positive value of x (to the left) results from a positive value of y (to the right). The flow through the cylinder port uncovered by the pilot valve can be treated as flow through an orifice. Let Δp be the pressure drop across the orifice. Thus, from (7.3.9) with p replaced by Δp, the volume flow rate through the cylinder port is given by

$$q_v = \frac{1}{\rho}q_m = \frac{1}{\rho}C_d A_o \sqrt{2\Delta p \rho} = C_d A_o \sqrt{2\Delta p/\rho} \tag{1}$$

where A_o is the uncovered area of the port, C_d is the discharge coefficient, and ρ is the mass density of the fluid. The area A_o is approximately equal to yD, where D is the port depth (into the page). If C_d, ρ, Δp, and D are taken to be constant, equation (1) can be written as

$$q_v = C_d D y \sqrt{2\Delta p/\rho} = By \tag{2}$$

where $B = C_d D\sqrt{2\Delta p/\rho}$.

Assuming that the rod's area is small compared to the piston area, the piston areas on the left and right sides are equal to A. The rate at which the piston pushes fluid out of the cylinder is $A\,dx/dt$. Conservation of volume requires the volume flow rate into the cylinder be equal to the volume flow rate out. Therefore,

$$q_v = A\frac{dx}{dt} \tag{3}$$

Figure 7.4.6 A hydraulic motor.
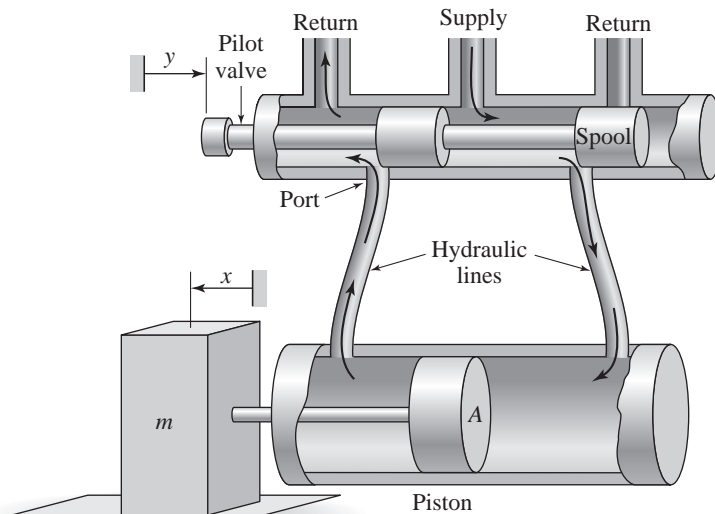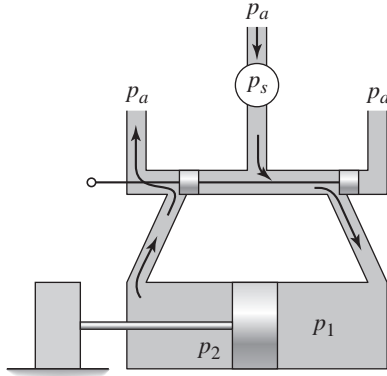
Figure 7.4.7 Pressures in a hydraulic motor.

Combining the last two equations gives the model for the servomotor:

$$\frac{dx}{dt} = \frac{B}{A}y \qquad (4)$$

This model predicts a constant piston velocity dx/dt if the pilot valve position y is held fixed.

The pressure drop Δp can be determined as follows. We assume that because of geometric symmetry, the pressure drop is the same across both the inlet and outlet valves. From Figure 7.4.7 we see that

$$\Delta p = (p_s + p_a) - p_1 = p_2 - p_a$$

and thus

$$p_1 - p_2 = p_s - 2\Delta p \qquad (5)$$

where p_1 and p_2 are the pressures on either side of the piston. The force on the piston is $A(p_1 - p_2)$, and from Newton's law, $m\ddot{x} = A(p_1 - p_2)$. Using the approximation $m\ddot{x} = 0$, we see that $p_1 = p_2$, and thus equation (5) shows that $\Delta p = p_s/2$. Therefore $B = C_d D \sqrt{p_s/\rho}$.

Equation (4) is accurate for many applications, but for high accelerations or large loads, this model must be modified because it neglects the effects of the inertia of the load and piston on the pressures on each side of the piston.

PUMP MODELS

Pump behavior, especially dynamic response, can be quite complicated and difficult to model. At our level of treatment, we will confine ourselves to obtaining linearized models based on the steady-state performance curves. Typical performance curves for a centrifugal pump are shown in Figure 7.4.8a, which relates the mass flow rate q_m through the pump to the pressure increase Δp in going from the pump inlet to its outlet, for a given pump speed s_j. For a given speed and given equilibrium values $(q_m)_e$ and $(\Delta p)_e$, we can obtain a linearized description as shown in part (b) of the figure. This linearized model consists of a straight line tangent to the pump curve at the equilibrium point, and can be expressed as

$$\delta q_m = -\frac{1}{r}\delta(\Delta p)$$

where δq_m and $\delta(\Delta p)$ are the deviations of q_m and Δp from their equilibrium values. Thus, $\delta q_m = q_m - (q_m)_e$ and $\delta(\Delta p) = \Delta p - (\Delta p)_e$.

Identification of the equilibrium values depends on the load connected downstream of the pump. Once this load is known, the resulting equilibrium flow rate of the system

Figure 7.4.8 (a) Performance
curves for a centrifugal pump.
(b) Linearized model.

q_m

s_3

s_2

s_1

Δp

(a)

q_m

$\delta q_m = -\dfrac{1}{r}\,\delta(\Delta p)$

$(q_m)_e$

Pump curve

$(\Delta p)_e$ Δp

(b)
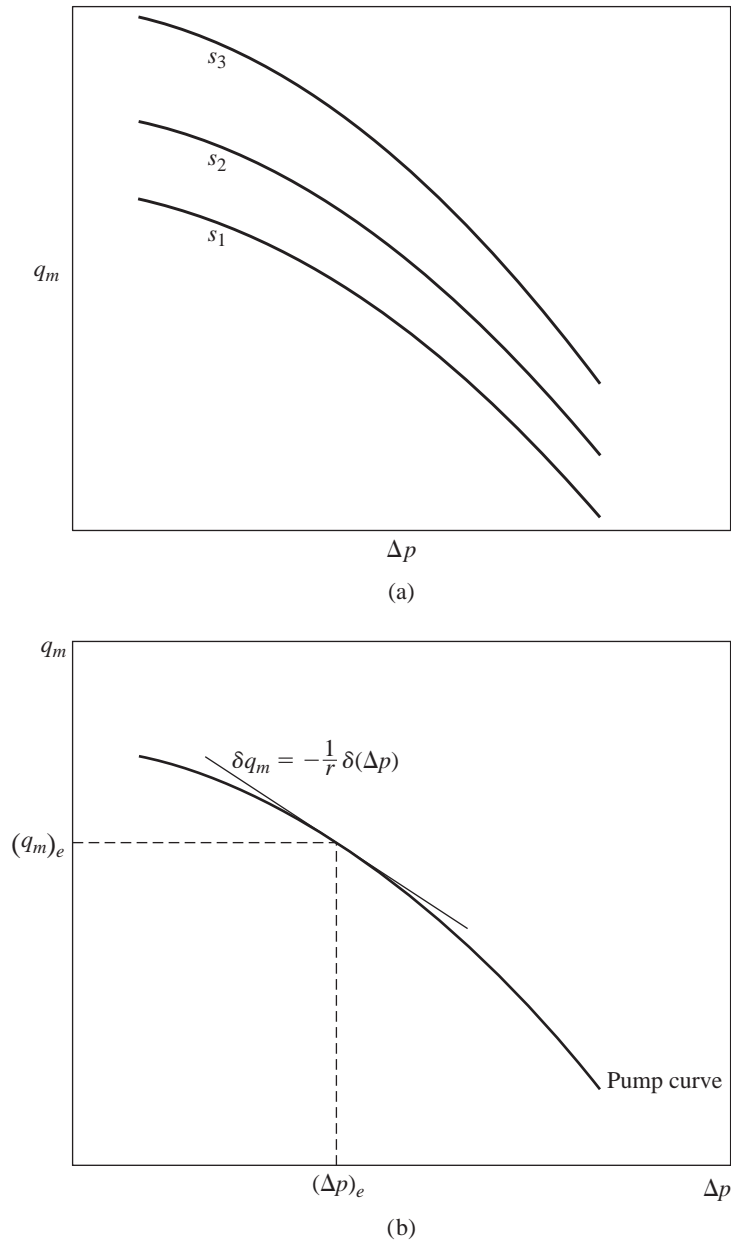
can be found as a function of Δp. When this function is graphed on the same plot as
the pump curve, the intersection of the two curves will establish the equilibrium point.
The required procedure is illustrated by the following example.

EXAMPLE 7.4.8

A Liquid-Level System with a Pump

■ Problem
Figure 7.4.9 shows a liquid-level system with a pump input and a drain whose linear resistance
is R_2. The inlet from the pump to the tank has a linear resistance R_1. Obtain a linearized model
of the liquid height h.
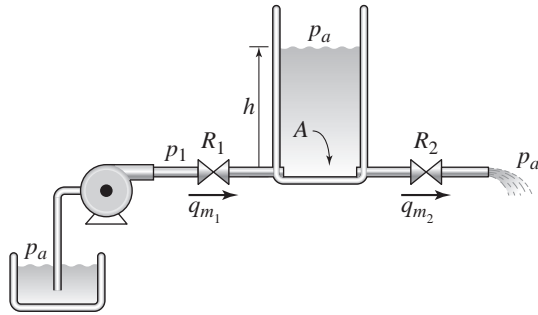
Figure 7.4.9 A liquid-level system with a pump.

■ Solution

Let $\Delta p = p_1 - p_a$. Denote the mass flow rates through each resistance as q_{m_1} and q_{m_2}. These flow rates are

$$q_{m_1} = \frac{1}{R_1}(p_1 - \rho g h - p_a) = \frac{1}{R_1}(\Delta p - \rho g h) \tag{1}$$

$$q_{m_2} = \frac{1}{R_2}(\rho g h + p_a - p_a) = \frac{1}{R_2}\rho g h \tag{2}$$

From conservation of mass,

$$\rho A \frac{dh}{dt} = q_{m_1} - q_{m_2} = \frac{1}{R_1}(\Delta p - \rho g h) - \frac{1}{R_2}\rho g h \tag{3}$$

At equilibrium, $q_{m_1} = q_{m_2}$, so from equation (3),

$$\frac{1}{R_1}(\Delta p - \rho g h) = \frac{1}{R_2}\rho g h$$

which gives

$$\rho g h = \frac{R_2}{R_1 + R_2}\Delta p \tag{4}$$

Substituting this into expression (2) we obtain an expression for the equilibrium value of the flow rate q_{m_2} as a function of Δp:

$$q_{m_2} = \frac{1}{R_1 + R_2}\Delta p \tag{5}$$

This is simply an expression of the series resistance law, which applies here because $\dot{h} = 0$ at equilibrium and thus the same flow occurs through R_1 and R_2.

When equation (5) is plotted on the same plot as the pump curve, as in Figure 7.4.10, the intersection gives the equilibrium values of q_{m_1} and Δp. A straight line tangent to the pump curve and having the slope $-1/r$ then gives the linearized model:

$$\delta q_{m_1} = -\frac{1}{r}\delta(\Delta p) \tag{6}$$

where δq_{m_1} and $\delta(\Delta p)$ are the deviations from the equilibrium values.

From equation (4),

$$\Delta p = \frac{R_1 + R_2}{R_2}\rho g h$$

and thus

$$\delta(\Delta p) = \frac{R_1 + R_2}{R_2}\rho g\, \delta h$$

Figure 7.4.10 Graphical solution of the pump model.
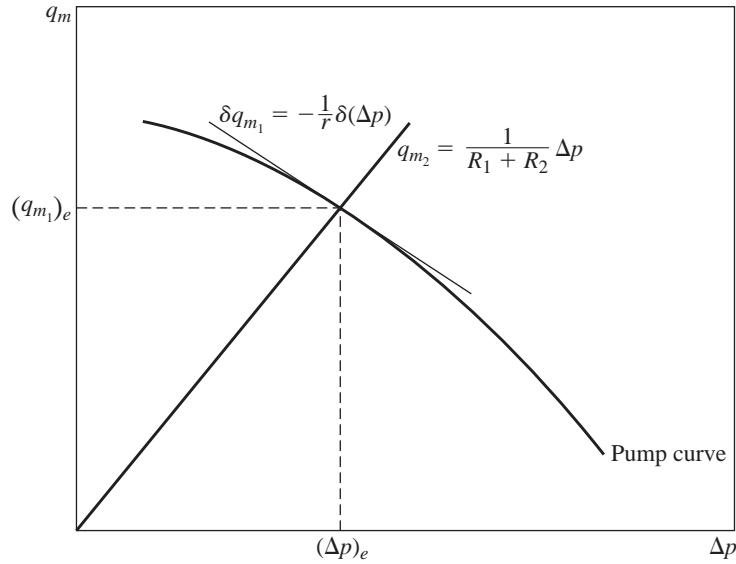
and from equation (6),

$$\delta q_{m_1} = -\frac{1}{r}\delta(\Delta p) = -\frac{1}{r}\frac{R_1 + R_2}{R_2}\rho g\,\delta h \qquad (7)$$

The linearized form of equation (3) is

$$\rho A\frac{d}{dt}\,\delta h = \delta q_{m_1} - \delta q_{m_2}$$

From equations (2) and (7),

$$\rho A\frac{d}{dt}\,\delta h = -\frac{1}{r}\frac{R_1 + R_2}{R_2}\rho g\,\delta h - \frac{\rho g}{R_2}\,\delta h$$

or

$$A\frac{d}{dt}\,\delta h = -\left(\frac{1}{r}\frac{R_1 + R_2}{R_2} + \frac{1}{R_2}\right)g\,\delta h$$

This is the linearized model, and it is of the form

$$\frac{d}{dt}\,\delta h = -b\,\delta h$$

where

$$b = \left(\frac{1}{r}\frac{R_1 + R_2}{R_2} + \frac{1}{R_2}\right)\frac{g}{A}$$

The equation has the solution $\delta h(t) = \delta h(0)e^{-bt}$. Thus if additional liquid is added to or taken from the tank so that $\delta h(0) \neq 0$, the liquid height will eventually return to its equilibrium value. The time to return is indicated by the time constant, which is $1/b$.

NONLINEAR SYSTEMS

Common causes of nonlinearities in hydraulic system models are a nonlinear resistance relation, such as due to orifice flow or turbulent flow, or a nonlinear capacitance relation, such as a tank with a variable cross section. If the liquid height is relatively constant, say

because of a liquid-level controller, we can analyze the system by linearizing the model. In cases where the height varies considerably, we must solve the nonlinear equation numerically.

| Liquid-Level System with an Orifice | **EXAMPLE 7.4.9** |

■ **Problem**

Consider the liquid-level system with an orifice, treated in Example 7.3.2. The model is

$$A\frac{dh}{dt} = q_{vi} - q_{v0} = q_{vi} - C_d A_o \sqrt{2gh}$$

Consider the case where $A = 2$ ft^2 and $C_d A_o \sqrt{2g} = 6$. Estimate the system's time constant for two cases: (i) the inflow rate is held constant at $q_{vi} = 12$ ft^3/sec and (ii) the inflow rate is held constant at $q_{vi} = 24$ ft^3/sec.

■ **Solution**

Substituting the given values, we obtain

$$2\frac{dh}{dt} = q_{vi} - q_{v0} = q_{vi} - 6\sqrt{h} \tag{1}$$

When the inflow rate is held constant at the value q_{ve}, the liquid height reaches an equilibrium value h_e that can be found from the preceding equation by setting dh/dt equal to zero. This gives $36h_e = q_{ve}^2$.

The two cases of interest to us are (i) $h_e = (12)^2/36 = 4$ ft and (ii) $h_e = (24)^2/36 = 16$ ft. Figure 7.4.11 is a plot of the flow rate $6\sqrt{h}$ through the orifice as a function of the height h. The two points corresponding to $h_e = 4$ and $h_e = 16$ are indicated on the plot.

In Figure 7.4.11 two straight lines are shown, each passing through one of the points of interest ($h_e = 4$ and $h_e = 16$), and having a slope equal to the slope of the curve at that point. The general equation for these lines is

$$q_{vo} = 6\sqrt{h} = 6\sqrt{h_e} + \left(\frac{dq_{vo}}{dh}\right)_e (h - h_e) = 6\sqrt{h_e} + 3h_e^{-1/2}(h - h_e)$$
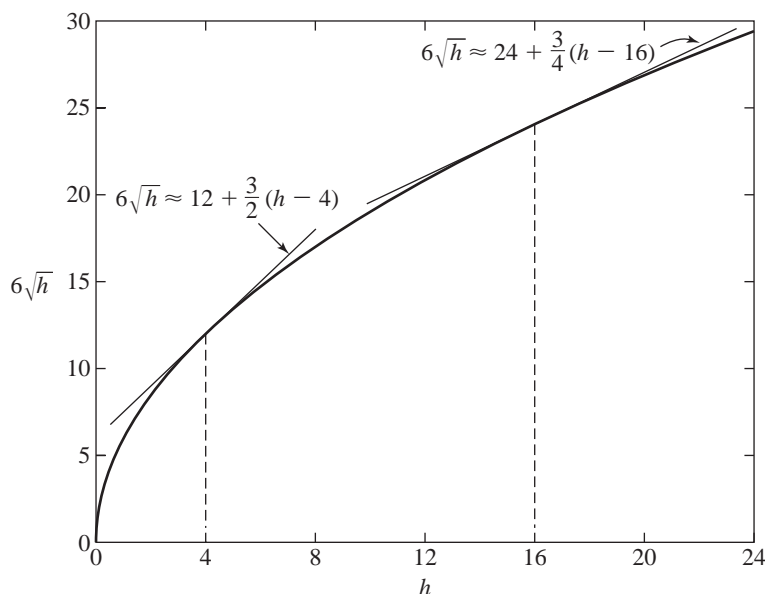
Figure 7.4.11 Linearized approximations of the resistance relation.

Substitute this into equation (1) to obtain

$$2\frac{dh}{dt} = q_{vi} - 6\sqrt{h_e} - 3h_e^{-1/2}(h - h_e) = q_{vi} - 3\sqrt{h_e} - 3h_e^{-1/2}h$$

The time constant of this linearized model is $2\sqrt{h_e}/3$, and is 4/3 sec for $h_e = 4$ and is 8/3 sec for $h_e = 16$. Thus, if the input rate q_{vi} is changed slightly from its equilibrium value of $q_{vi} = 12$, the liquid height will take about 4(4/3) or 16/3 sec to reach its new height.

If the input rate q_{vi} is changed slightly from its value of $q_{vi} = 24$, the liquid height will take about 4(8/3) or 32/3 seconds to reach its new height.

Note that the model's time constant depends on the particular equilibrium solution chosen for the linearization. Because the straight line is an approximation to the $6\sqrt{h}$ curve, we cannot use the linearized models to make predictions about the system's behavior far from the equilibrium point. However, despite this limitation, a linearized model is useful for designing a flow control system to keep the height near some desired value. If the control system works properly, the height will stay near the equilibrium value, and the linearized model will be accurate.

FLUID INERTANCE

We have defined fluid inertance I as

$$I = \frac{p}{dq_m/dt}$$

which is the ratio of the pressure difference over the rate of change of the mass flow rate. The inertance is the change in pressure required to produce a unit rate of change in mass flow rate. Thus inertance relates to fluid acceleration and kinetic energy, which are often negligible either because the moving fluid mass is small or because it is moving at a steady rate. There are, however, some cases where inertance may be significant. The effect known as *water hammer* is due partly to inertance. Also, as we will see in the following example, inertance can be significant in conduits that are long or that have small cross sections.

EXAMPLE 7.4.10

Figure 7.4.12 Derivation of pipe inertance expression.

Calculation of Inertance
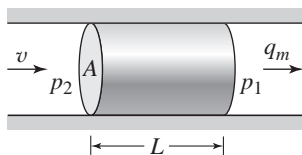
■ Problem

Consider fluid flow (either liquid or gas) in a nonaccelerating pipe (Figure 7.4.12). Derive the expression for the inertance of a slug of fluid of length L.

■ Solution

The mass of the slug is ρAL, where ρ is the fluid mass density. The net force acting on the slug due to the pressures p_1 and p_2 is $A(p_2 - p_1)$. Applying Newton's law to the slug, we have

$$\rho AL \frac{dv}{dt} = A(p_2 - p_1)$$

where v is the fluid velocity. The velocity v is related to the mass flow rate q_m by $\rho Av = q_m$. Using this to substitute for v, we obtain

$$L \frac{dq_m}{dt} = A(p_2 - p_1)$$

or

$$\frac{L}{A}\frac{dq_m}{dt} = p_2 - p_1$$

With $p = p_2 - p_1$, we obtain

$$\frac{L}{A} = \frac{p}{dq_m/dt}$$

Thus, from the definition of inertance I,

$$I = \frac{L}{A}$$

Note that the inertance is larger for longer pipes and for pipes with smaller cross section.

The significance of inertance in a given application is often difficult to assess. Often a model is developed by ignoring inertance effects at first, and then, if possible, the model is verified by experiment to see if the neglected inertance is significant.

7.5 PNEUMATIC SYSTEMS

The working medium in a pneumatic device is a compressible fluid, most commonly air. The availability of air is an advantage for pneumatic devices, because it can be exhausted to the atmosphere at the end of the device's work cycle, thus eliminating the need for return lines. On the other hand, because of the compressibility of the working fluid, the response of pneumatic systems can be slower and more oscillatory than that of hydraulic systems.

Because the kinetic energy of a gas is usually negligible, the inertance relation is not usually needed to develop a model. Instead, capacitance and resistance elements form the basis of most pneumatic system models.

Temperature, pressure, volume, and mass are functionally related for a gas. The model most often used to describe this relationship is the *perfect gas law,* which is a good model of gas behavior under normal pressures and temperatures. The law states that

$$pV = mR_gT \tag{7.5.1}$$

where p is the absolute pressure of the gas with volume V, m is the mass, T is its absolute temperature, and R_g is the gas constant that depends on the particular type of gas. The values of R_g for air are 1715 ft-lb/slug-°R and 287 N · m/kg · K.

If heat is added to a gas from its surroundings, some of this heat can do external work on its surroundings, and the rest can increase the internal energy of the gas by raising its temperature. The *specific heat* of a substance at a specified temperature is the ratio of two heat values, the amount of heat needed to raise the temperature of a unit mass of the substance by 1° divided by the heat needed to raise a unit mass of water 1° at the specified temperature. Because the pressure and volume of a gas can change as its temperature changes, two specific heats are defined for a gas: one at constant pressure (c_p), and one at constant volume (c_v).

The amount of energy needed to raise the temperature of 1 kg of water from 14.5°C to 15.5°C is 4186 J. In the British Engineering system, the British thermal unit (BTU) can be considered to be the energy needed to raise 1 pound of water (1/32.174 of a slug in mass units) 1 degree Fahrenheit.

The perfect gas law enables us to solve for one of the variables p, V, m, or T if the other three are given. Additional information is usually available in the form of a pressure-volume or "process" relation. The following process models are commonly used, where the subscripts 1 and 2 refer to the start and the end of the process, respectively. We assume the mass m of the gas is constant.

1. *Constant-Pressure Process* ($p_1 = p_2$). The perfect gas law thus implies that $V_2/V_1 = T_2/T_1$. If the gas receives heat from the surroundings, some of it raises the temperature and some expands the volume.

2. *Constant-Volume Process* ($V_1 = V_2$). Here, $p_2/p_1 = T_2/T_1$. When heat is added to the gas, it merely raises the temperature because no external work is done in a constant-volume process.

3. *Constant-Temperature Process* (*an* isothermal *process*) ($T_1 = T_2$). Thus, $p_2/p_1 = V_1/V_2$. Any added heat does not increase the internal energy of the gas because of the constant temperature. It only does external work.

4. *Reversible Adiabatic (Isentropic) Process.* This process is described by the relation

$$p_1 V_1^\gamma = p_2 V_2^\gamma \tag{7.5.2}$$

where $\gamma = c_p/c_v$. *Adiabatic* means that no heat is transferred to or from the gas. *Reversible* means the gas and its surroundings can be returned to their original thermodynamic conditions. Because no heat is transferred, any external work done by the gas changes its internal energy by the same amount. Thus its temperature changes; that is,

$$W = mc_v(T_1 - T_2)$$

where W is the external work. The work W is positive if work is done on the surroundings.

5. *Polytropic Process.* A process can be more accurately modeled by properly choosing the exponent n in the polytropic process

$$p\left(\frac{V}{m}\right)^n = \text{constant}$$

If the mass m is constant, this general process reduces to the previous processes if n is chosen as 0, ∞, 1, and γ, respectively, and if the perfect gas law is used.

PNEUMATIC CAPACITANCE

Fluid capacitance is the relation between stored mass and pressure. Specifically, fluid capacitance C is the ratio of the change in stored mass to the change in pressure, or

$$C = \frac{dm}{dp} \tag{7.5.3}$$

For a container of constant volume V with a gas density ρ, $m = \rho V$, and the capacitance equation may be written as

$$C = \frac{d(\rho V)}{dp} = V\frac{d\rho}{dp} \tag{7.5.4}$$

If the gas undergoes a polytropic process,

$$p\left(\frac{V}{m}\right)^n = \frac{p}{\rho^n} = \text{constant} \tag{7.5.5}$$

Differentiating this expression gives

$$\frac{d\rho}{dp} = \frac{\rho}{np} = \frac{m}{npV}$$

For a perfect gas, this shows the capacitance of the container to be

$$C = \frac{mV}{npV} = \frac{V}{nR_gT} \tag{7.5.6}$$

Note that the same container can have a different capacitance for different expansion processes, temperatures, and gases, because C depends on n, T, and R_g.

Capacitance of an Air Cylinder | **EXAMPLE 7.5.1**

■ Problem
Obtain the capacitance of air in a rigid cylinder of volume 0.03 m^3, if the cylinder is filled by an isothermal process. Assume the air is initially at room temperature, 293 K.

■ Solution
The filling of the cylinder can be modeled as an isothermal process if it occurs slowly enough to allow heat transfer to occur between the air and its surroundings. In this case, $n = 1$ in the polytropic process equation, and from (7.5.6) we obtain,

$$C = \frac{0.03}{1(287)(293)} = 3.57 \times 10^{-7} \text{ kg} \cdot \text{m}^2/\text{N}$$

Pressurizing an Air Cylinder | **EXAMPLE 7.5.2**

■ Problem
Air at temperature T passes through a valve into a rigid cylinder of volume V, as shown in Figure 7.5.1. The mass flow rate through the valve depends on the pressure difference $\Delta p = p_i - p$, and is given by an experimentally determined function:

$$q_{mi} = f(\Delta p) \tag{1}$$

Develop a dynamic model of the gage pressure p in the container as a function of the input pressure p_i. Assume the filling process is isothermal.

■ Solution
From conservation of mass, the rate of mass increase in the container equals the mass flow rate through the valve. Thus, if $p_i - p > 0$, from equation (1)

$$\frac{dm}{dt} = q_{mi} = f(\Delta p)$$

But

$$\frac{dm}{dt} = \frac{dm}{dp}\frac{dp}{dt} = C\frac{dp}{dt}$$
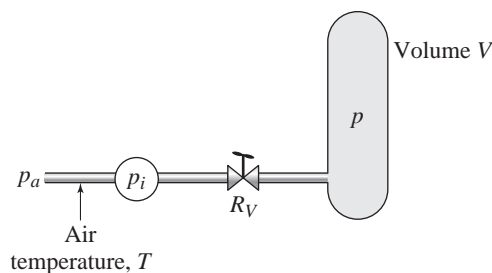
Figure 7.5.1 Pressurizing an air cylinder.

and thus,

$$C\frac{dp}{dt} = f(\Delta p) = f(p_i - p) \tag{2}$$

where the capacitance C is given by (7.5.6) with $n = 1$.

$$C = \frac{V}{R_g T}$$

If the function f is nonlinear, then the dynamic model is nonlinear.

PART II. THERMAL SYSTEMS

A *thermal system* is one in which energy is stored and transferred as thermal energy, commonly called heat. Although heat energy can play a role in pneumatic systems, we chose to treat pneumatic systems primarily as fluid systems because their dynamics is largely governed by fluid capacitance and fluid resistance. Examples of thermal systems include heating and cooling systems in buildings and mixing processes where heat must be added or removed to maintain an optimal reaction temperature.

Thermal systems operate because of temperature differences, as heat energy flows from an object with the higher temperature to an object with the lower temperature. Just as conservation of mass, fluid resistance, and fluid capacitance form the basis of fluid system models, so conservation of heat energy forms the basis of thermal system models, along with the concepts of thermal resistance and thermal capacitance. Thus, it is natural to study thermal systems along with fluid systems. We will see that thermal systems are also analogous to electric circuits, where conservation of charge plays the same role as conservation of heat, and voltage difference plays the same role as temperature difference.

7.6 THERMAL CAPACITANCE

For a mass m whose specific heat is c_p, the amount of heat energy E stored in the object at a temperature T is

$$E = mc_p(T - T_r) \tag{7.6.1}$$

where T_r is an arbitrarily selected reference temperature. As with gravitational potential energy, which is computed relative to an arbitrary height h_r as $mg(h - h_r)$, we can select the reference temperature T_r for convenience. This is because only the *change* in stored heat energy affects the dynamics of a thermal system, just as only the change in gravitational potential energy affects the dynamics of a mechanical system.

Thermal capacitance relates an object's temperature to the amount of heat energy stored. It is defined as

$$C = \frac{dE}{dT} \tag{7.6.2}$$

where E is the stored heat energy. If the temperature range is wide enough, c_p might vary considerably with temperature. However, if c_p is not a function of temperature, from (7.6.1) we have

$$C = mc_p = \rho V c_p \tag{7.6.3}$$

where ρ and V are the density and the volume of the mass m. The thermal capacitance C can be interpreted as the amount of heat energy required to raise the temperature of the object by $1°$. Thus, in SI, the units of C are J/$°$C or J/K. The FPS units of C are ft-lb/$°$F. Another common unit for C is BTU/$°$F.

The concept of thermal capacitance applies to fluids as well as solids. For example, at room temperature and atmospheric pressure, the ratio of the specific heat of water to that of air is 4.16. Thus, for the same *mass* of air and water, to raise the water temperature by $1°$ requires 4.16 times more energy than for air.

Temperature Dynamics of a Mixing Process | **EXAMPLE 7.6.1**

■ Problem

Liquid at a temperature T_i is pumped into a mixing tank at a constant volume flow rate q_v (Figure 7.6.1). The container walls are perfectly insulated so that no heat escapes through them. The container volume is V, and the liquid within is well mixed so that its temperature throughout is T. The liquid's specific heat and mass density are c_p and ρ. Develop a model for the temperature T as a function of time, with T_i as the input.

■ Solution

The amount of heat energy in the tank liquid is $\rho c_p V (T - T_r)$, where T_r is an arbitrarily selected reference temperature. From conservation of energy,

$$\frac{d\left[\rho c_p V (T - T_r)\right]}{dt} = \text{heat rate in} - \text{heat rate out} \qquad (1)$$

Liquid mass is flowing into the tank at the rate $\dot{m} = \rho q_v$. Thus heat energy is flowing into the tank at the rate

$$\text{heat rate in} = \dot{m} c_p (T_i - T_r) = \rho q_v c_p (T_i - T_r)$$

Similarly,

$$\text{heat rate out} = \rho q_v c_p (T - T_r)$$

Therefore, from equation (1), since ρ, c_p, V, and T_r are constants,

$$\rho c_p V \frac{dT}{dt} = \rho q_v c_p (T_i - T_r) - \rho q_v c_p (T - T_r) = \rho q_v c_p (T_i - T)$$

Cancel ρc_p and rearrange to obtain

$$\frac{V}{q_v} \frac{dT}{dt} + T = T_i$$
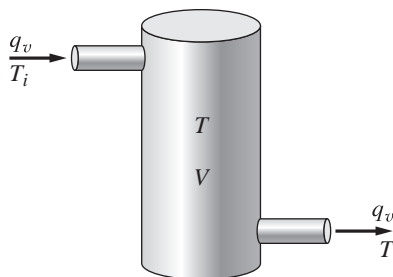
Figure 7.6.1 Temperature dynamics of a mixing process.

Note that T_r, ρ, and c_p do not appear in the final model form, so their specific numerical values are irrelevant to the problem. The system's time constant is V/q_v, and thus the liquid temperature T changes slowly if the tank volume V is large or if the inflow rate q_v is small, which agrees with our intuition.

7.7 THERMAL RESISTANCE

Heat energy is conserved, and thus heat in thermal system analysis plays the same role as charge in electrical systems. The flow of heat, called *heat transfer,* causes a change in an object's temperature. Heat transfer between two objects is caused by a difference in their temperatures. Thus temperature difference in thermal systems plays the same role as voltage difference in electrical systems, and so we utilize the concept of *thermal resistance* in a manner similar to electrical resistance.

CONDUCTION, CONVECTION, AND RADIATION

Heat transfer can occur by one or more modes: *conduction, convection,* and *radiation,* as illustrated by Figure 7.7.1. Temperature is a measure of the amount of heat energy in an object. Heat energy and thus temperature can be thought of as due to the kinetic energy of vibrating molecules. A higher temperature is an indication of higher molecule vibration velocity. Heat transfer by conduction occurs by diffusion of heat through a substance. This diffusion occurs by molecules transferring some of their kinetic energy to adjacent, slower molecules.

The mechanism for convection is due to fluid transport. This effect can be seen in boiling water and in thermal air currents. Convection also occurs within a fluid at the boundary of the fluid and a solid surface whose temperature is different from that of the fluid. Convective heat transfer might be due to forced convection, such as when a fluid is pumped past a surface, or natural (free) convection, which is caused by motion produced by density differences in the fluid.

Heat transfer by radiation occurs through infrared waves. Heat lamps are common examples of this type of transfer. Heating by solar radiation is another example.

NEWTON'S LAW OF COOLING

Newton's law of cooling is a linear model for heat flow rate as a function of temperature difference. The law, which is used for both convection and conduction models, is expressed as

$$q_h = \frac{1}{R}\Delta T \qquad (7.7.1)$$

where q_h is the heat flow rate, R is the thermal resistance, and ΔT is the temperature difference. In SI, q_h has the units of J/s, which is a watt (W). In the FPS system, the

Figure 7.7.1 Modes of heat transfer.
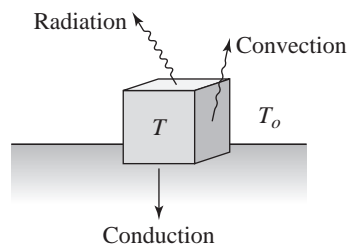
units of q_h are ft-lb/sec, but BTU/hr is also commonly used. For thermal resistance R the SI units are °C/W, and the FPS units are °F-sec/ft-lb.

For conduction through material of thickness L, an approximate formula for the conductive resistance is

$$R = \frac{L}{kA} \tag{7.7.2}$$

so that

$$q_h = \frac{kA}{L}\Delta T \tag{7.7.3}$$

where k is the *thermal* conductivity of the material and A is the surface area.

If convection occurs, we might need to analyze the system as a fluid as well as a thermal system. Fortunately many analytical and empirical results are available for common situations, and we can use them to obtain the necessary coefficients for our models. The thermal resistance for convection occurring at the boundary of a fluid and a solid is given by

$$R = \frac{1}{hA} \tag{7.7.4}$$

so that

$$q_h = hA\Delta T \tag{7.7.5}$$

where h is the so-called *film coefficient* or *convection coefficient* of the fluid-solid interface and A is the involved surface area. The film coefficient might be a complicated function of the fluid flow characteristics. For many cases of practical importance, the coefficient has been determined to acceptable accuracy, but a presentation of the results is lengthy and beyond the scope of this work. Standard references on heat transfer contain this information for many cases [Çengel, 2001].

When two bodies are in visual contact, radiation heat transfer occurs through a mutual exchange of heat energy by emission and absorption. Thermal radiation, such as solar energy, produces heat when it strikes a surface capable of absorbing it. The radiation can also be reflected or refracted, and all three mechanisms can occur at a single surface. A net exchange of heat energy occurs from the warmer to the colder body. The rate of this exchange depends on material properties and geometric factors affecting the relative visibility and the amount of surface area involved. The net heat transfer rate depends on the difference of the body temperatures raised to the fourth power (a consequence of the so-called *Stefan-Boltzmann* law).

$$q_h = \beta\left(T_1^4 - T_2^4\right) \tag{7.7.6}$$

The absolute body temperatures are T_1 and T_2, and β is a factor incorporating the other effects. Determining β, like the convection coefficient, is too involved to consider here, but many results are available in the literature.

The radiation model is nonlinear, and therefore we cannot define a specific thermal resistance. However, we can use a linearized model if the temperature change is not too large. Note that linear thermal resistance is a special case of the more general definition of thermal resistance:

$$R = \frac{1}{dq_h/dT} \tag{7.7.7}$$

Figure 7.7.2 (a) Conductive
heat transfer through a plate.
(b) Thermal model.
(c) Analogous electric circuit.
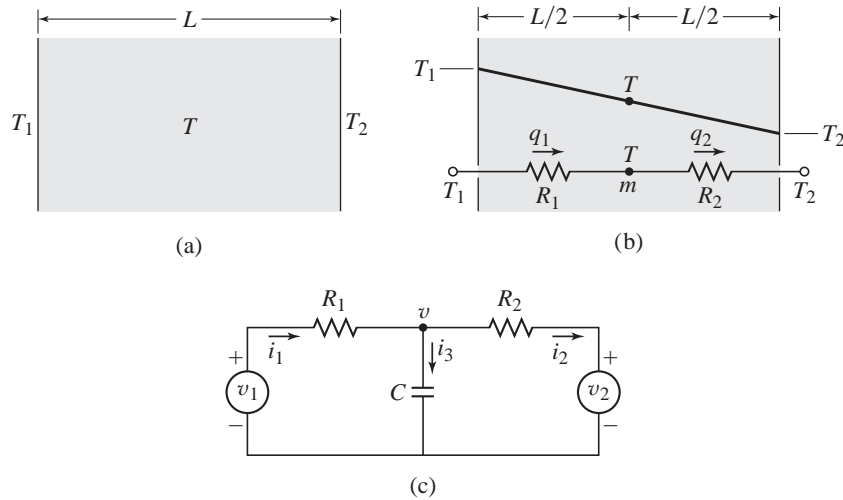
For example, suppose that T_2 is constant. Then from (7.7.6) and (7.7.7),

$$R = \frac{1}{dq_h/dT_1} = \frac{1}{4\beta T_1^3}$$

When this is evaluated at a specific temperature T_1, we can obtain a specific value for
the linearized radiation resistance.

HEAT TRANSFER THROUGH A PLATE

Consider a solid plate or wall of thickness L, as shown in cross section in Figure 7.7.2(a).
The temperatures of the objects (either solid or fluid) on each side of the plate are T_1
and T_2. If $T_1 > T_2$, heat will flow from the left side to the right side. The temperatures
T_1 and T_2 of the adjacent objects will remain constant if the objects are large enough.
(One can easily visualize this with a building; the outside air temperature is not af-
fected significantly by heat transfer through the building walls because the mass of the
atmosphere is so large.)

If the plate material is homogeneous, eventually the temperature distribution in
the plate will look like that shown in part (b) of the figure. This is the steady-state
temperature distribution. *Fourier's law of heat conduction* states that the heat transfer
rate per unit area within a homogeneous substance is directly proportional to the negative
temperature gradient. The proportionality constant is the thermal conductivity k. For the
case shown in Figure 7.7.2(b), the negative gradient is $(T_1 - T_2)/L$, and the heat transfer
rate is thus

$$q_h = \frac{kA(T_1 - T_2)}{L}$$

where A is the plate area in question. Comparing this with (7.7.1) shows that the thermal
resistance is given by (7.7.2), with $\Delta T = T_1 - T_2$.

Under transient conditions the temperature profile is not linear and must be obtained
by solving a partial differential equation (a so-called *distributed-parameter model*). To
obtain an ordinary differential equation (a *lumped-parameter model*), which is easier to
solve, we must select a point in the plate and use its temperature as the representative
temperature of the object. Under steady-state conditions, the average temperature is at
the center, and so for this reason we select as an educated guess the center temperature

as the representative temperature for the transient calculations. We therefore consider the entire mass m of the plate to be concentrated ("lumped") at the plate centerline, and consider conductive heat transfer to occur over a path of length $L/2$ between temperature T_1 and temperature T. Thus, the thermal resistance for this path is

$$R_1 = \frac{L/2}{kA}$$

Similarly, for the path from T to T_2, the thermal resistance is $R_2 = (L/2)/(kA)$. These resistances and the lumped mass m are represented in Figure 7.7.2b.

From Figure 7.7.2b we can derive the following model by applying conservation of heat energy. Assuming that $T_1 > T > T_2$, we obtain

$$mc_p \frac{dT}{dt} = q_1 - q_2 = \frac{1}{R_1}(T_1 - T) - \frac{1}{R_2}(T - T_2) \qquad (7.7.8)$$

The thermal capacitance is $C = mc_p$.

This system is analogous to the circuit shown in Figure 7.7.2c, where the voltages v, v_1, and v_2 are analogous to the temperatures T, T_1, and T_2. Note that the current i_1 is analogous to the heat flow rate into the mass m through the left-hand conductive path, and that current i_2 is analogous to the heat flow rate out of the mass m through the right-hand conductive path. The current i_3 is the net current into the capacitance ($i_3 = i_1 - i_2$) and increases the voltage v. The current i_3 is analogous to the net heat flow rate into the mass m, which increases the mass temperature T.

SERIES AND PARALLEL THERMAL RESISTANCES

Suppose the capacitance C in the circuit in Figure 7.7.2c is zero. This is equivalent to removing the capacitance to obtain the circuit shown in Figure 7.7.3a, and we can see immediately that the two resistances are in series. Therefore they can be combined by the series law: $R = R_1 + R_2$ to obtain the equivalent circuit shown in part (b) of the figure.

By analogy we would expect that thermal resistances would obey the same series law as electrical resistances. If the plate mass m is very small, its thermal capacitance C is also very small. In this case, the mass absorbs a negligible amount of heat energy, so the heat flow rate q_1 through the left-hand conductive path must equal the rate q_2 through the right-hand path. That is, if $C = 0$,

$$q_1 = \frac{1}{R_1}(T_1 - T) = q_2 = \frac{1}{R_2}(T - T_2) \qquad (7.7.9)$$

The solution of these equations is

$$T = \frac{R_2 T_1 + R_1 T_2}{R_1 + R_2}$$

$$q_1 = q_2 = \frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T_2}{R}$$

The latter solution shows that the resistances R_1 and R_2 are equivalent to the single resistance $R = R_1 + R_2$, which is the series resistance law.

Thus thermal resistances are in *series* if they pass the same heat flow rate; if so, they are equivalent to a single resistance equal to the sum of the individual resistances. It can also be shown that thermal resistances are in *parallel* if they have the temperature difference; if so, they are equivalent to a single resistance calculated by the

Figure 7.7.3 (a) Conductive heat transfer through a plate with negligible capacitance. (b) Analogous circuit.

(a) (b)
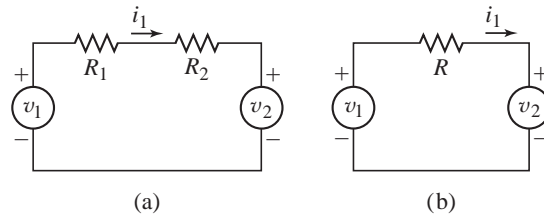
Figure 7.7.4 (a) Convective and conductive heat transfer through a plate with negligible capacitance. (b) Thermal model.

(a) (b)
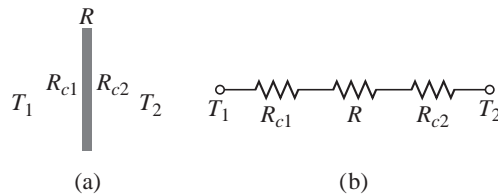
reciprocal formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots$$

We have seen that the resistances R_1 and R_2 in Figure 7.7.2b are in series if $C = 0$. This can occur if the mass m is negligible or if the value of c_p is so small that the product mc_p is negligible. However, by examining the left-hand side of (7.7.8), we see that $q_1 = q_2$ if either $C = 0$ or $dT/dt = 0$. Thus we conclude that the series resistance law also applies under steady-state conditions, where $dT/dt = 0$. So if $C = 0$ or $dT/dt = 0$ for the plate shown in Figure 7.7.2a, it can be represented as a pure conductive resistance of zero mass, as shown in Figure 7.7.4a, where $R = R_1 + R_2$.

If convection occurs on both sides of the plate, the convective resistances R_{c_1} and R_{c_2} are in series with the conductive resistance R, and the total resistance is given by $R + R_{c_1} + R_{c_2}$, as shown in part (b) of the figure.

In practice, to obtain a simpler model, the series resistance formula is sometimes used even in applications where the thermal capacitance is not small or where transient conditions exist, but you must be aware that the formula is an approximation in those situations.

EXAMPLE 7.7.1

Thermal Resistance of Building Wall

■ Problem
Engineers must be able to predict the rate of heat loss through a building wall to determine the heating system's requirements. The wall cross section shown in Figure 7.7.5 consists of four layers: an inner layer of plaster/lathe 10 mm thick, a layer of fiberglass insulation 125 mm thick, a layer of wood 60 mm thick, and an outer layer of brick 50 mm thick. For the given materials, the resistances for a wall area of 1 m² are $R_1 = 0.036$, $R_2 = 4.01$, $R_3 = 0.408$, and $R_4 = 0.038°$C/W.

Suppose that $T_i = 20°$C and $T_o = -10°$C. (a) Compute the total wall resistance for 1 m² of wall area, and compute the heat loss rate if the wall's area is 3 m by 5 m. (b) Find the temperatures T_1, T_2, and T_3, assuming steady-state conditions.
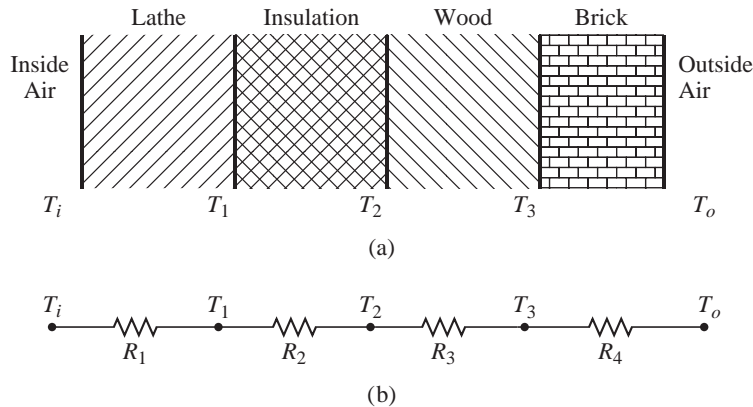
Figure 7.7.5 Heat transfer through a wall with four layers.

■ Solution

a.   The series resistance law gives

$$R = R_1 + R_2 + R_3 + R_4 = 0.036 + 4.01 + 0.408 + 0.038 = 4.492°\text{C/W}$$

which is the total resistance for 1 m^2 of wall area. The wall area is $3(5) = 15$ m^2, and thus the total heat loss is

$$q_h = 15\frac{1}{R}(T_i - T_o) = 15\frac{1}{4.492}(20 + 10) = 100.2 \text{ W}$$

This is the heat rate that must be supplied by the building's heating system to maintain the inside temperature at 20°C, if the outside temperature is -10°C.

b.   If we assume that the inner and outer temperatures T_i and T_o have remained constant for some time, then the heat flow rate through each layer is the same, q_h. Applying conservation of energy gives the following equations.

$$q_h = \frac{1}{R_1}(T_i - T_1) = \frac{1}{R_2}(T_1 - T_2) = \frac{1}{R_3}(T_2 - T_3) = \frac{1}{R_4}(T_3 - T_o)$$

The last three equations can be rearranged as follows:

$$(R_1 + R_2)T_1 - R_1 T_2 = R_2 T_i$$

$$R_3 T_1 - (R_2 + R_3)T_2 + R_2 T_3 = 0$$

$$-R_4 T_2 + (R_3 + R_4)T_3 = R_3 T_o$$

For the given values of T_i and T_o, the solution to these equations is $T_1 = 19.7596$, $T_2 = -7.0214$, and $T_3 = -9.7462$°C.

Parallel Resistances | **EXAMPLE 7.7.2**

■ Problem

A certain wall section is composed of a 15 cm by 15 cm glass block 8 cm thick. Surrounding the block is a 50 cm by 50 cm brick section, which is also 8 cm thick (see Figure 7.7.6). The thermal conductivity of the glass is $k = 0.81$ W/m · °C. For the brick, $k = 0.45$ W/m · °C. (a) Determine the thermal resistance of the wall section. (b) Compute the heat flow rate through (1) the glass, (2) the brick, and (3) the wall if the temperature difference across the wall is 30°C.

Figure 7.7.6 An example of parallel thermal resistances.

Brick →

Glass block
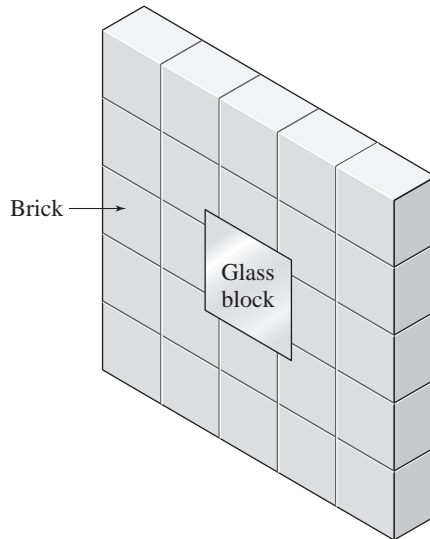
■ **Solution**

a. The resistances are found from (7.7.2):

$$R = \frac{L}{kA}$$

For the glass,

$$R_1 = \frac{0.08}{0.81(0.15)^2} = 4.39$$

For the brick,

$$R_2 = \frac{0.08}{0.45\left[(0.5)^2 - (0.15)^2\right]} = 0.781$$

Because the temperature difference is the same across both the glass and the brick, the resistances are in parallel, and thus their total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = 0.228 + 1.28 = 1.51$$

or $R = 0.633°$C/W.

b. The heat flow through the glass is

$$q_1 = \frac{1}{R_1}\Delta T = \frac{1}{4.39}30 = 6.83 \text{ W}$$

The heat flow through the brick is

$$q_2 = \frac{1}{R_2}\Delta T = \frac{1}{0.781}30 = 38.4 \text{ W}$$

Thus the total heat flow through the wall section is

$$q_h = q_1 + q_2 = 45.2 \text{ W}$$

This rate could also have been calculated from the total resistance as follows:

$$q_h = \frac{1}{R}\Delta T = \frac{1}{0.663}30 = 45.2 \text{ W}$$

EXAMPLE 7.7.3

Radial Conductive Resistance

■ Problem

Consider a cylindrical tube whose inner and outer radii are r_i and r_o. Heat flow in the tube wall can occur in the axial direction along the length of the tube and in the radial direction. If the tube surface is insulated, there will be no radial heat flow, and the heat flow in the axial direction is given by

$$q_h = \frac{kA}{L}\Delta T$$

where L is the length of the tube, ΔT is the temperature difference between the ends a distance L apart, and A is area of the solid cross section (see Figure 7.7.7a).

   If only the ends of the tube are insulated, then the heat flow will be entirely radial. Derive an expression for the conductive resistance in the radial direction.

■ Solution

As shown in Figure 7.7.7b the inner and outer temperatures are T_i and T_o, and are assumed to be constant along the length L of the tube. As shown in part (c) of the figure, from Fourier's law, the heat flow rate per unit area through an element of thickness dr is proportional to the negative of the temperature gradient dT/dr. Thus, assuming that the temperature inside the tube wall does not change with time, the heat flow rate q_h out of the section of thickness dr is the same as the heat flow into the section. Therefore,

$$\frac{q_h}{2\pi r L} = -k\frac{dT}{dr}$$

Thus,

$$q_h = -k\frac{dT}{dr}2\pi r L = -2\pi L k\frac{dT}{dr/r}$$

or

$$\int_{r_i}^{r_o} q_h\frac{dr}{r} = -2\pi L k \int_{T_i}^{T_o} dT$$

Because q_h is constant, the integration yields

$$q_h \ln\frac{r_o}{r_i} = -2\pi L k(T_o - T_i)$$

or

$$q_h = \frac{2\pi L k}{\ln(r_o/r_i)}(T_i - T_o)$$

The radial resistance is thus given by

$$R = \frac{\ln(r_o/r_i)}{2\pi L k} \tag{1}$$
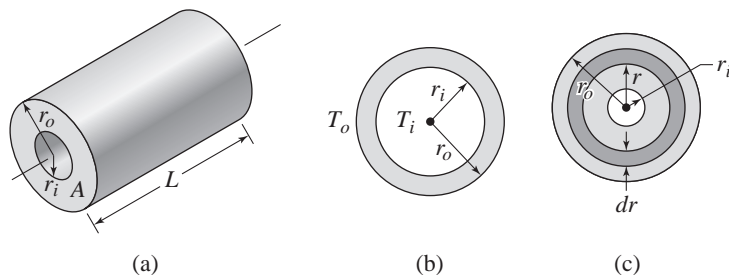
Figure 7.7.7 Radial conductive heat transfer.

(a)   (b)   (c)

EXAMPLE 7.7.4 | Heat Loss from Water in a Pipe

■ **Problem**

Water at 120°F flows in a copper pipe 6 ft long, whose inner and outer radii are 1/4 in. and 3/8 in. The temperature of the surrounding air is 70°F. Compute the heat loss rate from the water to the air in the radial direction. Use the following values. For copper, $k = 50$ lb/sec-°F. The convection coefficient at the inner surface between the water and the copper is $h_i = 16$ lb/sec-ft-°F. The convection coefficient at the outer surface between the air and the copper is $h_o = 1.1$ lb/sec-ft-°F.

■ **Solution**

Assuming that the temperature inside the pipe wall does not change with time, then the same heat flow rate occurs in the inner and outer convection layers and in the pipe wall. Thus the three resistances are in series and we can add them to obtain the total resistance. The inner and outer surface areas are

$$A_i = 2\pi r_i L = 2\pi \left(\frac{1}{4}\right)\left(\frac{1}{12}\right)6 = 0.785 \text{ ft}^2$$

$$A_o = 2\pi r_o L = 2\pi \left(\frac{3}{8}\right)\left(\frac{1}{12}\right)6 = 1.178 \text{ ft}^2$$

The inner convective resistance is

$$R_i = \frac{1}{h_i A_i} = \frac{1}{16(0.785)} = 0.08 \frac{\text{sec-°F}}{\text{ft-lb}}$$

$$R_o = \frac{1}{h_o A_o} = \frac{1}{1.1(1.178)} = 0.77 \frac{\text{sec-°F}}{\text{ft-lb}}$$

The conductive resistance of the pipe wall is

$$R_c = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi L k} = \frac{\ln\left(\frac{3/8}{1/4}\right)}{2\pi(6)(50)} = 2.15 \times 10^{-4} \frac{\text{sec-°F}}{\text{ft-lb}}$$

Thus the total resistance is

$$R = R_i + R_c + R_o = 0.08 + 2.15 \times 10^{-4} + 0.77 = 0.85 \frac{\text{sec-°F}}{\text{ft-lb}}$$

The heat loss from the pipe, assuming that the water temperature is a constant 120° along the length of the pipe, is

$$q_h = \frac{1}{R}\Delta T = \frac{1}{0.85}(120 - 70) = 59 \frac{\text{ft-lb}}{\text{sec}}$$

To investigate the assumption that the water temperature is constant, compute the thermal energy E of the water in the pipe, using the mass density $\rho = 1.94$ slug/ft^3 and $c_p = 25,000$ ft-lb/slug-°F:

$$E = mc_p T_i = \left(\pi r_i^2 L \rho\right)c_p T_i = 47,624 \text{ ft-lb}$$

Assuming that the water flows at 1 ft/sec, a slug of water will be in the pipe for 6 sec. During that time it will lose $59(6) = 354$ ft-lb of heat. Because this amount is very small compared to E, our assumption that the water temperature is constant is confirmed.

7.8 DYNAMIC MODELS OF THERMAL SYSTEMS

In this section, we discuss how to obtain lumped-parameter (ordinary differential equation) models of thermal systems containing one or more thermal capacitances, and how to obtain the value of thermal resistance experimentally.

Heat transfer occurs between two objects, but it will also occur *within* an object if the temperature varies with location in the object. To obtain an ordinary differential equation model of the temperature dynamics of an object, we must be able to assign a single temperature that is representative of the object. Sometimes it is difficult to assign such a representative temperature to a body or fluid because of its complex shape or motion and the resulting complex distribution of temperature throughout the object. When a single representative temperature cannot be assigned, several coupled lumped-parameter models or even a distributed-parameter model will be required.

THE BIOT CRITERION

For solid bodies immersed in a fluid, a useful criterion for determining the validity of the uniform-temperature assumption is based on the *Biot* number, defined as

$$N_B = \frac{hL}{k} \qquad (7.8.1)$$

where L is a representative dimension of the object, which is usually taken to be the ratio of the volume to the surface area of the body. For example, the ratio L for a sphere of radius r is $(4/3)\pi r^3/4\pi r^2 = r/3$. If the shape of the body resembles a plate, cylinder, or sphere, it is common practice to consider the object to have a single uniform temperature if N_B is small. Often, if $N_B < 0.1$, the temperature is taken to be uniform. The accuracy of this approximation improves if the inputs vary slowly.

The Biot number is the ratio of the convective heat transfer rate to the conductive rate. This can be seen by expressing N_B for a plate of thickness L as follows, using (7.7.1) through (7.7.3):

$$N_B = \frac{q_{\text{convection}}}{q_{\text{conduction}}} = \frac{hA\Delta T}{kA\Delta T/L} = \frac{hL}{k}$$

So the Biot criterion reflects the fact that if the conductive heat transfer rate is large relative to the convective rate, any temperature changes due to conduction within the object will occur relatively rapidly, and thus the object's temperature will become uniform relatively quickly.

Calculation of the ratio L depends on the surface area that is exposed to convection. For example, a cube of side length d has a value of $L = d^3/(6d^2) = d/6$ if all six sides are exposed to convection, whereas if four sides are insulated, the value is $L = d^3/(2d^2) = d/2$.

Quenching with Constant Bath Temperature

EXAMPLE 7.8.1

■ Problem

Hardness and other properties of metal can be improved by the rapid cooling that occurs during *quenching,* a process in which a heated object is placed into a liquid bath (see Figure 7.8.1). Consider a lead cube with a side length of $d = 20$ mm. The cube is immersed in an oil bath for which $h = 200$ W/(m$^2 \cdot$ °C). The oil temperature is T_b.

Figure 7.8.1 Quenching with a constant bath temperature.
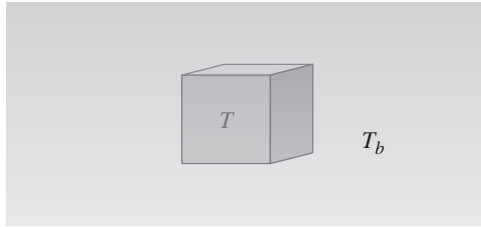
Thermal conductivity varies as function of temperature, but for lead the variation is relatively small (k for lead varies from 35.5 W/m · °C at 0°C to 31.2 W/m · °C at 327°C). The density of lead is 1.134×10^4 kg/m^3. Take the specific heat of lead to be 129 J/kg · °C.

(a) Show that temperature of the cube can be considered uniform; and (b) develop a model of the cube's temperature as a function of the liquid temperature T_b, which is assumed to be known.

■ **Solution**

a.   The ratio of volume of the cube to its surface area is

$$L = \frac{d^3}{6d^2} = \frac{d}{6} = \frac{0.02}{6}$$

and using an average value of 34 W/m · °C for k, we compute the Biot number to be

$$N_B = \frac{200(0.02)}{34(6)} = 0.02$$

which is much less than 0.1. According to the Biot criterion, we may treat the cube as a lumped-parameter system with a single uniform temperature, denoted T.

b.   If we assume that $T > T_b$, then the heat flows from the cube to the liquid, and from conservation of energy we obtain

$$C\frac{dT}{dt} = -\frac{1}{R}(T - T_b) \tag{1}$$

When deriving thermal system models, you must make an assumption about the relative values of the temperatures, and assign the heat flow direction consistent with that assumption. The specific assumption does not matter as long as you are consistent. Thus, although the bath temperature will be less than the cube temperature in the quenching application, you may still assume that $T_b > T$ and arrive at the correct model as long as you assign the heat flow direction to be *into* the cube. However, when making such assumptions, your physical insight is improved if you assume the most likely situation; the nature of the quenching process means that $T > T_b$, so this is the logical assumption to use.

The thermal capacitance of the cube is computed as

$$C = mc_p = \rho V c_p = 1.134 \times 10^4 (0.02)^3 (129) = 11.7 \text{ J/}°\text{C}$$

The thermal resistance R is due to convection, and is

$$R = \frac{1}{hA} = \frac{1}{200(6)(0.02)^2} = 2.08°\text{C} \cdot \text{s/J}$$

Thus the model is

$$11.7\frac{dT}{dt} = -\frac{1}{2.08}(T - T_b)$$

or

$$24.4\frac{dT}{dt} + T = T_b$$

The time constant is $\tau = RC = 24.4$ s. If the bath is large enough so that the cube's energy does not appreciably affect the bath temperature T_b, then when the cube is dropped into the bath, the temperature T_b acts like a step input. The cube's temperature will reach the temperature T_b in approximately $4\tau = 98$ s.

Figure 7.8.2 shows a circuit that is analogous to the thermal model of the quenching process. The voltages v and v_b are analogous to the temperatures T and T_b. The circuit model is

Figure 7.8.2 Electric circuit analogous to quenching with a constant bath temperature.

$$C\frac{dv}{dt} = \frac{1}{R}(v_b - v)$$

MULTIPLE THERMAL CAPACITANCES

When it is not possible to identify one representative temperature for a system, you must identify a representative temperature for each distinct thermal capacitance. Then, after identifying the resistance paths between each capacitance, apply conservation of heat energy to each capacitance. In doing so, you must arbitrarily but consistently assume that some temperatures are greater than others, to assign directions to the resulting heat flows. The order of the resulting model equals the number of representative temperatures.

Quenching with Variable Bath Temperature

EXAMPLE 7.8.2

■ Problem
Consider the quenching process treated in Example 7.8.1. If the thermal capacitance of the liquid bath is not large, the heat energy transferred from the cube will change the bath temperature, and we will need a model to describe its dynamics. Consider the representation shown in Figure 7.8.3. The temperature outside the bath is T_o, which is assumed to be known. The convective resistance between the cube and the bath is R_1, and the combined convective/conductive resistance of the container wall and the liquid surface is R_2. The capacitances of the cube and the liquid bath are C and C_b, respectively.

Figure 7.8.3 Quenching with a variable bath temperature.

a. Derive a model of the cube temperature and the bath temperature assuming that the bath loses no heat to the surroundings (that is, $R_2 = \infty$).
b. Obtain the model's characteristic roots and the form of the response.

■ Solution
a. Assume that $T > T_b$. Then the heat flow is out of the cube and into the bath. From conservation of energy for the cube,

$$C\frac{dT}{dt} = -\frac{1}{R_1}(T - T_b) \tag{1}$$

and for the bath,

$$C_b\frac{dT_b}{dt} = \frac{1}{R_1}(T - T_b) \tag{2}$$

Equations (1) and (2) are the desired model. Note that the heat flow rate in equation (2) must have a sign opposite to that in equation (1) because the heat flow *out* of the cube must be the same as the heat flow *into* the bath.

b. Applying the Laplace transform to equations (1) and (2) with zero initial conditions, we obtain

$$(R_1Cs + 1)T(s) - T_b(s) = 0 \tag{3}$$

$$(R_1C_bs + 1)T_b(s) - T(s) = 0 \tag{4}$$

Solving equation (3) for $T_b(s)$ and substituting into equation (4) gives

$$[(R_1C_bs + 1)(R_1Cs + 1) - 1]T(s) = 0$$

from which we obtain

$$R_1^2C_bCs^2 + R_1(C + C_b)s = 0$$

So the characteristic roots are

$$s = 0, \quad s = -\frac{C + C_b}{R_1CC_b}$$

Because equations (3) and (4) are homogeneous, the form of the response is

$$T(t) = A_1e^{0t} + B_1e^{-t/\tau} = A_1 + B_1e^{-t/\tau} \qquad \tau = \frac{R_1CC_b}{C + C_b}$$

$$T_b(t) = A_2e^{0t} + B_2e^{-t/\tau} = A_2 + B_2e^{-t/\tau}$$

where the constants A_1, A_2, B_1, and B_2 depend on the initial conditions. The two temperatures become constant after approximately 4τ. Note that $T(t) \to A_1$ and $T_b(t) \to A_2$ as $t \to \infty$. From physical insight we know that T and T_b will become equal as $t \to \infty$. Therefore, $A_2 = A_1$.

The final value of the temperatures, A_1, can be easily found from physical reasoning using conservation of energy. The initial energy in the system consisting of the cube and the bath is the thermal energy in both; namely, $CT(0) + C_bT_b(0)$. The final energy is expressed as $CA_1 + C_bA_1$, and is the same as the initial energy. Thus,

$$CT(0) + C_bT_b(0) = CA_1 + C_bA_1 \qquad \text{or} \qquad A_1 = \frac{CT(0) + C_bT_b(0)}{C + C_b} = A_2$$

Note also that $T(0) = A_1 + B_1$, and $T_b(0) = A_2 + B_2$. Thus, $B_1 = T(0) - A_1$ and $B_2 = T_b(0) - A_2$.

Figure 7.8.4 shows an electric circuit that is analogous to the quenching system of Figure 7.8.3 with $R_2 = \infty$. The voltages v and v_b are analogous to the temperatures T and T_b.

Figure 7.8.4 Electric circuit analogous to quenching with a variable bath temperature and infinite container resistance.
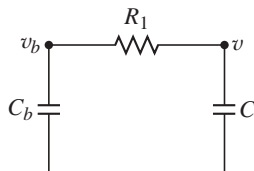
	EXAMPLE 7.8.3
Quenching with Heat Loss to the Surroundings	

■ **Problem**

Consider the quenching process treated in the previous example (Figure 7.8.3). (a) Derive a model of the cube temperature and the bath temperature assuming R_2 is finite. (b) Obtain the model's characteristic roots and the form of the response of $T(t)$, assuming that the surrounding temperature T_o is constant.

■ **Solution**

a. If R_2 is finite, then we must now account for the heat flow into or out of the container. Assume that $T > T_b > T_o$. Then the heat flows from the cube into the bath and then into the surroundings. From conservation of energy,

$$C\frac{dT}{dt} = -\frac{1}{R_1}(T - T_b) \tag{1}$$

and

$$C_b\frac{dT_b}{dt} = \frac{1}{R_1}(T - T_b) - \frac{1}{R_2}(T_b - T_o) \tag{2}$$

Equations (1) and (2) are the desired model.

b. Applying the Laplace transform with zero initial conditions, we obtain

$$(R_1Cs + 1)T(s) - T_b(s) = 0 \tag{3}$$

$$(R_1R_2C_bs + R_1 + R_2)T_b(s) - R_2T(s) = R_1T_o \tag{4}$$

Solving equation (3) for $T_b(s)$ and substituting into equation (4) gives the transfer function

$$\frac{T(s)}{T_o(s)} = \frac{1}{R_1R_2C_bCs^2 + [(R_1 + R_2)C + R_2C_b]s + 1} \tag{5}$$

The denominator gives the characteristic equation

$$R_1R_2C_bCs^2 + [(R_1 + R_2)C + R_2C_b]s + 1 = 0$$

So there will be two nonzero characteristic roots. If these roots are real, say $s = -1/\tau_1$ and $s = -1/\tau_2$, and if T_o is constant, the response will have the form

$$T(t) = Ae^{-t/\tau_1} + Be^{-t/\tau_2} + D$$

where the constants A and B depend on the initial conditions. Note that $T(t) \rightarrow D$ as $t \rightarrow \infty$. Applying the final value theorem to equation (5) gives $T(\infty) = T_o$ and thus $D = T_o$. We could have also obtained this result through physical reasoning.

Figure 7.8.5 shows an electric circuit that is analogous to the quenching system of Figure 7.8.3 with R_2 finite.
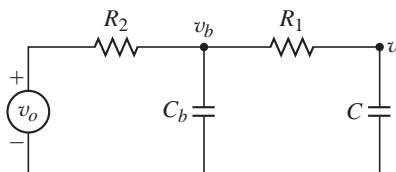
Figure 7.8.5 Electric circuit analogous to quenching with a variable bath temperature and finite container resistance.

EXPERIMENTAL DETERMINATION OF THERMAL RESISTANCE

The mass density ρ, the specific heat c_p, and the thermal conductivity k are accurately known for most materials, and thus we can obtain accurate values of the thermal capacitance C, and also of the conductive resistance L/kA, especially if the thermal capacitance of the conducting element is small. However, determination of the convective resistance is difficult to do analytically, and we must usually resort to experimentally determined values. In some cases, we may not be able to distinguish between the effects of conduction, convection, and radiation heat transfer, and the resulting model will contain a thermal resistance that expresses the aggregated effects.

EXAMPLE 7.8.4

Temperature Dynamics of a Cooling Object

■ Problem
Consider the experiment with a cooling cup of water described in Example 1.5.1 in Chapter 1. Water of volume 250 ml in a glass measuring cup was allowed to cool after being heated to 204°F. The surrounding air temperature was 70°F. The measured water temperature at various times is given in the table in Example 1.5.1. From that data we derived the following model of the water temperature as a function of time.

$$T = 129e^{-0.0007t} + 70 \qquad (1)$$

where T is in °F and time t is in seconds. Estimate the thermal resistance of this system.

■ Solution

Figure 7.8.6 Generic representation of a thermal system having a single capacitance and a single resistance.

We model the cup and water as the object shown in Figure 7.8.6. We assume that convection has mixed the water well so that the water has the same temperature throughout. Let R be the aggregated thermal resistance due to the combined effects of (1) conduction through the sides and bottom of the cup, (2) convection from the water surface and from the sides of the cup into the air, and (3) radiation from the water to the surroundings. Assume that the air temperature T_o is constant and select it as the reference temperature. The heat energy in the water is

$$E = \rho V c_p (T - T_o)$$

From conservation of heat energy

$$\frac{dE}{dt} = -\frac{1}{R}(T - T_o)$$

or, since ρ, V, c_p, and T_o are constant,

$$\rho V c_p \frac{dT}{dt} = -\frac{1}{R}(T - T_o)$$

The water's thermal capacitance is $C = \rho V c_p$ and the model can be expressed as

$$RC\frac{dT}{dt} + T = T_o \qquad (2)$$

The model's complete response is

$$T(t) = T(0)e^{-t/RC} + \left(1 - e^{-t/RC}\right)T_o = [T(0) - T_o]e^{-t/RC} + T_o$$

Comparing this with equation (1), we see that

$$RC = \frac{1}{0.0007} = 1429 \text{ sec}$$

or

$$R = \frac{1429}{C} \frac{°F}{\text{ft-lb}}$$

where $C = \rho V c_p$. Because the temperature data was given in °F, we must convert the volume of 250 ml to ft³. Note that $V = 250$ ml $= 2.5 \times 10^{-4}$ m³, so that $V = 2.5 \times 10^{-4}(3.28 \text{ ft/m})^3 \text{m}^3 = 8.82 \times 10^{-3}$ ft³. Using the values at room temperature for water, we have $\rho = 1.942$ slug/ft³, $c_p = 25{,}000$ ft-lb/slug-°F, and thus $C = 423$ ft-lb-sec/°F. Therefore the aggregated thermal resistance is

$$R = \frac{1429}{C} = 3.37 \frac{°F}{\text{ft-lb}}$$

The usefulness of this result is that this value of R can be used to predict the temperature dynamics of the water/cup system under somewhat different conditions. For example, we can use it to estimate the temperature of a different amount of water if we also know how much the surface area changes. This is because the thermal resistance is inversely proportional to area, as can be seen from (7.7.2) and (7.7.4). Suppose we double the water volume to 500 ml, so that the new value of the thermal capacitance becomes $C = 2(423) = 846$ ft-lb-sec/°F. Suppose the surface area of the new volume is 5/3 that of the smaller volume. Then we estimate the new value of R to be $R = (3/5)(3.37) = 2.02$°F/ft-lb. Thus the model of the new water mass is given by equation (2) with $RC = (2.02)(846) = 1709$ sec.

| Temperature Sensor Response | **EXAMPLE 7.8.5** |

■ Problem

A thermocouple can be used to measure temperature. The electrical resistance of the device is a function of the temperature of the surrounding fluid. By calibrating the thermocouple and measuring its resistance, we can determine the temperature. Because the thermocouple has mass, it has thermal capacitance, and thus its temperature change (and electrical resistance change) will lag behind any change in the fluid temperature.

Estimate the response time of a thermocouple suddenly immersed in a fluid. Model the device as a sphere of copper constantin alloy, whose diameter is 2 mm, and whose properties are $\rho = 8920$ kg/m³, $k = 19$ W/m·°C, and $c_p = 362$ J/kg·°C. Take the convection coefficient to be $h = 200$ W/m³.

■ Solution

First compute the Biot number N_B to see if a lumped-parameter model is sufficient. For a sphere of radius r,

$$L = \frac{V}{A} = \frac{(4/3)\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{0.001}{3} = 3.33 \times 10^{-4}$$

The Biot number is

$$N_B = \frac{hL}{k} = \frac{200(3.33 \times 10^{-4})}{19} = 0.0035$$

which is much less than 0.1. So we can use a lumped-parameter model.

Applying conservation of heat energy to the sphere, we obtain the model:

$$c_p \rho V \frac{dT}{dt} = hA(T_o - T)$$

where T is the temperature of the sphere and T_o is the fluid temperature. The time constant of this model is

$$\tau = \frac{c_p \rho}{h} \frac{V}{A} = \frac{362(8920)}{200} 3.33 \times 10^{-4} = 5.38 \text{ s}$$

The thermocouple temperature will reach 98% of the fluid temperature within $4\tau = 21.5$ s.

EXAMPLE 7.8.6

State-Variable Model of Wall Temperature Dynamics

■ Problem

Consider the wall shown in cross section in Figure 7.7.5 and treated in Example 7.7.1. In that example the thermal capacitances of the layers were neglected. We now want to develop a model that includes their effects. Neglect any convective resistance on the inside and outside surfaces.

■ Solution

We lump each thermal mass at the centerline of its respective layer and assign half of the layer's thermal resistance to the heat flow path on the left and half to the path on the right side of the lumped mass. The representation is shown in Figure 7.8.7a. Let

$$R_a = \frac{R_1}{2} \qquad R_b = \frac{R_1}{2} + \frac{R_2}{2}$$

$$R_c = \frac{R_2}{2} + \frac{R_3}{2} \qquad R_d = \frac{R_3}{2} + \frac{R_4}{2} \qquad R_e = \frac{R_4}{2}$$

An equivalent electrical circuit is shown in part (b) for those who benefit from such an analogy.

For thermal capacitance C_1, conservation of energy gives

$$C_1 \frac{dT_1}{dt} = \frac{T_i - T_1}{R_a} - \frac{T_1 - T_2}{R_b}$$

For C_2,

$$C_2 \frac{dT_2}{dt} = \frac{T_1 - T_2}{R_b} - \frac{T_2 - T_3}{R_c}$$

For C_3,

$$C_3 \frac{dT_3}{dt} = \frac{T_2 - T_3}{R_c} - \frac{T_3 - T_4}{R_d}$$

Figure 7.8.7 (a) A wall model with four capacitances. (b) Analogous electric circuit.

(a)

(b)
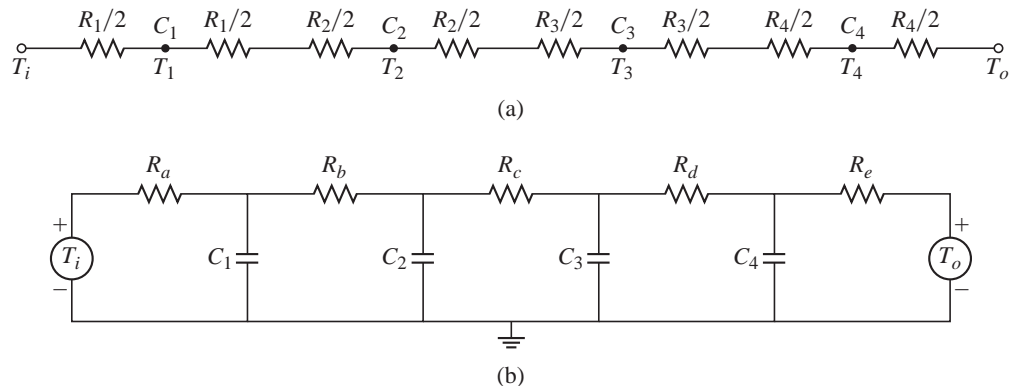
Finally, for C_4,

$$C_4 \frac{dT_4}{dt} = \frac{T_3 - T_4}{R_d} - \frac{T_4 - T_0}{R_e}$$

These four equations may be put into state variable form as follows.

$$\frac{d\mathbf{T}}{dt} = \mathbf{AT} + \mathbf{Bu}$$

where

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} T_i \\ T_o \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_{42} \end{bmatrix}$$

where

$$a_{11} = -\frac{R_a + R_b}{C_1 R_a R_b} \qquad a_{12} = \frac{1}{C_1 R_b}$$

$$a_{21} = \frac{1}{C_2 R_b} \qquad a_{22} = -\frac{R_b + R_c}{C_2 R_b R_c} \qquad a_{23} = \frac{1}{C_2 R_c}$$

$$a_{32} = \frac{1}{C_3 R_c} \qquad a_{33} = -\frac{R_c + R_d}{C_3 R_c R_d} \qquad a_{34} = \frac{1}{C_3 R_d}$$

$$a_{43} = \frac{1}{C_4 R_d} \qquad a_{44} = -\frac{R_d + R_e}{C_4 R_d R_e}$$

$$b_{11} = \frac{1}{C_1 R_a} \qquad b_{42} = \frac{1}{C_4 R_e}$$

In Section 7.9 we will use MATLAB to solve these equations.

PART III. MATLAB AND SIMULINK APPLICATIONS

Sections 7.9 and 7.10 show how MATLAB and Simulink can be used to solve problems involving fluid and thermal systems.

7.9 MATLAB APPLICATIONS

Fluid and thermal system models are often nonlinear. When a differential equation is nonlinear, we often have no analytical solution to use for checking our numerical results. In such cases, we can use our physical insight to guard against grossly incorrect results. We can also check the equation singularities that might affect the numerical procedure. Finally, we can sometimes use an approximation to replace the nonlinear equation with a linear one that can be solved analytically. Although the linear approximation does not give the exact answer, it can be used to see if our numerical answer is "in the ball park." Example 7.9.1 illustrates this approach.

Lumped-parameter models of thermal systems often need several lumped masses in order to represent the dynamics well. This requires a higher-order model that is difficult to solve in closed form and must be solved numerically. Such models naturally occur in state-variable form and are therefore easily handled with the MATLAB state-variable functions. Example 7.9.2 deals with heat transfer through a multilayered wall. It shows how to obtain the response of a state-variable model that is subjected to two different inputs and whose initial conditions are nonzero.

EXAMPLE 7.9.1

Figure 7.9.1 Draining of a spherical tank.

Liquid Height in a Spherical Tank
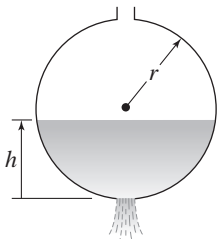
■ Problem
Figure 7.9.1 shows a spherical tank for storing water. The tank is filled through a hole in the top and drained through a hole in the bottom. The following model for the liquid height h is developed in the Problems 7.38 and 7.39.

$$\pi(2Rh - h^2)\frac{dh}{dt} = -C_d A_o \sqrt{2gh} \tag{1}$$

For water, $C_d = 0.6$ is a common value.

Use MATLAB to solve this equation to determine how long it will take for the tank to empty if the initial height is 9 ft. The tank has a radius of $R = 5$ ft and has a 1-in.-diameter hole in the bottom. Use $g = 32.2$ ft/sec^2. Discuss how to check the solution.

■ Solution
With $C_d = 0.6$, $R = 5$, $g = 32.2$, and $A_o = \pi(1/24)^2$, equation (1) becomes

$$\frac{dh}{dt} = -\frac{0.0334\sqrt{h}}{10h - h^2} \tag{2}$$

We can use our physical insight to guard against grossly incorrect results. We can also check the preceding expression for dh/dt for singularities. The denominator does not become zero unless $h = 0$ or $h = 10$, which correspond to a completely empty and a completely full tank. So we will avoid singularities if $0 < h(0) < 10$.

We can use the following approximation to estimate the time to empty. Replace h on the right side of equation (2) with its average value, namely, $(9 - 0)/2 = 4.5$ feet. This gives $dh/dt = -0.00286$, whose solution is $h(t) = h(0) - 0.00286t = 9 - 0.00286t$. According to this equation, if $h(0) = 9$, the tank will be empty at $t = 9/0.00286 = 3147$ sec, or 52 min. We will use this value as a "reality check" on our answer.

The function file based on this equation is

```
function hdot = height(t,h)
hdot = -(0.0334*sqrt(h))/(10*h-h^2);
```

The file is called as follows, using the `ode45` solver.

```
[t, h] = ode45(@height, [0, 2475], 9);
plot(t,h),xlabel('Time (sec)'),ylabel('Height (ft)')
```

The resulting plot is shown in Figure 7.9.2. Note how the height changes more rapidly when the tank is nearly full or nearly empty. This is to be expected because of the effects of the tank's shape. The tank empties in 2475 sec, or 41 min. This value is not grossly different from our rough estimate of 52 min, so we should feel comfortable accepting the numerical results.

The value of the final time of 2475 sec was found by increasing the final time until the plot showed that the height became zero. You could use a `while` loop to do this, by increasing the final time in the loop while calling `ode45` repeatedly.
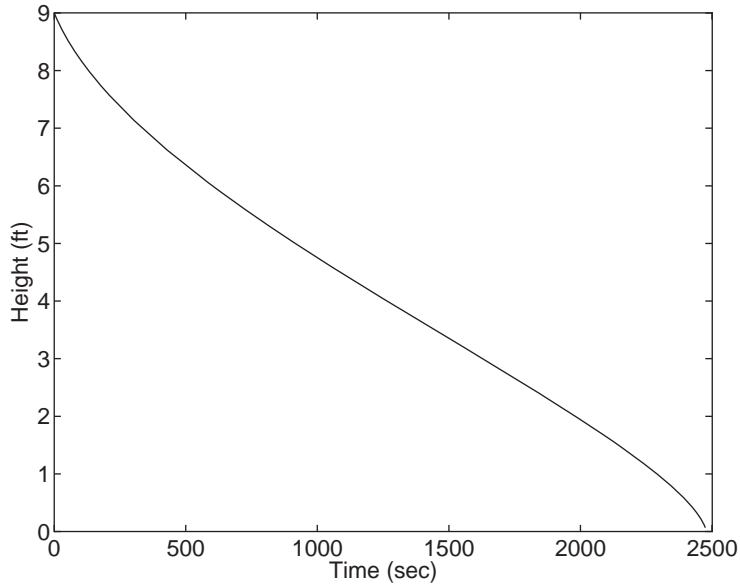
Figure 7.9.2 Plot of liquid height in a draining spherical tank.

Heat Transfer Through a Wall | **EXAMPLE 7.9.2**

■ **Problem**

Consider the wall cross section shown in Figure 7.9.3. The temperature model was developed in Example 7.8.6. Use the following values and plot the temperatures versus time for the case where the inside temperature is constant at $T_i = 20°C$ and the outside temperature T_o decreases linearly from $5°C$ to $-10°C$ in 1 h. The initial wall temperatures are $10°C$.

The resistance values in $°C/W$ are

$$R_a = 0.018 \qquad R_b = 2.023 \qquad R_c = 2.204$$

$$R_d = 0.223 \qquad R_e = 0.019$$

The capacitance values in $J/°C$ are

$$C_1 = 8720 \qquad C_2 = 6210$$

$$C_3 = 6637 \qquad C_4 = 2.08 \times 10^4$$

■ **Solution**

The model was developed in Example 7.8.6.

The given information shows that the outside temperature is described by

$$T_o(t) = 5 - 15t \quad 0 \le t \le 3600 \text{ s}$$
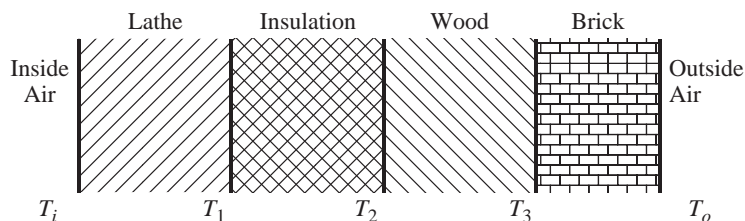
Figure 7.9.3 Heat transfer through a wall with four layers.

The following MATLAB program creates the required plots. Recall that the total response is the sum of the forced and the free responses.

```
% htwall.m Heat transfer thru a multilayer wall.
% Resistance and capacitance data.
Ra = 0.018; Rb = 2.023; Rc = 2.204; Rd = 0.223; Re = 0.019;
C1 = 8720; C2 = 6210; C3 = 6637; C4 = 20800;
% Compute the matrix coefficients.
a11 = -(Ra+Rb)/(C1*Ra*Rb); a12 = 1/(C1*Rb);
a21 = 1/(C2*Rb); a22 = -(Rb+Rc)/(C2*Rb*Rc); a23 = 1/(C2*Rc);
a32 = 1/(C3*Rc); a33 = -(Rc+Rd)/(C3*Rc*Rd); a34 = 1/(C3*Rd);
a43 = 1/(C4*Rd); a44 = -(Rd+Re)/(C4*Rd*Re);
b11 = 1/(C1*Ra); b42 = 1/(C4*Re);
% Define the A and B matrices.
A = [a11,a12,0,0; a21,a22,a23,0; 0,a32,a33,a34; 0,0,a43,a44];
B = [b11,0; 0,0; 0,0; 0,b42];
% Define the C and D matrices.
% The outputs are the four wall temperatures.
C = eye(4);
D = zeros(size(B));
% Create the LTI model.
sys = ss(A,B,C,D);
% Create the time vector for 1 hour (3600 seconds).
t = (0:1:3600);
% Create the input vector.
u = [20*ones(size(t));(5-15*ones(size(t)).*t/3600)];
% Compute the forced response.
[yforced,t] = lsim(sys,u,t);
% Compute the free response.
[yfree,t] = initial(sys,[10,10,10,10],t);
% Plot the response along with the outside temperature.
plot(t,yforced+yfree,t,u(2,:))
% Compute the time constants.
tau =(-1./real(eig(A)))/60
```

The plot is shown in Figure 7.9.4. Note how T_1 follows the inside temperature, while T_4 follows the outside temperature, as expected. The time constants are 2.6, 6, 24, and 117 min.

From the plot note that T_1 reaches steady state in less than 1000 s, but the dominant time constant is 117 min, or 7020 s. The discrepancy is explained by examining the transfer functions for numerator dynamics. After running htwall, which puts the matrices A and B in the workspace, the following MATLAB session can be used to obtain the transfer functions for T_1 and T_2. The C and D matrices need to be replaced to obtain only T_1 and T_2 as the outputs.

```
≫C1 = [1,0,0,0]; D1=[0,0];
≫sys1 = ss(A,B,C1,D1);
≫C2 = [0,1,0,0]; D2 = [0,0];
≫sys2 = ss(A,B,C2,D2);
≫tf1 = tf(sys1)
≫tf2 = tf(sys2)
```
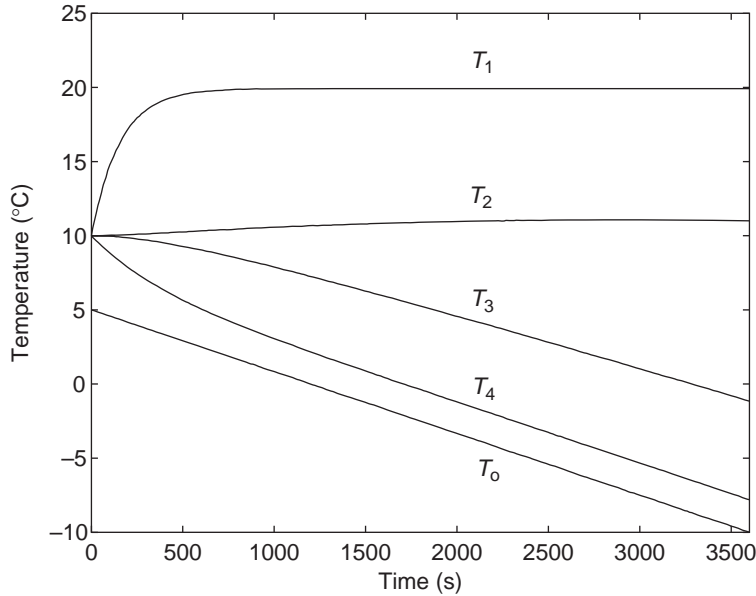
Figure 7.9.4 Wall temperatures as functions of time.

This session produces four transfer functions. Two of them are

$$\frac{T_1(s)}{T_i(s)} = \frac{6.371 \times 10^{-3}s^3 + 2.321 \times 10^{-5}s^2 + 1.545 \times 10^{-8}s + 1.758 \times 10^{-12}}{P(s)}$$

$$\frac{T_2(s)}{T_i(s)} = \frac{5.071 \times 10^{-7}s^2 + 1.77 \times 10^{-9}s + 9.622 \times 10^{-13}}{P(s)}$$

where the denominator is

$$P(s) = s^4 + 0.01007s^3 + 2.583 \times 10^{-5}s^2 + 1.585 \times 10^{-8}s + 1.765 \times 10^{-12}$$

The transfer function $T_1(s)/T_i(s)$ has higher order numerator dynamics than $T_2(s)/T_i(s)$. This accounts for the faster response of T_1.

7.10 SIMULINK APPLICATIONS

One potential disadvantage of a graphical interface such as Simulink is that to simulate a complex system, the diagram can become rather large, and therefore somewhat cumbersome. Simulink, however, provides for the creation of *subsystem blocks,* which play a role analogous to subprograms in a programming language. A subsystem block is actually a Simulink program represented by a single block. A subsystem block, once created, can be used in other Simulink programs. In this section, we introduce subsystem blocks, and also show how to use the Relay block, which is an example of something that is tedious to program in MATLAB. We also introduce the Fcn block.

SUBSYSTEM BLOCKS

You can create a subsystem block in one of two ways, by dragging the Subsystem block from the block library to the model window, or by first creating a Simulink model and then "encapsulating" it within a bounding box. We will illustrate the latter method.

We will create a subsystem block for the liquid-level system shown in Figure 7.10.1, in which the resistances are nonlinear and obey the following signed-square-root relation:

$$q = \frac{1}{R}\text{SSR}(\Delta p)$$

where q is the mass flow rate, R is the resistance, Δp is the pressure difference across the resistance, and

$$\text{SSR}(u) = \begin{cases} \sqrt{u} & \text{if } u \geq 0 \\ -\sqrt{|u|} & \text{if } u < 0 \end{cases}$$

Note that we can express the SSR(u) function in MATLAB as follows: `sgn(u)*sqrt(abs(u))`.

The model of the system in Figure 7.10.1 is the following:

$$\rho A \frac{dh}{dt} = q + \frac{1}{R_l}\text{SSR}(p_l - p) - \frac{1}{R_r}\text{SSR}(p - p_r)$$

where p_l and p_r are the *gage* pressures at the left and right-hand sides, A is the bottom area, q is a mass flow rate, and $p = \rho g h$. Note that the atmospheric pressure p_a cancels out of the model because of the use of gage pressure.

First construct the Simulink model shown in Figure 7.10.2. The oval blocks are input and outport ports (In 1 and Out 1), which are available in the Ports and Subsystems library. When entering the gains in each of the four Gain blocks, note that you can use MATLAB variables and expressions. Before running the program we will assign values to these variables in the MATLAB Command window. Enter the gains for the four Gain

Figure 7.10.1 A liquid-level system.
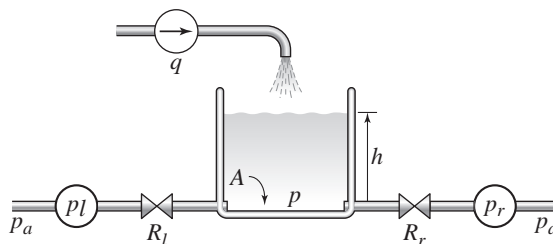
Figure 7.10.2 Simulink model of the system shown in Figure 7.10.1.
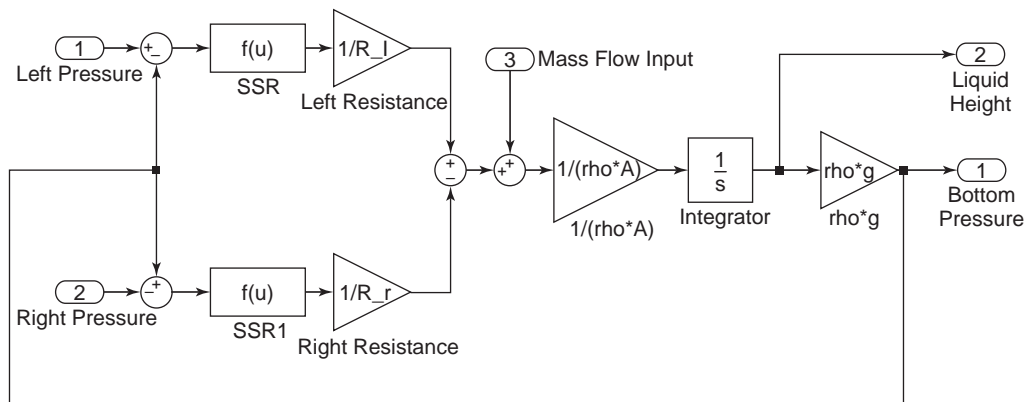
blocks using the expressions shown in the block. You may also use a variable as the Initial condition of the Integrator block. Name this variable h0.

The SSR blocks are examples of the Fcn block in the User-Defined Functions Library. Double-click on the block and enter the MATLAB expression sgn(u)* sqrt(abs(u)). Note that the Fcn block requires you to use the variable u. The output of the Fcn block must be a scalar, as is the case here, and you cannot perform matrix operations in the Fcn block, but these also are not needed here. (Alternatives to the Fcn block are the Math Function block and the MATLAB Fcn block, which are discussed in Section 8.8.) Save the model and give it a name, such as Tank.

Now create a "bounding box" surrounding the diagram. Do this by placing the mouse cursor in the upper left, holding the mouse button down, and dragging the expanding box to the lower right to enclose the entire diagram. Then choose **Create Subsystem** from the **Edit** menu. Simulink will then replace the diagram with a single block having as many input and output ports as required, and will assign default names. You can resize the block to make the labels readable (see Figure 7.10.3). You can view or edit the subsystem by double-clicking on it.

Suppose we want to create a simulation of the system shown in Figure 7.10.4, where the mass inflow rate q_1 is a step function. To do this, create the Simulink model shown in Figure 7.10.5. The square blocks are Constant blocks from the Sources library. These give constant inputs. The larger rectangular blocks are two subsystem blocks of the type just created. To insert them into the model, first open the Tank subsystem model, select **Copy** from the **Edit** menu, then paste it twice into the new model window. Connect the input and output ports and edit the labels as shown. Then double-click on the Tank 1 subsystem block, set the gain 1/R_l equal to 0, the gain 1/R_r equal to 1/R_1, and the gain 1/rho*A equal to 1/rho*A_1. Set the Initial condition of the integrator to h10. Note that setting the gain 1/R_l equal to 0 is equivalent to $R\_l = \infty$, which represents the fact that there is no inlet on the left-hand side.

Then double-click on the Tank 2 subsystem block, set the gain 1/R_l equal to 1/R_1, the gain 1/R_r equal to 1/R_2, and the gain 1/rho*A equal to 1/rho*A_2. Set the Initial condition of the integrator to h20. For the Step block, set the Step time to 0, the Initial value to 0, the Final value to the variable q_1, and the Sample time to 0. Save the model using a name other than Tank.
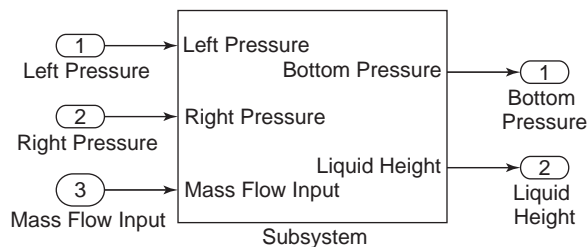
Figure 7.10.3 The Subsystem block.
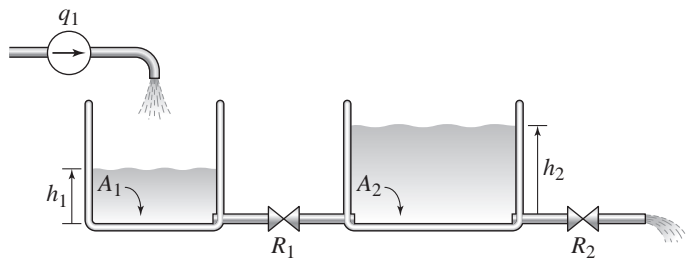
Figure 7.10.4 A liquid-level system with two tanks.

Figure 7.10.5 Simulink model of the system shown in Figure 7.10.4.
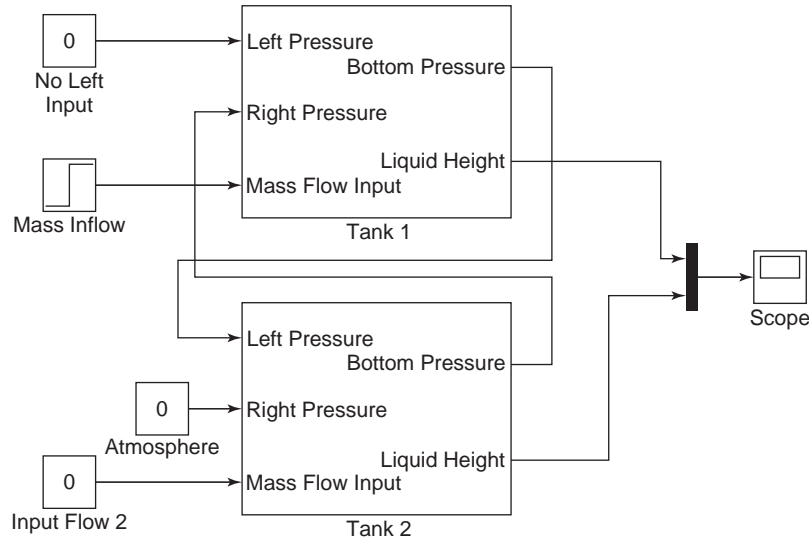
Figure 7.10.6 The relay function. (a) The case where *On* > *Off*. (b) The case where *On* < *Off*.
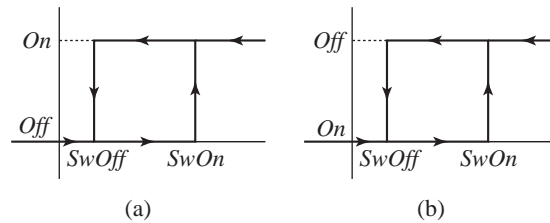
Before running the model, in the Command window assign numerical values to the variables. As an example, you may type the following values for water, in U.S. Customary units, in the Command window.

```
>>A_1 = 2;A_2 = 5;rho = 1.94;g = 32.2;
>>R_1 = 20;R_2 = 50;q_1 = 0.3;h10 = 1;h20 = 10;
```

After selecting a simulation Stop time, you may run the simulation. The Scope will display the plots of the heights h_1 and h_2 versus time.

SIMULATION OF THERMAL SYSTEMS

Home heating systems are controlled by a thermostat, which measures the room temperature and compares it with the desired temperature set by the user. Suppose the user selects 70°F. A typical thermostat would switch the heating system on whenever the inside temperature drops below 69° and switch the system off whenever the temperature is above 71°. The 2° temperature difference is the thermostat's *band,* and different thermostat models might use a different value for the band.

The thermostat is an example of a *relay*. The Simulink Relay block in the Discontinuities library is an example of something that is tedious to program in MATLAB but is easy to implement in Simulink. Figure 7.10.6a is a graph of the logic of a relay. The relay switches the output between two specified values, named *On* and *Off* in the figure. Simulink calls these values Output when on and Output when off. When the relay output is *On,* it remains *On* until the input drops below the value of the Switch off point parameter, named *SwOff* in the figure. When the relay output is *Off,* it remains *Off* until the input exceeds the value of the Switch on point parameter, named *SwOn* in the

figure. The Switch on point parameter value must be greater than or equal to the Switch off point value. Note that the value of *Off* need not be zero. Note also that the value of *Off* need not be less than the value of *On*. As we will see in Example 7.10.1, it is sometimes necessary to use this case. The case where $Off > On$ is shown in Figure 7.10.6b.

| Thermostatic Control of Temperature | **EXAMPLE 7.10.1** |

■ Problem
(a) Develop a Simulink model of a thermostatic control system in which the temperature model is

$$RC\frac{dT}{dt} + T = Rq + T_a(t)$$

where T is the room air temperature in °F, T_a is the ambient (outside) air temperature in °F, time t is measured in hours, q is the input from the heating system in ft-lb/hr, R is the thermal resistance, and C is the thermal capacitance. The thermostat switches q on at the value q_{max} whenever the temperature drops below $69°$, and switches q to $q = 0$ whenever the temperature is above $71°$. The value of q_{max} indicates the heat output of the heating system.

Run the simulation for the case where $T(0) = 70°$ and $T_a(t) = 50 + 10\sin(\pi t/12)$. Use the values $R = 5 \times 10^{-5}$ °F-hr/lb-ft and $C = 4 \times 10^4$ lb-ft/°F. Plot the temperatures T and T_a versus t on the same graph, for $0 \le t \le 24$ hr. Do this for two cases: $q_{max} = 4 \times 10^5$ and $q_{max} = 8 \times 10^5$ lb-ft/hr. Investigate the effectiveness of each case. (b) The integral of q over time is the energy used. Plot $\int q\,dt$ versus t and determine how much energy is used in 24 hr for the case where $q_{max} = 8 \times 10^5$.

■ Solution
The model can be arranged as follows:

$$\frac{dT}{dt} = \frac{1}{RC}[Rq + T_a(t) - T]$$

The Simulink model is shown in Figure 7.10.7, where $1/RC = 0.5$ and the gain labeled R has the value $R = 5 \times 10^{-5}$, which is entered as 5e-5. The output of the Relay block is $q(t)$. The input $T_a(t)$ is produced with the Sine block. For the Sine block, set the Amplitude to 10, the Bias to 50, the Frequency to pi/12, the Phase to 0, and the Sample time to 0. The second Integrator block and Scope 2 were included to compute and display $\int q\,dt$ versus t.

For the Relay block, set the Switch on point to 71, the Switch off point to 69, the Output when on to 0, and the Output when off to the variable *qmax*. This corresponds to Figure 7.10.6a. Set the simulation stop time to 24.
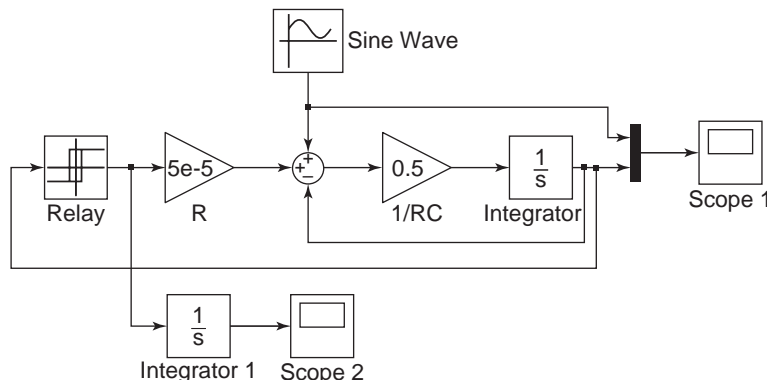
Figure 7.10.7 Simulink model of a temperature control system.

You can set the value of *qmax* by editing the Relay block or by setting its value in the MATLAB Command window before running the simulation, for example, by typing `qmax = 4e+5` before the first simulation.

The simulation results show that when $q_{max} = 4 \times 10^5$, the system is unable to keep the temperature from falling below 69°. When $q_{max} = 8 \times 10^5$, the temperature stays within the desired band. The plot of $\int q\, dt$ versus t for this case shows that the energy used at the end of 24 hr is 9.6158×10^6 ft-lb. This value can be obtained by exporting the output of the second integrator to the workspace.

7.11 CHAPTER REVIEW

Part I of this chapter treated fluid systems, which can be divided into hydraulics and pneumatics. Hydraulics is the study of systems in which the fluid is incompressible; that is, its density stays approximately constant over a range of pressures. Pneumatics is the study of systems in which the fluid is compressible. Hydraulics and pneumatics share a common modeling principle: conservation of mass. It forms the basis of all our models of such systems.

Modeling pneumatic systems also requires application of thermodynamics, because the temperature of a gas can change when its pressure changes. Thus pneumatics provides a bridge to the treatment of thermal systems, which is the subject of Part II of the chapter. Thermal systems are systems that operate due to temperature differences. They thus involve the flow and storage of heat energy, and conservation of heat energy forms the basis of our thermal models.

Now that you have finished this chapter, you should be able to

1. Apply conservation of mass to model simple hydraulic and pneumatic systems.
2. Derive expressions for the capacitance of simple hydraulic and pneumatic systems.
3. Determine the appropriate resistance relation to use for laminar, turbulent, and orifice flow.
4. Develop a dynamic model of hydraulic and pneumatic systems containing one or more capacitances.
5. Determine the appropriate thermal resistance relation to use for conduction, convection, and radiation heat transfer.
6. Develop a model of a thermal process having one or more thermal storage compartments.
7. Apply MATLAB and Simulink to solve fluid and thermal system models.

REFERENCE

[Çengel, 2001] Y. A. Çengel and R. H. Turner, *Fundamentals of Thermal-Fluid Sciences,* McGraw-Hill, NY, 2001.

PROBLEMS

Section 7.1 Conservation of Mass

7.1   For the hydraulic system shown in Figure P7.1, given $A_1 = 10$ in.^{2}, $A_2 = 30$ in.^{2}, and $mg = 60$ lb, find the force f_1 required to lift the mass m a distance $x_2 = 6$ in. Also find the distance x_1 and the work done by the force f_1.

Figure P7.1
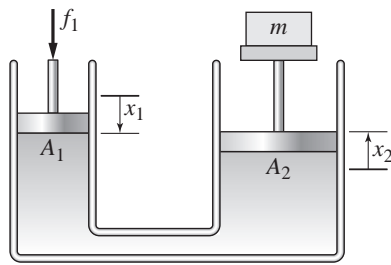
Figure P7.4
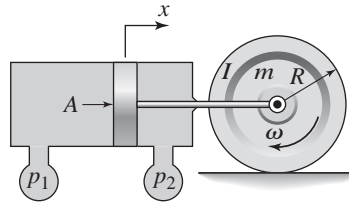
7.2 Refer to the water storage and supply system shown in Figure 7.1.2. The cylindrical tank has a radius of 11 ft, and the water height is initially 5 ft. Find the water height after 5 hr if 1000 gallons per minute are pumped out of the well and 800 gallons per minute are withdrawn from the tank. Note that 1 gallon is 0.13368 ft^3.

7.3 Consider the piston and mass shown in Figure 7.1.4a. Suppose there is dry friction acting between the mass m and the surface. Find the minimum area A of the piston required to move the mass against the friction force μmg, where $\mu = 0.6$, $mg = 1000$ N, $p_1 = 3 \times 10^5$ Pa, and $p_2 = 10^5$ Pa.

7.4 In Figure P7.4 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Given $p_1 - p_2 = 3 \times 10^5$ Pa, $A = 0.005$ m^2, $R = 0.4$ m, $m = 100$ kg, and $I = 7$ kg $\cdot$ m^2, determine the angular velocity $\omega(t)$ of the cylinder assuming that it starts from rest.

7.5 Refer to Figure 7.1.4a, and suppose that $p_1 - p_2 = 10$ lb/$in.^2$, $A = 3$ $in.^2$, and $mg = 600$ lb. If the mass starts from rest at $x(0) = 0$, how far will it move in 0.5 sec, and how much hydraulic fluid will be displaced?

7.6 Pure water flows into a mixing tank of volume $V = 300$ m^3 at the constant volume rate of 10 m^3/s. A solution with a salt concentration of s_i kg/m^3 flows into the tank at a constant volume rate of 2 m^3/s. Assume that the solution in the tank is well mixed so that the salt concentration in the tank is uniform. Assume also that the salt dissolves completely so that the volume of the mixture remains the same. The salt concentration s_o kg/m^3 in the outflow is the same as the concentration in the tank. The input is the concentration $s_i(t)$, whose value may change during the process, thus changing the value of s_o. Obtain a dynamic model of the concentration s_o.

7.7 Consider the mixing tank treated in Problem 7.6. Generalize the model to the case where the tank's volume is V m^3. For quality control purposes, we want to adjust the output concentration s_o by adjusting the input concentration s_i. How much volume should the tank have so that the change in s_o lags behind the change in s_i by no more than 20 s?

Section 7.2 Fluid Capacitance

7.8 Derive the expression for the fluid capacitance of the cylindrical tank shown in Figure P7.8.

7.9 Derive the expression for the capacitance of the container shown in Figure P7.9.

Figure P7.8
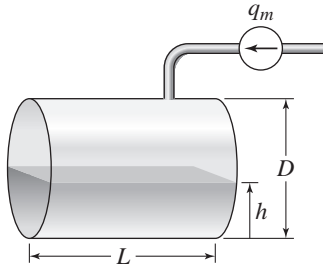
Figure P7.9
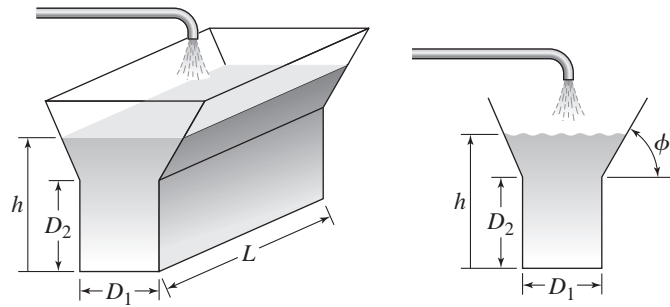
7.10   Consider the cylindrical tank shown in Figure P7.8. Derive the dynamic model of the height h, assuming that the input mass flow rate is $q_m(t)$.

7.11   Consider the tank shown in Figure P7.9. Derive the dynamic model of the height h, assuming that the input mass flow rate is $q_m(t)$.

Section 7.3 Fluid Resistance

7.12   Air flows in a certain cylindrical pipe 1 m long with an inside diameter of 1 mm. The pressure difference between the ends of the pipe is 0.1 atm. Compute the laminar resistance, the Reynolds number, the entrance length, and the mass flow rate. Comment on the accuracy of the resistance calculation. For air use $\mu = 1.58 \times 10^{-5}$ N · s/m^2 and $\rho = 1.2885$ kg/m^3.

7.13   Derive the expression for the linearized resistance due to orifice flow near a reference height h_r.

7.14   Consider the cylindrical container treated in Example 7.3.1. Suppose the outlet flow is turbulent. Derive the dynamic model of the system (a) in terms of the gage pressure p at the bottom of the tank and (b) in terms of the height h.

7.15   A certain tank has a bottom area $A = 20$ m^2. The liquid level in the tank is initially 5 m. When the outlet is opened, it takes 200 s to empty by 98%.
   a.   Estimate the value of the linear resistance R.
   b.   Find the steady-state height if the inflow is $q = 3$ m^3/s.

7.16   A certain tank has a circular bottom area $A = 20$ ft^2. It is drained by a pipe whose linear resistance is $R = 150$ m^{-1}sec^{-1}. The tank contains water whose mass density is 1.94 slug/ft^3.
   a.   Estimate how long it will take for the tank to empty if the water height is initially 30 ft.
   b.   Suppose we dump water into the tank at a rate of 0.1 ft^3/sec. If the tank is initially empty and the outlet pipe remains open, find the steady-state height and the time to reach one-third that height, and estimate how long it will take to reach the steady-state height.

7.17   The water inflow rate to a certain tank was kept constant until the water height above the orifice outlet reached a constant level. The inflow rate was then measured, and the process repeated for a larger inflow rate. The data are given in the table. Find the effective area $C_d A_o$ for the tank's outlet orifice.

Inflow rate (liters/min)	Liquid height (cm)
98	30
93	27
91	24
86	21
81	18
75	15
68	12
63	9
56	6
49	3

7.18   In the system shown in Figure P7.18, a component such as a valve has been inserted between the two lengths of pipe. Assume that turbulent flow exists throughout the system. Use the resistance relation 7.3.3. (a) Find the total turbulent resistance. (b) Develop a model for the behavior of the liquid height h, with the mass flow rate q_{mi} as the input.
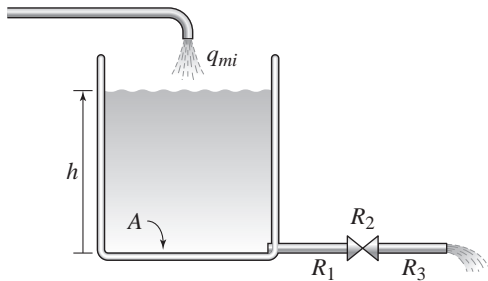
Figure P7.18

7.19   The cylindrical tank shown in Figure 7.3.3 has a circular bottom area A. The mass inflow rate from the flow source is $q_{mi}(t)$, a given function of time. The flow through the outlet is *turbulent,* and the outlet discharges to atmospheric pressure p_a. Develop a model of the liquid height h.

Section 7.4 Dynamic Models of Hydraulic Systems

7.20   In the liquid level system shown in Figure P7.20, the resistances R_1 and R_2 are linear, and the input is the pressure source p_s. Obtain the differential equation model for the height h, assuming that $h > D$.
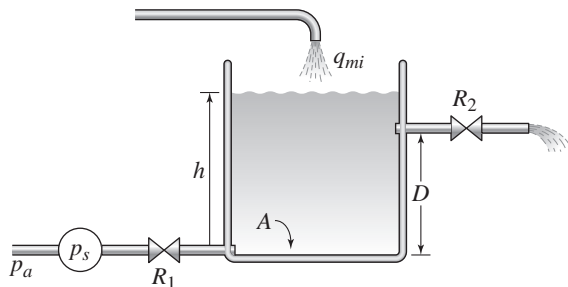
Figure P7.20

7.21   The water height in a certain tank was measured at several times with no inflow applied. See Figure 7.3.3. Assume that laminar flow exists in the outlet pipe. The data are given in the table. The tank's bottom area is $A = 6 \text{ ft}^2$.

   a.   Estimate the laminar resistance R.

b. Suppose the initial height is known to be exactly 20.2 ft. How does this change the results of part (a)?

Time (sec)	Height (ft)
0	20.2
300	17.26
600	14.6
900	12.4
1200	10.4
1500	9.0
1800	7.6
2100	6.4
2400	5.4

7.22 Derive the model for the system shown in Figure P7.22. The flow rate q_{mi} is a mass flow rate and the resistances are linear.

Figure P7.22
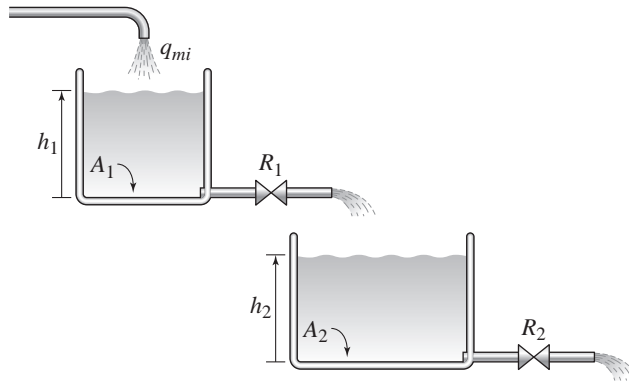
7.23 (a) Develop a model of the two liquid heights in the system shown in Figure P7.23. The inflow rate $q_{mi}(t)$ is a mass flow rate. (b) Using the values $R_1 = R$, $R_2 = 3R$, $A_1 = A$, and $A_2 = 4A$, find the transfer function $H_2(s)/Q_{mi}(s)$.

Figure P7.23
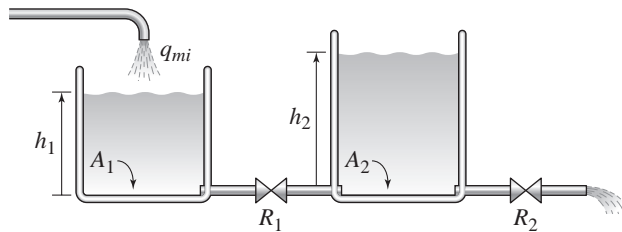
7.24 Consider Example 7.4.3. Suppose that $R_1 = R$, $R_2 = 3R$, $A_1 = A$, and $A_2 = 2A$. Find the transfer function $H_1(s)/Q_{mi}(s)$ and the characteristic roots.

7.25 Design a piston-type damper using an oil with a viscosity at 20°C of $\mu = 0.9$ kg/(m · s). The desired damping coefficient is 2000 N · s/m. See Figure 7.4.4.

7.26 For the damper shown in Figure 7.4.4, assume that the flow through the hole is turbulent, and neglect the term $m\ddot{y}$. Develop a model of the relation between the force f and $\dot{x}$, the relative velocity between the piston and the cylinder.

7.27 An electric motor is sometimes used to move the spool valve of a hydraulic motor. In Figure P7.27 the force f is due to an electric motor acting through a rack-and-pinion gear. Develop a model of the system with the load displacement y as the output and the force f as the input. Consider two cases: (a) $m_1 = 0$ and (b) $m_1 \neq 0$.
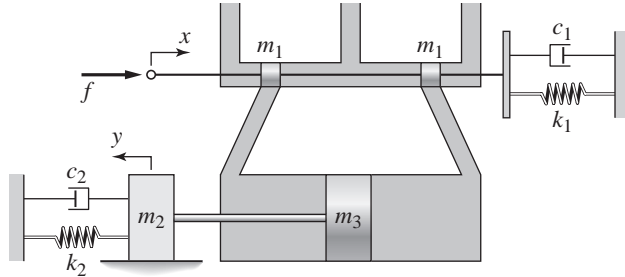
Figure P7.27

7.28 In Figure P7.28 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Develop a dynamic model of the axle's translation x, with the pressures p_1 and p_2 as the inputs.
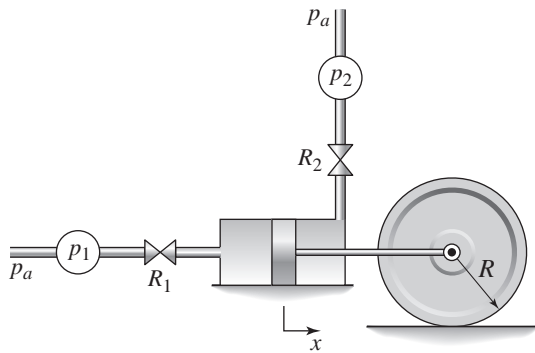
Figure P7.28

7.29 Figure P7.29 shows a pendulum driven by a hydraulic piston. Assuming small angles θ and a concentrated mass m a distance L_1 from the pivot, derive the equation of motion with the pressures p_1 and p_2 as inputs.

7.30 Figure P7.30 shows an example of a *hydraulic accumulator,* which is a device for reducing pressure fluctuations in a hydraulic line or pipe. The fluid density is ρ, the plate mass is m, and the plate area is A. Develop a dynamic model of the pressure p with the pressures p_1 and p_2 as the given inputs. Assume that $m\ddot{x}$ of the plate is small, and that the hydrostatic pressure $\rho g h$ is small.

7.31 Design a hydraulic accumulator of the type shown in Figure P7.30. The liquid volume in the accumulator should increase by 30 in.^{3} when the pressure p increases by 1.5 lb/in.^{2}. Determine suitable values for the plate area A and the spring constant k.

7.32 Consider the liquid-level system treated in Example 7.4.8 and shown in Figure 7.4.9. The pump curve and the line for the steady-state flow through both valves are shown in Figure P7.32. It is known that the bottom area of the tank is 2 m^2 and the outlet resistance is $R_2 = 400$ 1/(m $\cdot$ s). (a) Compute the pump resistance R_1 and the steady-state height. (b) Derive a linearized dynamic model of the height deviation δh in the tank.

Figure P7.29

p_a

p_2

R_2

A

p_1

p_a

R_1

p_3

p_4

x

L_2

L_1

m

θ

g

Figure P7.30

k

x

A

h

p_1

R

p

R

p_2

Figure P7.32

Pump curve, $s = 1500$ rpm

Series valve line

50

40

30

q_m (kg/s)

20

Slope $= -\dfrac{1}{700}$

10

0

0   10^4 2×10^4 3×10^4 4×10^4 5×10^4
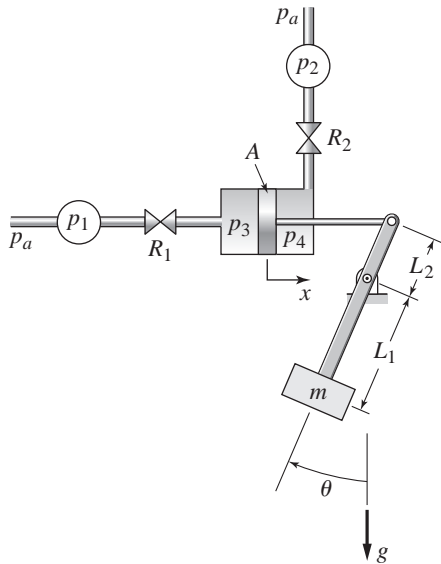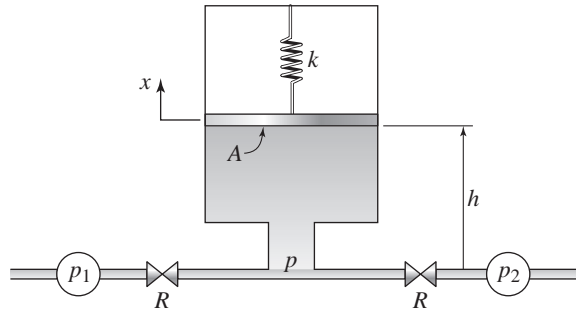
Δp (N/m^2)
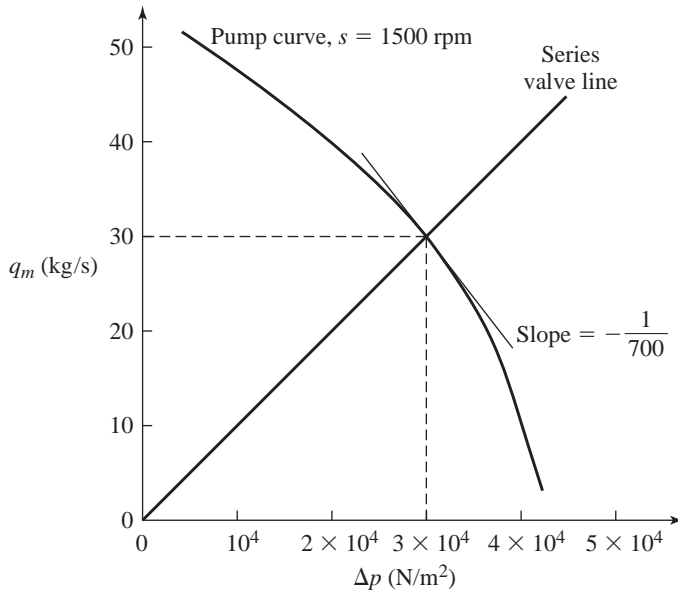
7.33   Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34   Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35   Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.
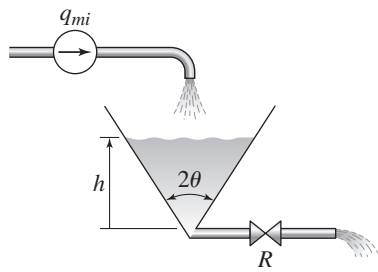
Figure P7.33   **Figure P7.34**   **Figure P7.35**
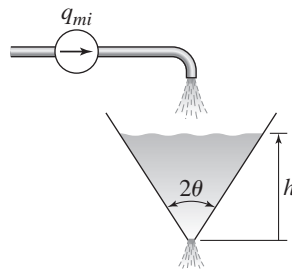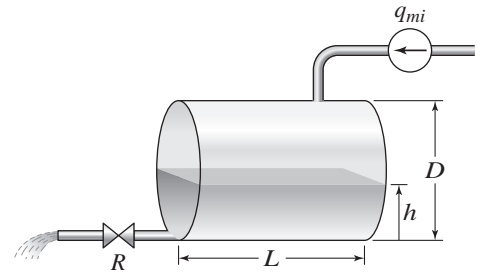
7.36 A certain tank contains water whose mass density is 1.94 slug/ft^3. The tank's circular bottom area is $A = 100$ ft^2. It is drained by an orifice in the bottom. The effective cross-sectional area of the orifice is $C_d A_o = 0.5$ ft^2. A pipe dumps water into the tank at the *volume* flow rate q_v.

a. Derive the model for the tank's height h with the input q_v.

b. Compute the steady-state height if the input flow rate is $q_v = 5$ ft^3/sec.

c. Estimate the tank's time constant when the height is near the steady-state height.

7.37 (a) Derive the expression for the fluid capacitance of the conical tank shown in Figure P7.37. The cone angle θ is a constant and should appear in your answer as a parameter. (b) Derive the dynamic model of the liquid height h. The mass inflow rate is $q_{mi}(t)$. The resistance R is linear.

7.38 (a) Determine the capacitance of a spherical tank of radius R, shown in Figure P7.38. (b) Obtain a model of the pressure at the bottom of the tank, given the mass flow rate q_{mi}.

Figure P7.37
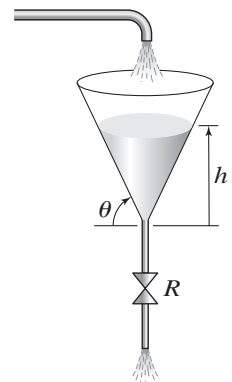
Figure P7.38 A spherical tank.

(a)   (b)

7.39 Obtain the dynamic model of the liquid height h in a spherical tank of radius R, shown in Figure P7.39. The mass inflow rate through the top opening is q_{mi} and the orifice resistance is R_o.
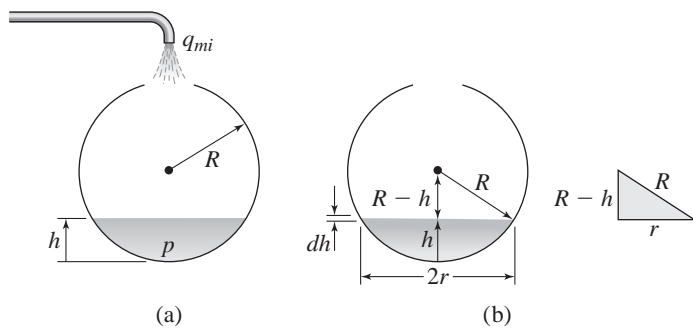
Figure P7.39 A spherical tank with an orifice resistance.
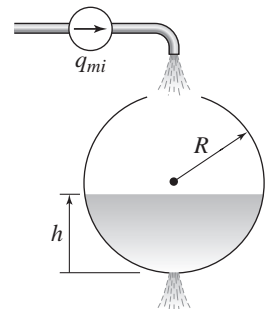
Section 7.5 Pneumatic Systems

7.40 A rigid container has a volume of 20 ft^3. The air inside is initially at 70°F. Find the pneumatic capacitance of the container for an isothermal process.

7.41 Consider the pneumatic system treated in Example 7.5.2. Derive the linearized model for the case where $p_i < p$.

7.42   Figure P7.42 shows two rigid tanks whose pneumatic capacitances are C_1 and C_2. The variables δp_i, δp_1, and δp_2 are small deviations around a reference steady-state pressure p_{ss}. The pneumatic lines have linearized resistances R_1 and R_2. Assume an isothermal process. Derive a model of the pressures δp_1 and δp_2 with δp_i as the input.

Figure P7.42
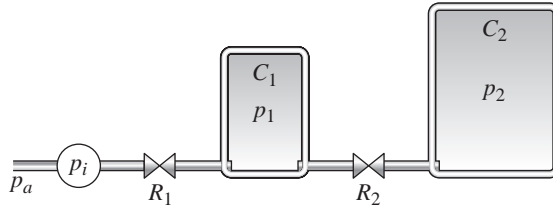
Section 7.6 Thermal Capacitance

7.43   (a) Compute the thermal capacitance of 250 ml of water, for which $\rho = 1000 \text{ kg/m}^3$ and $c_p = 4.18 \times 10^3 \text{ J/kg} \cdot °\text{C}$. Note that 1 ml $= 10^{-6} \text{ m}^3$. (b) How much energy does it take to raise the water temperature from room temperature ($20°\text{C}$) to $99°\text{C}$ (just below boiling).

7.44   A certain room measures 15 ft by 10 ft by 8 ft. (a) Compute the thermal capacitance of the room air using $c_p = 6.012 \times 10^3 \text{ ft-lb/slug-}°\text{F}$ and $\rho = 0.0023 \text{ slug/ft}^3$. (b) How much energy is required to raise the air temperature from $68°\text{F}$ to $72°\text{F}$, neglecting heat transfer to the walls, floor, and ceiling?

7.45   Liquid initially at $20°\text{C}$ is pumped into a mixing tank at a constant volume flow rate of $0.5 \text{ m}^3/\text{s}$. See Figure 7.6.1. At time $t = 0$ the temperature of the incoming liquid suddenly is changed to $80°\text{C}$. The tank walls are perfectly insulated. The tank volume is 12 m^3, and the liquid within is well-mixed so that its temperature is uniform throughout, and denoted by T. The liquid's specific heat and mass density are c_p and ρ. Given that $T(0) = 20°\text{C}$, develop and solve a dynamic model for the temperature T as a function of time.

Section 7.7 Thermal Resistance

7.46   The copper shaft shown in Figure P7.46 consists of two cylinders with the following dimensions: $L_1 = 10 \text{ mm}$, $L_2 = 5 \text{ mm}$, $D_1 = 2 \text{ mm}$, and $D_2 = 1.5 \text{ mm}$. The shaft is insulated around its circumference so that heat transfer occurs only in the axial direction. (a) Compute the thermal resistance of each section of the shaft and of the total shaft. Use the following value for the conductivity of copper: $k = 400 \text{ W/m} \cdot °\text{C}$. (b) Compute the heat flow rate in the axial direction if the temperature difference across the endpoints of the shaft is $30°\text{C}$.

Figure P7.46
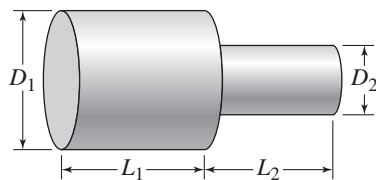
7.47 A certain radiator wall is made of copper with a conductivity $k = 47$ lb/sec-°F at 212°F. The wall is 3/16 in. thick and has circulating water on one side with a convection coefficient $h_1 = 85$ lb/sec-ft-°F. A fan blows air over the other side, which has a convection coefficient $h_2 = 15$ lb/sec-ft-°F. Find the thermal resistance of the radiator on a square foot basis.

7.48 A particular house wall consists of three layers and has a surface area of 30 m^2. The inside layer is 10 mm thick and made of plaster board with a thermal conductivity of $k = 0.2$ W/(m·°C). The middle layer is made of fiberglass insulation with $k = 0.04$ W/(m·°C). The outside layer is 20 mm thick and made of wood siding with $k = 0.1$ W/(m·°C). The inside temperature is 20°C, and the convection coefficient for the inside wall surface is $h_i = 40$ W/(m^2·°C). The convection coefficient for the outside wall surface is $h_o = 70$ W/(m^2·°C). How thick must the insulation layer be so that the heat loss is no greater than 400 W if the outside temperature is −20°C?

7.49 A certain wall section is composed of a 12 in. by 12 in. brick area 4 in. thick. Surrounding the brick is a 36 in. by 36 in. concrete section, which is also 4 in. thick. The thermal conductivity of the brick is $k = 0.086$ lb/sec-°F. For the concrete, $k = 0.02$ lb/sec-°F. (a) Determine the thermal resistance of the wall section. (b) Compute the heat flow rate through (1) the concrete, (2) the brick, and (3) the wall section if the temperature difference across the wall is 40°F.

7.50 Water at 120°F flows in an iron pipe 10 ft long, whose inner and outer radii are 1/2 in. and 3/4 in. The temperature of the surrounding air is 70°F. (a) Assuming that the water temperature remains constant along the length of the pipe, compute the heat loss rate from the water to the air in the radial direction, using the following values. For iron, $k = 10.1$ lb/sec-°F. The convection coefficient at the inner surface between the water and the iron is $h_i = 16$ lb/sec-ft-°F. The convection coefficient at the outer surface between the air and the iron is $h_o = 1.1$ lb/sec-ft-°F. (b) Suppose the water is flowing at 0.5 ft/sec. Check the validity of the constant-temperature assumption. For water, $\rho = 1.94$ slug/ft^3 and $c_p = 25{,}000$ ft-lb/slug-°F.

Section 7.8 Dynamic Models of Thermal Systems

7.51 Consider the water pipe treated in Example 7.7.4. Suppose now that the water is not flowing. The water is initially at 120°F. The copper pipe is 6 ft long, with inner and outer radii of 1/4 in. and 3/8 in. The temperature of the surrounding air is constant at 70°F. Neglect heat loss from the ends of the pipe, and use the following values. For copper, $k = 50$ lb/sec-°F. The convection coefficient at the inner surface between the water and the copper is now different because the water is standing. Use $h_i = 6$ lb/sec-ft-°F. The convection coefficient at the outer surface between the air and the copper is $h_o = 1.1$ lb/sec-ft-°F. Develop and solve a dynamic model of the water temperature $T(t)$ as a function of time.

7.52 A steel tank filled with water has a volume of 1000 ft^3. Near room temperature, the specific heat for water is $c = 25{,}000$ ft-lb/slug-°F, and its mass density is $\rho = 1.94$ slug/ft^3.

   a. Compute the thermal capacitance C_1 of the water in the tank.

b. Denote the total thermal resistance (convective and conductive) of the tank's steel wall by R_1. The temperature of the air surrounding the tank is T_o. The tank's water temperature is T_1. Assume that the thermal capacitance of the steel wall is negligible. Derive the differential equation model for the water's temperature, with T_o as the input.

7.53 Consider the tank of water discussed in Problem 7.52. A test was performed in which the surrounding air temperature T_o was held constant at 70°F. The tank's water temperature was heated to 90° and then allowed to cool. The following data show the tank's water temperature as a function of time. Use these data to estimate the value of the thermal resistance R_1.

Time t (sec)	Water temperature T_1 (°F)
0	90
500	82
1000	77
1500	75
2000	73
2500	72
3000	71
4000	70

7.54 The oven shown in Figure P7.54 has a heating element with appreciable capacitance C_1. The other capacitance is that of the oven air C_2. The corresponding temperatures are T_1 and T_2, and the outside temperature is T_o. The thermal resistance of the heater-air interface is R_1; that of the oven wall is R_2. Develop a model for T_1 and T_2, with input q_i, the heat flow rate delivered to the heater mass.

7.55 A simplified representation of the temperature dynamics of two adjacent masses is shown in Figure P7.55. The mass with capacitance C_2 is perfectly insulated on all sides except one, which has a convective resistance R_2. The thermal capacitances of the masses are C_1 and C_2, and their representative uniform temperatures are T_1 and T_2. The thermal capacitance of the surroundings is very large and the temperature is T_o. (a) Develop a model of the behavior of T_1 and T_2. (b) Discuss what happens if the thermal capacitance C_2 is very small.

Figure P7.54
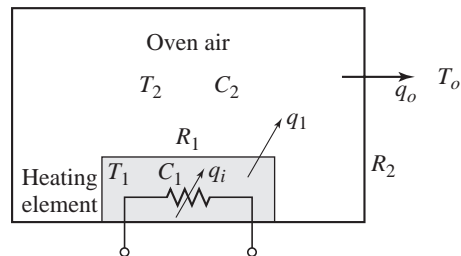
Figure P7.55
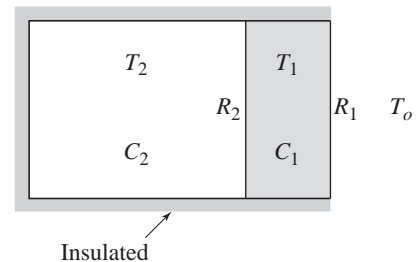
7.56 A metal sphere 25 mm in diameter was heated to 95°C, and then suspended in air at 22°C. The mass density of the metal is 7920 kg/m³, its specific heat

at 30°C is $c_p = 500$ J/(kg · °C), and its thermal conductivity at 30°C is 400 W/(m · °C). The following sphere temperature data were measured as the sphere cooled.

t (s)	T (°C)	t (s)	T (°C)	t (s)	T(°C)
0	95	120	85	540	67
15	93	135	84	600	65
30	92	180	82	660	62
45	90	240	79	720	61
60	89	300	76	780	59
75	88	360	73	840	57
90	87	420	71	900	56
105	86	480	69	960	54

a. Assume that the sphere's heat loss rate is due entirely to convection. Estimate the convection coefficient h.

b. Compute the Biot number and discuss the accuracy of the lumped-parameter model used in part (a).

c. Discuss whether some of the heat loss rate could be due to radiation. Give a numerical reason for your answer.

7.57 A copper sphere is to be quenched in an oil bath whose temperature is 50°C. The sphere's radius is 30 mm, and the convection coefficient is $h = 300$ W/(m^2 · °C). Assume the sphere and the oil properties are constant. These properties are given in the following table. The sphere's initial temperature is 400°C.

Property	Sphere	Oil
Density ρ (kg/m^3)	8900	7900
Specific heat c_p J/(kg · °C)	385	400
Thermal conductivity k [W/(m · °C)]	400	—

Assume that the volume of the oil bath is large enough so that its temperature does not change when the sphere is immersed in the bath. Obtain the dynamic model of the sphere's temperature T. How long will it take for T to reach 130°C?

7.58 Consider the quenching process discussed in Problem 7.57. Suppose the oil bath volume is 0.1 m^3. Neglect any heat loss to the surroundings and develop a dynamic model of the sphere's temperature and the bath temperature. How long will it take for the sphere temperature to reach 130°C?

Section 7.9 MATLAB Applications

7.59 Consider Example 7.7.1. The MATLAB left division operator can be used to solve the set of linear algebraic equations $\mathbf{AT} = \mathbf{b}$ as follows: T = A\b. Use this method to write a script file to solve for the three steady-state temperatures T_1, T_2, and T_3, given values for the resistances and the temperatures T_i and T_o. Use the results of Example 7.7.1 to test your file.

Figure P7.60

(a)   (b)
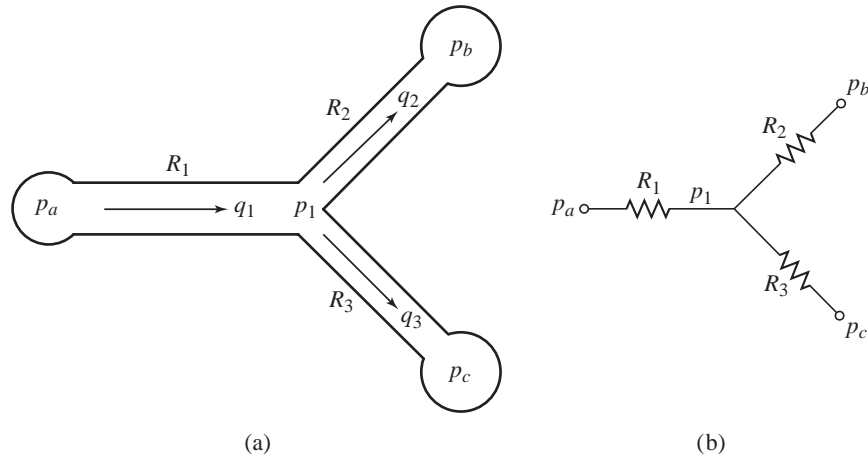
7.60 Fluid flows in pipe networks can be analyzed in a manner similar to that used for electric resistance networks. Figure P7.60a shows a network with three pipes, which is analogous to the electrical network shown in part (b) of the figure. The volume flow rates in the pipes are q_1, q_2, and q_3. The pressures at the pipe ends are p_a, p_b, and p_c. The pressure at the junction is p_1.

a. Assuming that the linear resistance relation applies, we have

$$q_1 = \frac{1}{R_1}(p_a - p_1)$$

Obtain the equations for q_2 and q_3.

b. Note that conservation of mass gives $q_1 = q_2 + q_3$. Set up the equations in a matrix form $\mathbf{Aq} = \mathbf{b}$ suitable for solving for the three flow rates q_1, q_2, and q_3, and the pressure p_1, given the values of the pressures p_a, p_b, and p_c, and the values of the resistances R_1, R_2, and R_3. Find the expressions for matrix $\mathbf{A}$ and the column vector $\mathbf{b}$.

c. Use MATLAB to solve the matrix equations obtained in part (b) for the case: $p_a = 30$ psi, $p_b = 25$ psi, and $p_c = 20$ psi. Use the resistance values $R_1 = 10{,}000$, $R_2 = R_3 = 14{,}000$ 1/(ft-sec). These values correspond to fuel oil flowing through pipes 2 ft long, with 2 in. and 1.4 in. diameters, respectively. The units of the answers should be ft^3/sec for the flow rates, and lb/ft^2 for pressure.

7.61 The equation describing the water height h in a spherical tank with a drain at the bottom is

$$\pi(2rh - h^2)\frac{dh}{dt} = -C_d A_o \sqrt{2gh}$$

Suppose the tank's radius is $r = 3$ m and that the circular drain hole has a radius of 2 cm. Assume that $C_d = 0.5$, and that the initial water height is $h(0) = 5$ m. Use $g = 9.81$ m/s^2.

a. Use an approximation to estimate how long it takes for the tank to empty.

b. Use MATLAB to solve the nonlinear equation and plot the water height as a function of time until $h(t)$ is not quite zero.

7.62 The following equation describes a certain dilution process, where $y(t)$ is the concentration of salt in a tank of fresh water to which salt brine is being added.

$$\frac{dy}{dt} + \frac{2}{10 + 2t} y = 4$$

Suppose that $y(0) = 0$.

a. Use MATLAB to solve this equation for $y(t)$ and to plot $y(t)$ for $0 \le t \le 10$.

b. Check your results by using an approximation that converts the differential equation into one having constant coefficients.

7.63 A tank having vertical sides and a bottom area of 100 ft^2 is used to store water. To fill the tank, water is pumped into the top at the rate given in the following table. Use MATLAB to solve for and plot the water height $h(t)$ for $0 \le t \le 10$ min.

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Flow Rate (ft³/min)	0	80	130	150	150	160	165	170	160	140	120

7.64 A cone-shaped paper drinking cup (like the kind used at water fountains) has a radius R and a height H. If the water height in the cup is h, the water volume is given by

$$V = \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3$$

Suppose that the cup's dimensions are $R = 1.5$ in. and $H = 4$ in.

a. If the flow rate from the fountain into the cup is 2 in.^{3}/sec, use MATLAB to determine how long will it take to fill the cup to the brim.

b. If the flow rate from the fountain into the cup is given by $2(1 - e^{-2t})$ in.^{3}/sec, use MATLAB to determine how long will it take to fill the cup to the brim.

Section 7.10 Simulink Applications

7.65 Refer to Figure 7.10.1. Assume that the resistances obey the linear relation, so that the mass flow q_l through the left-hand resistance is $q_l = (p_l - p)/R_l$, with a similar linear relation for the right-hand resistance.

a. Create a Simulink subsystem block for this element.

b. Use the subsystem block to create a Simulink model of the system discussed in Example 7.4.3 and shown in Figure 7.4.3a. Assume that the mass inflow rate q_{mi} is a step function.

c. Use the Simulink model to obtain plots of $h_1(t)$ and $h_2(t)$ for the following parameter values: $A_1 = 2$ m^2, $A_2 = 5$ m^2, $R_1 = 400$ 1/(m · s), $R_2 = 600$ 1/(m · s), $\rho = 1000$ kg/m^3, $q_{mi} = 50$ kg/s, $h_1(0) = 1.5$ m, and $h_2(0) = 0.5$ m.

7.66 Use Simulink to solve Problem 7.61(b).

7.67 Use Simulink to solve Problem 7.63.

7.68 Use Simulink to solve Problem 7.64. Plot $h(t)$ for both parts (a) and (b).

7.69 Refer to Example 7.10.1. Use the simulation with $q = 8 \times 10^5$ to compare the energy consumption and the thermostat cycling frequency for the two temperature bands (69°, 71°) and (68°, 72°).

7.70   Consider the liquid-level system shown in Figure 7.3.3. Suppose that the height h is controlled by using a relay to switch the flow rate q_{mi} between the values 0 and 50 kg/s. The flow rate is switched on when the height is less than 4.5 m and is switched off when the height reaches 5.5 m. Create a Simulink model for this application using the values $A = 2 \text{ m}^2$, $R = 400 \text{ 1/(m} \cdot \text{s)}$, $\rho = 1000 \text{ kg/m}^3$, and $h(0) = 1$ m. Obtain a plot of $h(t)$.

System Analysis in the Frequency Domain

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Sketch frequency response plots for a given transfer function, and use the plots or the transfer function to determine the steady-state response to a sinusoidal input.

2. Compute the frequencies at which resonance occurs, and determine a system's bandwidth.

3. Analyze vibration isolation systems and the effects of base motion and rotating unbalance.

4. Determine the steady-state response to a periodic input, given the Fourier series description of the input.

5. Estimate the form of a transfer function and its parameter values, given frequency response data.

6. Use MATLAB as an aid in the preceding tasks.

The term *frequency response* refers to how a system responds to a periodic input, such as a sinusoid. An input $f(t)$ is periodic with a period P if $f(t + P) = f(t)$ for all values of time t, where P is a constant called the *period*. Periodic inputs are commonly found in many applications. The most common perhaps is ac voltage, which is sinusoidal. For the common ac frequency of 60 Hz, the period is $P = 1/60$ s. Rotating unbalanced machinery produces periodic forces on the supporting structures, internal combustion engines produce a periodic torque, and reciprocating pumps produce hydraulic and pneumatic pressures that are periodic.

Frequency response analysis focuses on harmonic inputs, such as sines and cosines. A sine function has the form $A \sin \omega t$, where A is its *amplitude* and ω is its frequency in radians per unit time. Note that a cosine is simply a sine shifted by $90°$ or $\pi/2$ rad, as

$\cos \omega t = \sin(\omega t + \pi/2)$. Not all periodic inputs are sinusoidal or cosinusoidal, but an important result of mathematics, called the *Fourier series,* enables us to represent a periodic function as the sum of a constant term plus a series of cosine and sine terms of different frequencies and amplitudes. Thus we will be able to apply the results of this chapter to any such periodic input.

The transfer function is central to understanding and applying frequency response methods. The transfer function enables us to obtain a concise graphical description of a system's frequency response. This graphical description is called a *frequency response plot*. We analyze the response of first-order systems to sinusoidal inputs in Section 8.1. In Section 8.2 we generalize the method to higher-order systems. Section 8.3 discusses several phenomena and applications of frequency response, including *beating, resonance,* the effect of base motion and rotating unbalance, and instrument design.

In Section 8.4 we introduce the important concept of *bandwidth,* which enables us to develop a concise quantitative description of a system's frequency response, much like the time constant characterizes the step response. This section also shows how to analyze the response due to general periodic inputs that can be described with a Fourier series.

Experiments involving frequency response can often be used to determine the form of the transfer function of a system and to estimate the numerical values of the parameters in the transfer function. Section 8.5 gives some examples of this process. MATLAB has several useful functions for obtaining and for analyzing frequency response. These are treated in Section 8.6. ■

8.1 FREQUENCY RESPONSE OF FIRST-ORDER SYSTEMS

In this chapter we will frequently need to obtain the magnitude and angle of complex numbers. A complex number N can be represented in *rectangular* form as $N = x + jy$, where x is the real part and y is the imaginary part. The number can be plotted in two dimensions with x as the abscissa (on the horizontal axis) and y as the ordinate (on the vertical axis). We can think of the number as a two-dimensional vector whose head is at the point (x, y) and whose tail is at the origin. The vector length is the magnitude M of the number and the vector angle ϕ is measured counterclockwise from the positive real axis. In this form, the magnitude and angle of the number can be calculated from $|N| = \sqrt{x^2 + y^2}$. The complex conjugate of N is $x - yj$. See Figure 8.1.1a.

Another form is the *complex exponential* form:

$$N = Me^{j\phi} = M(\cos \phi + j \sin \phi)$$

Note that the complex conjugate of N is $Me^{-j\phi}$.

PRODUCTS AND RATIOS OF COMPLEX NUMBERS

The complex exponential form can be used to show that the magnitude and angle of a number consisting of products and ratios of complex numbers $N_1, N_2, \ldots,$ can be calculated as follows.

$$N = \frac{N_1 N_2}{N_3 N_4} = \frac{M_1 e^{j\phi_1} M_2 e^{j\phi_2}}{M_3 e^{j\phi_3} M_4 e^{j\phi_4}} = \frac{M_1 M_2}{M_3 M_4} e^{j(\phi_1 + \phi_2 - \phi_3 - \phi_4)}$$

Thus $N = |N|e^{j\phi}$, where

$$|N| = \frac{M_1 M_2}{M_3 M_4} \qquad \phi = \phi_1 + \phi_2 - \phi_3 - \phi_4$$

So the magnitudes combine as products and ratios, and the angles combine as sums and differences.

For example, consider the number

$$N = \frac{-2 - 5j}{3 + 4j}$$

The magnitude of N can be calculated by computing the ratio of the magnitudes of the numerator and denominator, as follows:

$$|N| = \left| \frac{-2 - 5j}{3 + 4j} \right| = \frac{|-2 - 5j|}{|3 + 4j|} = \frac{\sqrt{(-2)^2 + (-5)^2}}{\sqrt{3^2 + 4^2}} = \frac{\sqrt{29}}{5}$$

The angle of N, denoted by $\angle N$, is the difference between the angle of the numerator and the angle of the denominator. These angles are shown in Figure 8.1.1b, which is a vector representation of the complex numbers in the numerator and denominator. From this diagram we can see that the angles are given by

$$\angle(-2 - 5j) = 180° + \tan^{-1} \frac{5}{2} = 180° + 68° = 248°$$

$$\angle(3 + 4j) = \tan^{-1} \frac{4}{3} = 53°$$

Thus

$$\angle N = 248° - 53° = 195°$$

and

$$N = \frac{\sqrt{29}}{5} \angle 195° = \frac{\sqrt{29}}{5}(\cos 195° + j \sin 195°) = -1.04 + 0.28j$$
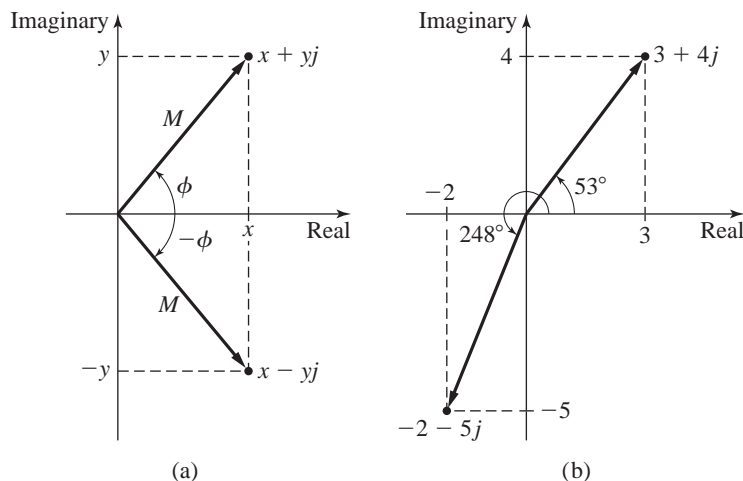
Figure 8.1.1 Vector representation of complex numbers.

COMPLEX NUMBERS AS FUNCTIONS OF FREQUENCY

In our applications in this chapter, complex numbers will be functions of a frequency ω, as for example,

$$N = \frac{-2 - j5\omega}{3 + j4\omega}$$

but the same methods apply for obtaining the magnitude and the angle of N, which will then be functions of ω. Thus,

$$|N| = \frac{\sqrt{2^2 + 5^2\omega^2}}{\sqrt{3^2 + 4^2\omega^2}} = \frac{\sqrt{4 + 25\omega^2}}{\sqrt{9 + 16\omega^2}}$$

$$\phi = \angle N = \angle(-2 - j5\omega) - \angle(3 + j4\omega)$$
$$= 180° + \tan^{-1}\left(\frac{5\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)$$

Once a value is given for ω, we can compute $|N|$ and ϕ.

FREQUENCY RESPONSE PROPERTIES

The methods of this chapter pertain to *any stable, linear, time-invariant (LTI)* system. The basic frequency response property of such systems is summarized in Table 8.1.1 and Figure 8.1.2. Any linear, time-invariant system, stable or not, has a transfer function, say $T(s)$. If a sinusoidal input of frequency ω is applied to such a system, and if the system is *stable,* the transient response eventually disappears, leaving a steady-state response that is sinusoidal with the same frequency ω as the input, but with a different amplitude and shifted in time relative to the input.

To prove this result, suppose the system transfer function $T(s)$ is of order n, the output is $x(t)$, and the input is $f(t) = A \sin \omega t$. Then

$$X(s) = F(s)T(s) = \frac{A\omega}{s^2 + \omega^2}T(s)$$

Table 8.1.1 Frequency response of a stable LTI system.

The transfer function $T(s)$ with s replaced by $j\omega$ is called the *frequency transfer function*. Therefore, the frequency transfer function is a complex function of ω, and it has a magnitude and an angle, just as any complex number. If the system is *stable,* the magnitude M of the frequency transfer function $T(j\omega)$ is the ratio of the sinusoidal steady-state output amplitude over a sinusoidal input amplitude. The phase shift of the steady-state output relative to the input is the angle of $T(j\omega)$. Thus, denoting the input by $A \sin \omega t$ and the steady-state output by $B \sin(\omega t + \phi)$, we have

$$|T(j\omega)| = \frac{B}{A} \equiv M(\omega) \tag{1}$$

$$\phi(\omega) = \angle T(j\omega) \tag{2}$$

$$B = AM(\omega) \tag{3}$$

where $|T(j\omega)|$ and $\angle T(j\omega)$ denote the magnitude and angle of the complex number $T(j\omega)$. Because of equation (1), M is called the *amplitude ratio* or the *magnitude ratio*.

These results are illustrated in Figure 8.1.2.

Figure 8.1.2 Frequency response of a stable linear system.

Sinusoidal input	Stable, linear system	Steady-state response
$A \sin \omega t$	$T(s)$	$B \sin(\omega t + \phi)$

$$M = \frac{B}{A} = |T(j\omega)|$$

$$\phi = \measuredangle T(j\omega)$$
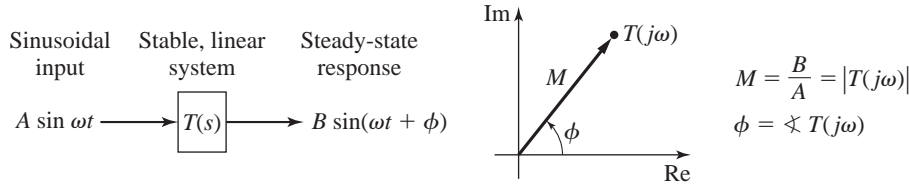
which can be expressed as a partial-fraction expansion.

$$\frac{A\omega}{s^2 + \omega^2} T(s) = \frac{C_1}{s + j\omega} + \frac{C_2}{s - j\omega} + \cdots$$

The factors containing $s + j\omega$ and $s - j\omega$ correspond to the term $s^2 + \omega^2$, while the remaining terms in the expansion correspond to the factors introduced by the denominator of $T(s)$. If the system is *stable,* all these factors will be negative or have negative real parts. Thus the response will have the form

$$x(t) = C_1 e^{-j\omega t} + C_2 e^{j\omega t} + \sum_{i=1}^{m} D_i e^{-\alpha_i t} \sin(\beta_i t + \phi_i) + \sum_{i=m+1}^{n} D_i e^{-\alpha_i t}$$

where the roots of $T(s)$ are $s_i = -\alpha_i \pm \beta_i j$ for $i = 1, \ldots, m$ and $s_i = -\alpha_i$ for $i = 1 + m, \ldots, n$. The steady-state response is thus given by

$$x_{ss}(t) = C_1 e^{-j\omega t} + C_2 e^{j\omega t}$$

We can evaluate C_1 and C_2 as follows. Note that $T(j\omega) = |T(j\omega)|e^{j\phi}$, where $\phi = \angle T(j\omega)$.

$$C_1 = T(s)\frac{A\omega}{s^2 + \omega^2}(s + j\omega)\bigg|_{s=-j\omega} = -\frac{A}{2j}T(-j\omega) = -\frac{A}{2j}|T(j\omega)|e^{-j\phi}$$

$$C_2 = T(s)\frac{A\omega}{s^2 + \omega^2}(s - j\omega)\bigg|_{s=j\omega} = \frac{A}{2j}T(j\omega) = \frac{A}{2j}|T(j\omega)|e^{j\phi}$$

The steady-state response can thus be expressed as

$$x_{ss}(t) = -\frac{A}{2j}|T(j\omega)|e^{-j\phi}e^{-j\omega t} + \frac{A}{2j}|T(j\omega)|e^{j\phi}e^{j\omega t}$$

$$= |T(j\omega)|A\frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j}$$

or

$$x_{ss}(t) = |T(j\omega)|A \sin(\omega t + \phi)$$

because

$$\frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} = \sin(\omega t + \phi)$$

Thus at steady state, the output amplitude is $|T(j\omega)|A$ and the phase shift is $\phi = \angle T(j\omega)$.

A numerical example will help to clarify these concepts.

FREQUENCY RESPONSE OF $\tau \dot{y} + y = f(t)$

We can generalize the results of Example 8.1.1 as follows. Consider the linear model having a time constant τ.

$$\tau \dot{y} + y = f(t) \tag{8.1.1}$$

The transfer function is

$$T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s + 1} \tag{8.1.2}$$

If the input is sinusoidal, $f(t) = A \sin \omega t$, the response is

$$y(t) = \frac{A\omega\tau}{1 + \omega^2 \tau^2} \left(e^{-t/\tau} - \cos \omega t + \frac{1}{\omega\tau} \sin \omega t \right) \tag{8.1.3}$$

For $t \geq 4\tau$, the transient response has essentially disappeared, and we can express the steady-state response as

$$y_{ss}(t) = \frac{A}{1 + \omega^2 \tau^2} (\sin \omega t - \omega\tau \cos \omega t) = B \sin(\omega t + \phi) \tag{8.1.4}$$

where we have used the identity

$$B \sin(\omega t + \phi) = B \cos \phi \sin \omega t + B \sin \phi \cos \omega t$$

Comparing this with (8.1.3), and noting that A, τ, and ω are positive, we see that

$$B \cos \phi = \frac{A}{1 + \omega^2 \tau^2} > 0$$

$$B \sin \phi = -\frac{A\omega\tau}{1 + \omega^2 \tau^2} < 0$$

Using the identity, $\cos^2 \phi + \sin^2 \phi = 1$, we obtain

$$B^2 = (B \cos \phi)^2 + (B \sin \phi)^2 = \left(\frac{A\omega\tau}{1 + \omega^2 \tau^2} \right)^2 + \left(\frac{A}{1 + \omega^2 \tau^2} \right)^2$$

or

$$B^2 = \frac{A^2}{1 + \omega^2 \tau^2}$$

These equations allow for two solutions for B and ϕ. One solution corresponds to $B > 0$ and the other to $B < 0$. Either solution is acceptable, but if we think of B as an amplitude of oscillation, then the $B > 0$ solution is more appropriate. If we choose the solution where $B > 0$, then $\cos \phi > 0$ and $\sin \phi < 0$, which means that ϕ is in the fourth quadrant. Thus,

$$B = +\frac{A}{\sqrt{1 + \omega^2 \tau^2}}$$

and the amplitude ratio $M = B/A$ is

$$M = \frac{B}{A} = \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \tag{8.1.5}$$

The phase angle is

$$\phi = \tan^{-1} \frac{\sin \phi}{\cos \phi} = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau) \tag{8.1.6}$$

Table 8.1.2 Frequency response of the model $\tau \dot{y} + y = f(t)$.

$$M = \frac{|Y|}{|F|} = \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \tag{1}$$

$$\phi = -\tan^{-1}(\omega\tau) \tag{2}$$

Thus the steady-state response is sinusoidal with the same frequency ω as the input, and its amplitude decreases as ω increases. The response is retarded in phase relative to the input, and this retardation increases with the frequency ω.

RESULTS FROM THE TRANSFER FUNCTION

If we substitute $s = j\omega$ into the transfer function (8.1.2), we obtain

$$T(j\omega) = \frac{1}{1 + j\omega\tau}$$

The magnitude is

$$|T(j\omega)| = \left| \frac{1}{1 + j\omega\tau} \right| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

and the angle is

$$\phi = \angle T(j\omega) = -\angle(1 + j\omega\tau) = -\tan^{-1}(\omega\tau)$$

These results are summarized in Table 8.1.2 and are identical to (8.1.5) and (8.1.6), thus proving the results of Table 8.1.1 for the model $\tau \dot{y} + y = f(t)$.

Frequency Response of a Mass with Damping | **EXAMPLE 8.1.1**

■ **Problem**

Consider a mass subjected to a sinusoidal applied force $f(t)$. The mass is $m = 0.2$ kg and the damping constant is $c = 1$ N·s/m. If v is the speed of the mass, then the equation of motion is $0.2\dot{v} + v = f(t)$ where $f(t) = \sin \omega t$ and ω is the oscillation frequency of the applied force. The initial speed is $v(0) = 0$. Find the total response for two cases: (a) $\omega = 15$ rad/s and (b) $\omega = 60$ rad/s.

■ **Solution**

This equation is of the form of (8.1.1) with $\tau = 0.2$ and $y = v$. From (8.1.3) the response is

$$v(t) = \frac{0.2\omega}{1 + (0.2)^2\omega^2} \left(e^{-5t} - \cos \omega t + \frac{1}{0.2\omega} \sin \omega t \right)$$

The transient response contains the exponential e^{-5t}, which is essentially zero for $t > 4/5$ s. The steady-state response is

$$v_{ss}(t) = \frac{0.2\omega}{1 + (0.2)^2\omega^2} \left(-\cos \omega t + \frac{1}{0.2\omega} \sin \omega t \right) = \frac{1}{\sqrt{1 + (0.2)^2\omega^2}} \sin(\omega t + \phi)$$

where from (8.1.6), $\phi = -\tan^{-1}(0.2\omega)$.

The steady-state response can be seen to oscillate at the input frequency ω with an amplitude of $1/\sqrt{1 + 0.04\omega^2}$, and a phase shift of ϕ relative to the input. Since the phase shift is negative,

Figure 8.1.3 Response of the model $0.2\dot{v} + v = \sin\omega t$: (a) $\omega = 15$ and (b) $\omega = 60$.

(a)

(b)
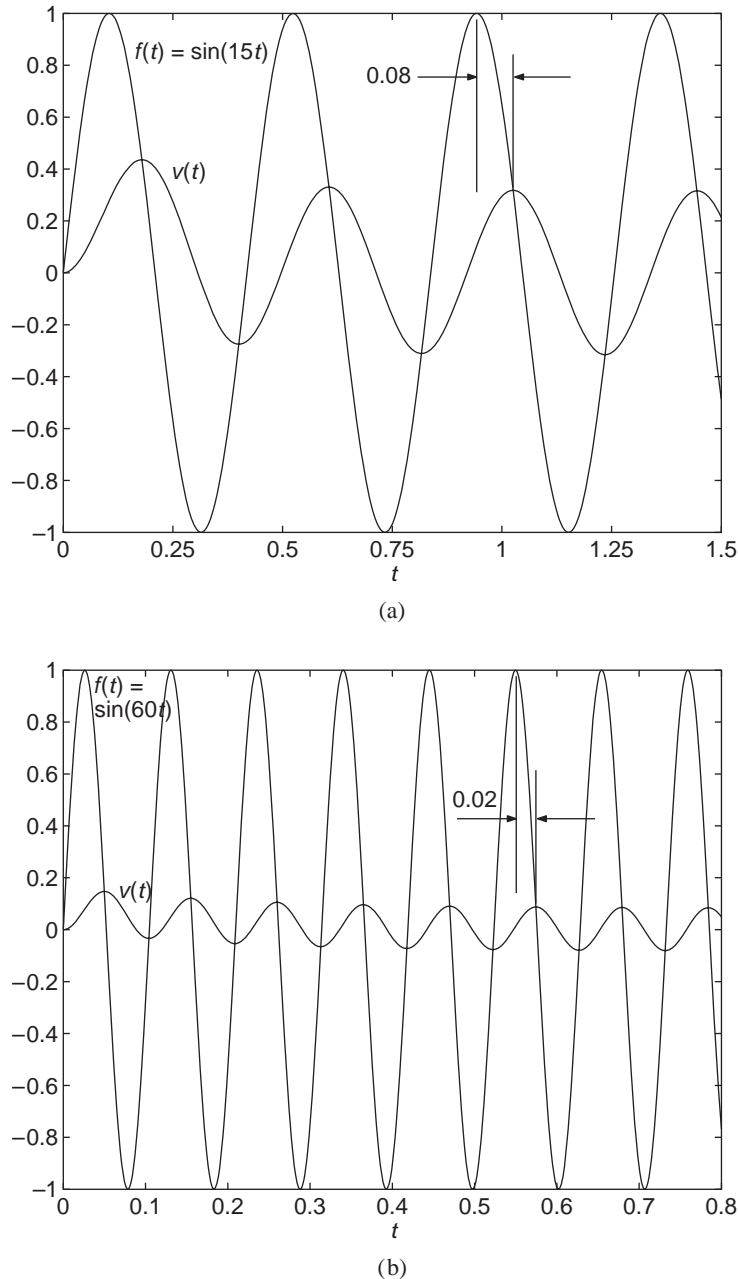
a peak in the input is followed by a peak in the response a time $|\phi|/\omega$ later. If ϕ is in radians and ω is in radians per second, the time shift $|\phi|/\omega$ will be in seconds.

Figure 8.1.3(a) shows the total response for the case where $\omega = 15$. Part (b) shows the case where $\omega = 60$. Note that the transient response has in both cases essentially disappeared by $t = 4/5$ as predicted. Note also that the amplitude of the response is much smaller at the higher frequency. This is because the inertia of the mass prevents it from reacting to a rapidly changing input. Although the time shift is also smaller at the higher frequency, this does not indicate a faster response, because the peak attained is much smaller.

| Circuit Response to a Step-Plus-Cosine Input | **EXAMPLE 8.1.2** |

■ Problem

The model of the voltage v_2 across the capacitor in a series RC circuit having an input voltage v_1 is

$$RC\frac{dv_2}{dt} + v_2 = v_1(t)$$

Suppose that $RC = 0.02$ s and that the applied voltage consists of a step function plus a cosine function: $v_1(t) = 5u_s(t) + 3\cos 80t$. Obtain the circuit's steady-state response.

■ Solution

We can apply the principle of superposition here to separate the effects of the step input from those of the cosine input. Note that if only a step voltage of 5 were applied, the steady-state response to the step would be 5 (this can be shown by setting $dv_2/dt = 0$ in the model to see that $v_2 = v_1$ at steady state). Next find the steady-state response due to the *cosine* function $3\cos 80t$. To do this, we can use the same formulas developed for the amplitude and phase shift due to a sine function, but express the response in terms of a cosine function. From (8.1.5) and (8.1.6) with $A = 3$,

$$B = AM = \frac{3}{\sqrt{1 + (80)^2(0.02)^2}} = 1.59$$

$$\phi = -\tan^{-1}[(80)(0.02)] = -\tan^{-1}(1.6) = -1.012 \text{ rad}$$

The steady-state response is $v_2(t) = 5 + 1.59\cos(80t - 1.012)$. Thus at steady state, the voltage oscillates about the mean value 5 V with an amplitude of 1.59 and a frequency of 80 rad/s.

THE LOGARITHMIC PLOTS

Inspecting (8.1.5) reveals that the steady-state amplitude of the output decreases as the frequency of the input increases. The larger τ is, the faster the output amplitude decreases with frequency. At a high frequency, the system's "inertia" prevents it from closely following the input. The larger τ is, the more sluggish is the system response. This also produces an increasing phase lag as ω increases. Figure 8.1.4 shows the magnitude ratio and phase curves for two values of τ. In both cases, the magnitude ratio is close to 1 at low frequencies and approaches 0 as the frequency increases. The rate of decrease is greater for systems having larger time constants. The phase angle is close to 0 at low frequencies and approaches $-90°$ as the frequency increases.

Logarithmic scales are usually used to plot the frequency response curves. There are two reasons for using logarithmic scales. The curves of M and ϕ versus ω can be sketched more easily if logarithmic axes are used because they enable us to add or subtract the magnitude plots of simple transfer functions to sketch the plot for a transfer function composed of the product and ratio of simpler ones. Logarithmic scales also can display a wider variation in numerical values. When plotted using logarithmic scales, the frequency response plots are frequently called *Bode* plots, after H. W. Bode, who applied these techniques to the design of amplifiers.

Keep in mind the following basic properties of logarithms:

$$\log(xy) = \log x + \log y \qquad \log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log x^n = n\log x$$

Figure 8.1.4 Magnitude ratio and phase angle of the model $\tau \dot{y} + y = f(t)$ for $\tau = 2$ and $\tau = 20$.
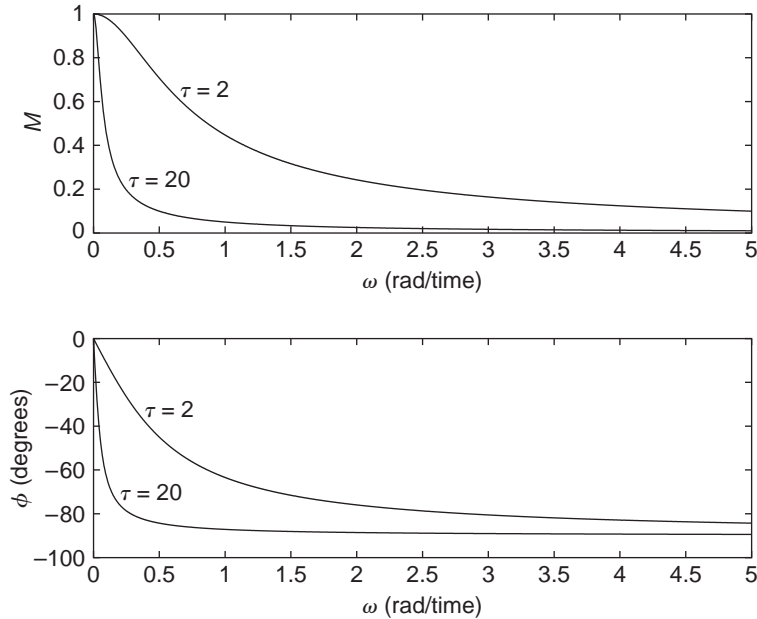
When using logarithmic scales the amplitude ratio M is specified in decibel units, denoted dB. (The decibel is named for Alexander Graham Bell.) The relationship between a number M and its decibel equivalent m is

$$m = 10 \log M^2 = 20 \log M \text{ dB} \tag{8.1.7}$$

where the logarithm is to the base 10. For example, the $M = 10$ corresponds to 20 dB; the $M = 1$ corresponds to 0 dB; numbers less than 1 have negative decibel values. It is common practice to plot $m(\omega)$ in decibels versus $\log \omega$. For easy reference, $\phi(\omega)$ is also plotted versus $\log \omega$, and it is common practice to plot ϕ in degrees.

LOGARITHMIC PLOT FOR $\tau \dot{y} + y = f(t)$

From equation (1) of Table 8.1.2,

$$M = \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \tag{8.1.8}$$

So we have

$$\begin{aligned}
m(\omega) &= 20 \log \frac{1}{\sqrt{1 + \tau^2 \omega^2}} \\
&= 20 \log(1) - 10 \log(1 + \tau^2 \omega^2) \\
&= -10 \log(1 + \tau^2 \omega^2) \tag{8.1.9}
\end{aligned}$$

Figure 8.1.5 shows the logarithmic plots for the two systems whose plots with rectilinear axes were given in Figure 8.1.4. Note that the shape of the m versus $\log \omega$ curve is very different from the shape of the M versus ω curve. The logarithmic plots can be confusing for beginners; take time to study them. Remember that $m = 0$ corresponds to a magnitude ratio of $M = 1$. Positive values of m correspond to $M > 1$, which means that the system *amplifies* the input. Negative values of m correspond to $M < 1$, which

means the system *attenuates* the input. When you need to obtain values of M from a plot of m versus $\log \omega$, use the relation

$$M = 10^{m/20} \qquad (8.1.10)$$

For example, $m = 50$ corresponds to $M = 316.2$; $m = -15$ corresponds to $M = 0.1778$; and $m = -3.01$ corresponds to $M = 1/\sqrt{2} = 0.7071$.

To sketch the logarithmic plot of m versus $\log \omega$, we approximate $m(\omega)$ in three frequency ranges. For $\tau \omega \ll 1$, $1 + \tau^2 \omega^2 \approx 1$, and (8.1.9) gives

$$m(\omega) \approx -10 \log 1 = 0 \qquad (8.1.11)$$

For $\tau \omega \gg 1$, $1 + \tau^2 \omega^2 \approx \tau^2 \omega^2$, and (8.1.9) gives

$$m(\omega) \approx -10 \log \tau^2 \omega^2 = -20 \log \tau \omega = -20 \log \tau - 20 \log \omega \qquad (8.1.12)$$

This gives a straight line versus $\log \omega$. This line is the high-frequency asymptote. Its slope is -20 dB/decade, where a *decade* is any $10:1$ frequency range. At $\omega = 1/\tau$, (8.1.12) gives $m(\omega) = 0$. This is useful for plotting purposes but does not represent the true value of m at that point because (8.1.12) was derived assuming $\tau \omega \gg 1$. For $\omega = 1/\tau$, (8.1.9) gives $m(\omega) = -10 \log 2 = -3.01$. Thus, at $\omega = 1/\tau$, $m(\omega)$ is 3.01 dB below the low-frequency asymptote given by (8.1.11). The low-frequency and high-frequency asymptotes meet at $\omega = 1/\tau$, which is the *breakpoint frequency*. It is also called the *corner frequency*. The plot is shown in Figure 8.1.6 (upper plot).

The phase angle is given by $\phi = -\tan^{-1}(\omega \tau)$, and the curve of ϕ versus ω is constructed as follows. For $\omega \tau \ll 1$, this equation gives $\phi(\omega) \approx \tan^{-1}(0) = 0°$. For $\omega \tau = 1$, $\phi(\omega) = -\tan^{-1}(1) = -45°$, and for $\omega \tau \gg 1$, $\phi(\omega) \approx -\tan^{-1}(\infty) = -90°$. Because the phase angle is negative, the output "lags" behind the input sine wave. Such a system is called a *lag* system. The $\phi(\omega)$ curves are easily sketched using these facts and are shown in Figure 8.1.6 (lower plot).

Figure 8.1.6 Asymptotes and corner frequency $\omega = 1/\tau$ of the model $1/(\tau s + 1)$.
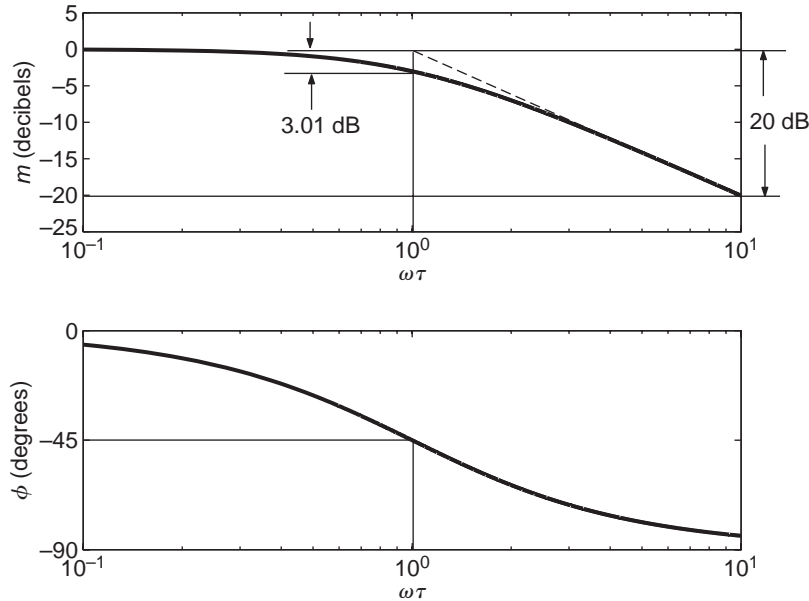
EXAMPLE 8.1.3 | A Low-Pass Filter

■ **Problem**

A series RC circuit is shown in Figure 8.1.7, where the input is the voltage v_s and the output is the voltage v_o. The two amplifiers serve to isolate the circuit from the loading effects of adjacent elements. Describe its frequency response characteristics.

■ **Solution**

The circuit model obtained in Chapter 3 has the following transfer function.

$$T(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

Comparing this with the transfer function given by (8.1.2), $1/(\tau s + 1)$, we see that $\tau = RC$. Thus the breakpoint frequency is $1/\tau = 1/RC$. This means that the circuit filters out sinusoidal voltage inputs whose frequency is higher than $1/RC$, and the higher the frequency, the greater is the filtering. Inputs having frequencies lower than $1/RC$ pass through the circuit with almost no loss of amplitude and with little phase shift. This is because $T(j\omega) \approx 1$ and $\phi \approx 0$ for $\omega < 1/RC$. The circuit is called a *low-pass filter* for this reason. It can be used to filter out unwanted high-frequency components of the input voltage, such as 60-Hz interference from nearby ac equipment.

Figure 8.1.7 Series RC circuit configured as a low-pass filter.
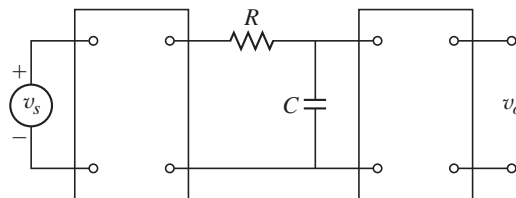
A High-Pass Filter | **EXAMPLE 8.1.4**

■ Problem

Consider the series RC circuit shown in Figure 8.1.8, where the output voltage is taken to be across the resistor. The input is the voltage v_s and the output is the voltage v_o. Obtain the circuit's frequency response plots, and interpret the circuit's effect on the input.

■ Solution

The input impedance between the voltage v_s and the current i is found from the series law.

$$\frac{V_s(s)}{I(s)} = R + \frac{1}{Cs}$$

Thus,

$$V_o(s) = I(s)R = \frac{V_s(s)}{R + 1/Cs}R = \frac{RCs}{RCs + 1}V_s(s)$$

The transfer function is

$$T(s) = \frac{V_o(s)}{V_s(s)} = \frac{RCs}{RCs + 1} = \frac{\tau s}{\tau s + 1}$$

where $\tau = RC$.

The frequency transfer function and magnitude ratio are

$$T(j\omega) = \frac{\tau\omega j}{1 + \tau\omega j}$$

$$M = \left|\frac{\tau\omega j}{1 + \tau\omega j}\right| = \frac{\tau\omega}{\sqrt{1 + (\tau\omega)^2}}$$

$$m(\omega) = 20\log|\tau| + 20\log|j\omega| - 20\log|1 + \tau\omega j|$$
$$= 20\log|\tau| + 20\log\omega - 20\log\sqrt{1 + (\tau\omega)^2}$$
$$= 20\log|\tau| + 20\log\omega - 10\log\left[1 + (\tau\omega)^2\right]$$

At frequencies where $\omega \ll 1/\tau$, the last term on the right is negligible, and thus the low-frequency asymptote is described by

$$m(\omega) = 20\log|\tau| + 20\log\omega, \qquad \omega \ll \frac{1}{\tau}$$

It can be sketched by noting that it has a slope of 20 dB/decade and it passes through the point $m = 0$, $\omega = 1/\tau$. The asymptote is shown by the dashed line in Figure 8.1.9 (upper plot). At high frequencies where $\omega \gg 1/\tau$, the slope of -20 due to the term $\tau s + 1$ in the denominator
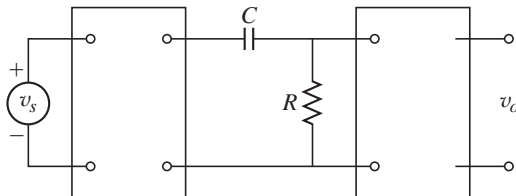
Figure 8.1.8 Series RC circuit configured as a high-pass filter.

Figure 8.1.9 Asymptotes and corner frequency $\omega = 1/\tau$ of the model $\tau s/(\tau s + 1)$.
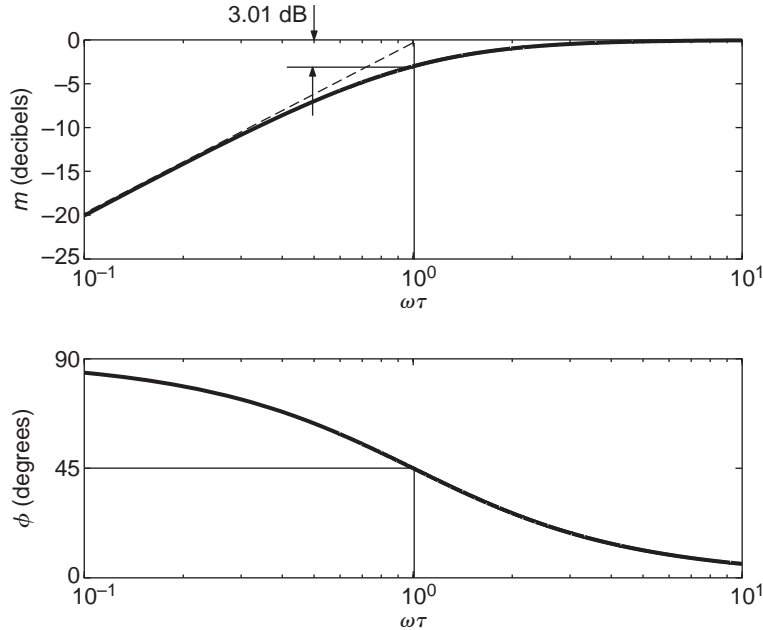
cancels the slope of $+20$ due to the term s in the numerator. Therefore, at high frequencies, $m(\omega)$ has a slope of approximately zero, and

$$m(\omega) \approx 20 \log |\tau| + 20 \log \omega - 20 \log |\tau \omega|$$

$$= 20 \log |\tau| + 20 \log \omega - 20 \log |\tau| - 20 \log \omega = 0 \quad \text{for } \omega \gg \frac{1}{\tau}$$

At $\omega = 1/\tau$, the denominator term $\tau s + 1$ contributes -3 dB. The composite curve for $m(\omega)$ is obtained by "blending" the low-frequency and high-frequency asymptotes through this point, as shown in Figure 8.1.9 (upper plot). A similar technique can be used to sketch $\phi(\omega)$.

The phase angle is

$$\phi(\omega) = \angle \tau + \angle(j\omega) - \angle(1 + \tau\omega j) = 0° + 90° - \tan^{-1}(\omega\tau)$$

For $\omega \ll 1/\tau$, the low-frequency asymptote is $\phi(\omega) \approx 90° - \tan^{-1}(0) = 90°$. For $\omega = 1/\tau$, $\phi(\omega) = 90° - \tan^{-1}(1) = 45°$. For $\omega \gg 1/\tau$, the high-frequency asymptote is $\phi(\omega) \approx 90° - \tan^{-1}(\infty) = 90° - 90° = 0°$. The result is sketched in Figure 8.1.9 (lower plot).

The log magnitude plot shows that the circuit passes signals with frequencies above $\omega = 1/\tau$ with little attenuation, and thus it is called a *high-pass* filter. It is used to remove dc and low-frequency components from a signal. This is desirable when we wish to study high-frequency components whose small amplitudes would be indiscernible in the presence of large-amplitude, low-frequency components. Such a circuit is incorporated into oscilloscopes for this reason. The voltage scale can then be selected so that the signal components to be studied will fill the screen.

EXAMPLE 8.1.5

Frequency Response of a Differentiating Circuit

■ **Problem**
The op-amp differentiator analyzed in Chapter 6 and shown in Figure 8.1.10a has the transfer function

$$\frac{V_o(s)}{V_s(s)} = -RCs$$
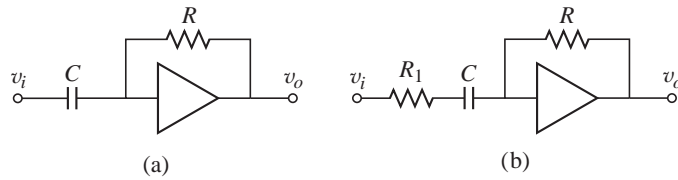
Figure 8.1.10 (a) Op-amp differentiator. (b) Modified op-amp differentiator.

It is susceptible to high-frequency noise, because the derivative of a rapidly changing signal is difficult to compute and produces an exaggerated output. In practice, this problem is often solved by filtering out high-frequency signals either with a low-pass filter inserted in series with the differentiator or by using a redesigned differentiator, such as the one shown in Figure 8.1.10b. Its transfer function is

$$\frac{V_o(s)}{V_s(s)} = -\frac{RCs}{R_1Cs + 1}$$

Analyze its frequency response characteristics.

■ **Solution**

Note that if a circuit is a pure differentiator, then $v_o = \dot{v}_s$, or

$$\frac{V_o(s)}{V_s(s)} = s$$

So the slope of $m(\omega)$ curve for a pure differentiator is 20 dB/decade. The transfer function of the redesigned differentiator can be rearranged as follows:

$$\frac{V_o(s)}{V_s(s)} = -\frac{RCs}{R_1Cs + 1} = -\frac{R}{R_1}\frac{R_1Cs}{R_1Cs + 1}$$

Thus, the frequency response plot of this transfer function is similar to that of the high-pass filter shown in Figure 8.1.9 (upper plot) with $\tau = R_1C$, except for the factor $-R/R_1$. We will neglect the factor -1, which can be eliminated by using a series inverter. The circuit's corner frequency is $\omega_c = 1/R_1C$, and the circuit acts like the ideal differentiator for frequencies up to about $\omega = 1/R_1C$. For higher frequencies, the magnitude ratio curve has zero slope rather than the 20 dB/decade slope required for differentiation, and thus the circuit does not differentiate the high-frequency signals, but merely amplifies them with a gain of R/R_1. Because the amplitudes of noisy high-frequency signals are generally small, this amplification effect is negligible. For $\omega < 1/R_1C$, the circuit's gain is RC. Thus, after choosing a convenient value for C, the two resistors are selected as follows: R_1 is used to set the cutoff frequency $1/R_1C$, and R is used to set the gain RC.

A COMMON FORM HAVING NUMERATOR DYNAMICS

Another example of numerator dynamics is the transfer function

$$T(s) = K\frac{\tau_1s + 1}{\tau_2s + 1} \tag{8.1.13}$$

An example of this form is the transfer function of the lead compensator circuit analyzed in Chapter 6, and shown again in Figure 8.1.11a. Its transfer function is

$$T(s) = \frac{V_o(s)}{V_s(s)} = \frac{R_1R_2Cs + R_2}{R_1R_2Cs + R_1 + R_2} \tag{8.1.14}$$

(a)   (b)
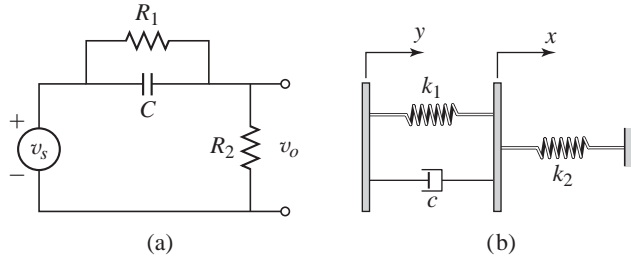
which can be rearranged as (8.1.13) where

$$\tau_1 = R_1 C \qquad \tau_2 = \frac{R_1 R_2 C}{R_1 + R_2} \qquad K = \frac{\tau_2}{\tau_1} \tag{8.1.15}$$

A mechanical example of this form is shown in Figure 8.1.11b. The equation of motion is

$$c\dot{x} + (k_1 + k_2)x = c\dot{y} + k_1 y$$

With y as the input, the transfer function is

$$T(s) = \frac{X(s)}{Y(s)} = \frac{cs + k_1}{cs + k_1 + k_2} \tag{8.1.16}$$

which can be rearranged as (8.1.13), where

$$\tau_1 = \frac{c}{k_1} \qquad \tau_2 = \frac{c}{k_1 + k_2} \qquad K = \frac{\tau_2}{\tau_1} \tag{8.1.17}$$

EXAMPLE 8.1.6

A Model with Numerator Dynamics

■ Problem
Find the steady-state response of the following system:

$$\dot{y} + 5y = 4\dot{g} + 12g$$

if the input is $g(t) = 20 \sin 4t$.

■ Solution
First obtain the transfer function.

$$T(s) = \frac{Y(s)}{G(s)} = \frac{4s + 12}{s + 5} = 4\frac{s + 3}{s + 5}$$

Here $\omega = 4$, so we substitute $s = 4j$ to obtain

$$T(j\omega) = 4\frac{3 + j\omega}{5 + j\omega} = 4\frac{3 + 4j}{5 + 4j}$$

Then,

$$M = |T(j\omega)| = 4\frac{|3 + 4j|}{|5 + 4j|} = 4\frac{\sqrt{3^2 + 4^2}}{\sqrt{5^2 + 4^2}} = 3.123$$

The phase angle is found as follows:

$$\phi = \angle T(j\omega) = \angle \left(4\frac{3+j\omega}{5+j\omega} \right) = \angle 4 + \angle(3+j\omega) - \angle(5+j\omega)$$

$$= 0° + \tan^{-1}\frac{\omega}{3} - \tan^{-1}\frac{\omega}{5}$$

Substitute $\omega = 4$ to obtain

$$\phi = \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{4}{5} = 0.253 \text{ rad}$$

Thus the steady-state response is

$$y_{ss}(t) = 20M \sin(4t + \phi) = 62.46 \sin(4t + 0.253)$$

Substituting $s = j\omega$ into the transfer function (8.1.13) gives

$$T(j\omega) = K\frac{\tau_1\omega j + 1}{\tau_2\omega j + 1} \tag{8.1.18}$$

Thus,

$$M(\omega) = |K|\frac{\sqrt{(\tau_1\omega)^2 + 1}}{\sqrt{(\tau_2\omega)^2 + 1}}$$

$$m(\omega) = 20\log|K| + 10\log[(\tau_1\omega)^2 + 1] - 10\log[(\tau_2\omega)^2 + 1] \tag{8.1.19}$$

Thus, the plot of $m(\omega)$ can be obtained by subtracting the plot of $\tau_2 s + 1$ from that of $\tau_1 s + 1$. The scale is then adjusted by $20\log|K|$. The sketches in Figure 8.1.12 are for $K = 1$, so that $20\log K = 0$.

The term $\tau_1 s + 1$ causes the curve to break upward at $\omega = 1/\tau_1$. The term $\tau_2 s + 1$ causes the curve to break downward at $\omega = 1/\tau_2$. If $1/\tau_1 > 1/\tau_2$, the composite curve looks like Figure 8.1.12a, and the system is a low-pass filter. If $1/\tau_1 < 1/\tau_2$, it is a high-pass filter (Figure 8.1.12b). The plots of $m(\omega)$ were obtained by using only the asymptotes of the terms $\tau_1 s + 1$ and $\tau_2 s + 1$ without using the 3-dB corrections at the corner frequencies $1/\tau_1$, and $1/\tau_2$. This sketching technique enables the designer to understand the system's general behavior quickly.

From (8.1.18), the phase angle is

$$\phi(\omega) = \angle K + \angle(\tau_1\omega j + 1) - \angle(\tau_2\omega j + 1)$$

$$= \angle K + \tan^{-1}(\tau_1\omega) - \tan^{-1}(\tau_2\omega) \tag{8.1.20}$$

where $\angle K = 0$ if $K > 0$. The plot can be found by combining the plots of $\phi(\omega)$ for K, $\tau_1 s + 1$, and $\tau_2 s + 1$, using the low-frequency, corner frequency, and high-frequency values of $0°$, $45°$, and $90°$ for the terms of the form $\tau s + 1$, supplemented by evaluations of (8.1.20) in the region between the corner frequencies. This sketching technique is more accurate when the corner frequencies are far apart.

$$\tau_1 < \tau_2$$

Slope $= -20$ dB/decade

m (dB)

$-20 \log \left(\dfrac{\tau_1}{\tau_2}\right)$

$\omega = 1/\tau_2$

$\omega = 1/\tau_1$

$\log \omega$

(a)

$$\tau_1 > \tau_2$$

$20 \log \left(\dfrac{\tau_1}{\tau_2}\right)$

Slope $= 20$ dB/decade

m (dB)

0

$\omega = 1/\tau_1$

$\omega = 1/\tau_2$

$\log \omega$

(b)
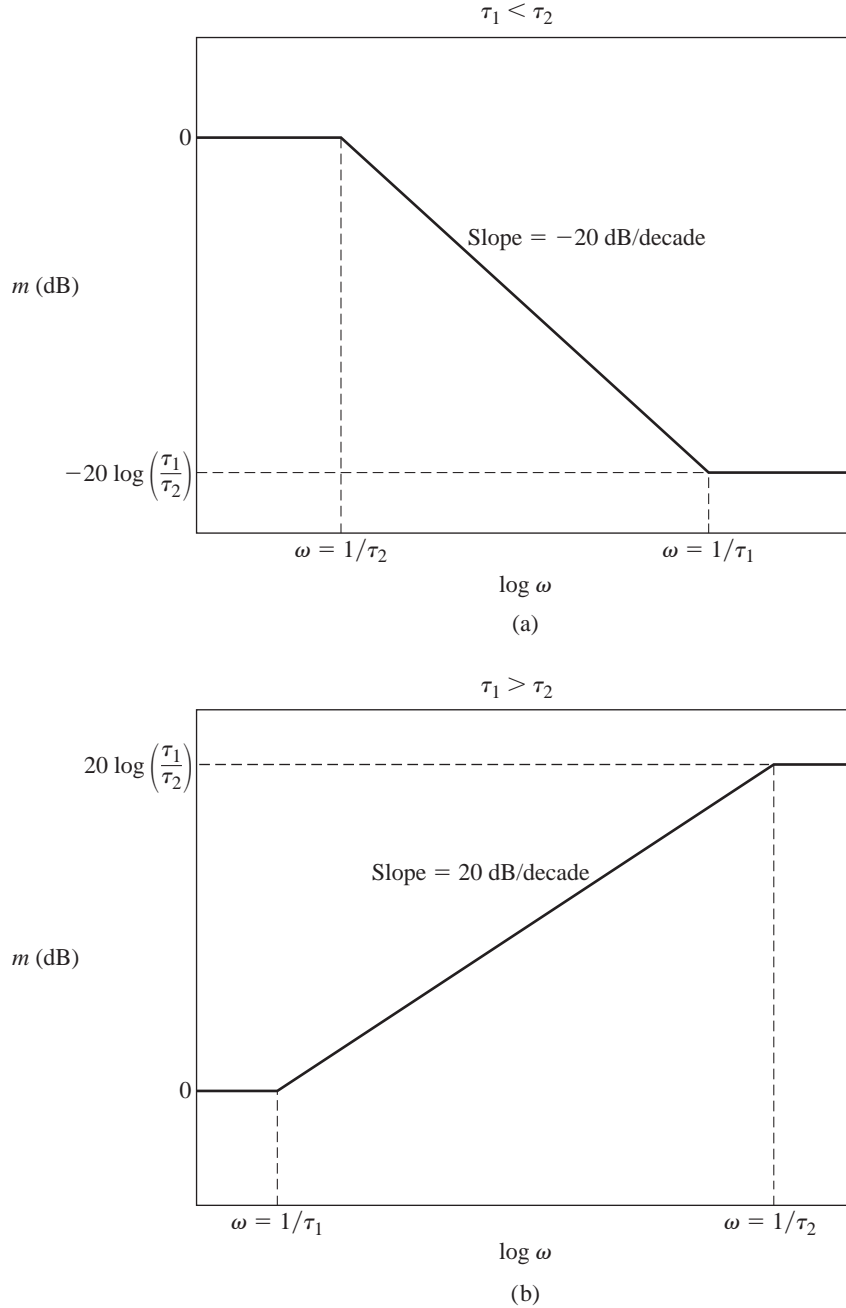
8.2 FREQUENCY RESPONSE OF HIGHER-ORDER SYSTEMS

The results of Section 8.1 can be easily extended to a stable, invariant, linear system of any order. The general form of a transfer function is

$$T(s) = K \frac{N_1(s) N_2(s) \dots}{D_1(s) D_2(s) \dots} \tag{8.2.1}$$

where K is a constant real number. In general, if a complex number $T(j\omega)$ consists of products and ratios of complex factors, such that

$$T(j\omega) = K \frac{N_1(j\omega)N_2(j\omega)\ldots}{D_1(j\omega)D_2(j\omega)\ldots} \tag{8.2.2}$$

where K is constant and real, then from the properties of complex numbers

$$|T(j\omega)| = \frac{|K||N_1(j\omega)||N_2(j\omega)|\ldots}{|D_1(j\omega)||D_2(j\omega)|\ldots} \tag{8.2.3}$$

In decibel units, this implies that

$$\begin{aligned}
m(\omega) &= 20\log|T(j\omega)| \\
&= 20\log|K| + 20\log|N_1(j\omega)| + 20\log|N_2(j\omega)| + \cdots \\
&\quad - 20\log|D_1(j\omega)| - 20\log|D_2(j\omega)| - \cdots
\end{aligned} \tag{8.2.4}$$

That is, when expressed in logarithmic units, multiplicative factors in the numerator of the transfer function are summed, while those in the denominator are subtracted. We can use this principle graphically to add or subtract the contribution of each term in the transfer function to obtain the plot for the overall system transfer function.

For the form (8.2.1), the phase angle is

$$\begin{aligned}
\phi(j\omega) &= \angle T(j\omega) = \angle K + \angle N_1(j\omega) + \angle N_2(j\omega) + \cdots \\
&\quad - \angle D_1(j\omega) - D_2(j\omega) - \cdots
\end{aligned} \tag{8.2.5}$$

The phase angles of multiplicative factors in the numerator are summed, while those in the denominator are subtracted. This enables us to build the composite phase angle plot from the plots for each factor.

COMMON TRANSFER FUNCTION FACTORS

Most transfer functions occur in the form given by (8.2.1). In addition, the factors $N_i(s)$ and $D_i(s)$ usually take the forms shown in Table 8.2.1. We have already obtained the frequency response plots for form 3, which is called a *lead* term because $\phi > 0$. The effect of a multiplicative constant K, which is form 1, is to shift the m curve up by $20\log|K|$. If $K > 0$, the phase plot is unchanged because the angle of a positive number is $0°$. If $K < 0$, the phase plot is shifted down by $180°$ because the angle of a negative number is $-180°$.

We will now develop the plots for form 2. Form 4 will be treated later in this section. For form 2

$$T(s) = s^n \tag{8.2.6}$$

Table 8.2.1 Common factors in the transfer function form: $T(s) = K \dfrac{N_1(s)N_2(s)\ldots}{D_1(s)D_2(s)\ldots}$.

Factor $N_i(s)$ or $D_i(s)$
1. Constant, K
2. s^n
3. $\tau s + 1$
4. $s^2 + 2\zeta\omega_n s + \omega_n^2 = \left[\left(\dfrac{s}{\omega_n}\right)^2 + 2\zeta\dfrac{s}{\omega_n} + 1\right]\omega_n^2, \quad \zeta < 1$

we have

$$m = 20 \log |j\omega|^n = 20n \log |j\omega| = 20n \log \omega \tag{8.2.7}$$

$$\phi = \angle(j\omega)^n = n\angle(j\omega) = n90° \tag{8.2.8}$$

So the slope of the m curve is constant at $20n$ dB/decade.

Consider a second-order system whose transfer function is

$$T(s) = \frac{K}{s(\tau s + 1)} \tag{8.2.9}$$

An example is the displacement of a mass with a damper but no spring. If the applied force is $f(t)$, the equation of motion is $m\ddot{x} + c\dot{x} = f(t)$ or

$$\frac{m}{c}\ddot{x} + \dot{x} = \frac{1}{c}f(t)$$

Thus

$$T(s) = \frac{X(s)}{F(s)} = \frac{1/c}{s[(m/c)s + 1]}$$

Thus $K = 1/c$ and $\tau = m/c$.

The three basic forms in this transfer function are K, s, and $\tau s + 1$. Because the s term is in the denominator, it shifts the composite m curve upward for $\omega < 1$ and shifts it down for $\omega > 1$. The composite m curve follows that of the s term until $\omega \approx 1/\tau$, when the $(\tau s + 1)$ term begins to have an effect. For $\omega > 1/\tau$, the composite slope is -40 dB/decade. The s term contributes a constant $-90°$ to the ϕ curve. The results are shown in Figure 8.2.1.

Figure 8.2.1 Semilog plots of log magnitude ratio and phase angle of the model $1/s(\tau s + 1)$.
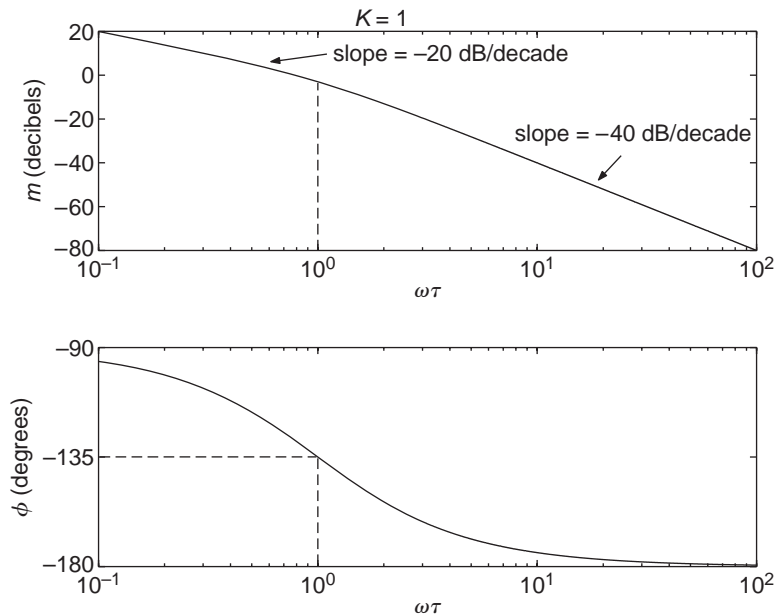
TWO REAL ROOTS

The denominator of a second-order transfer function with two real roots can be written as the product of two first-order factors like form 3. For example,

$$T(s) = \frac{1}{2s^2 + 14s + 20} = \frac{1}{2(s+2)(s+5)} = \frac{0.05}{(0.5s+1)(0.2s+1)}$$

where the time constants are $\tau_1 = 0.5$ and $\tau_2 = 0.2$.

Consider the second-order model

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Its transfer function is

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{8.2.10}$$

If the system is *overdamped*, both roots are real and distinct, and we can write $T(s)$ as

$$T(s) = \frac{1/k}{(m/k)s^2 + (c/k)s + 1} = \frac{1/k}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{8.2.11}$$

where τ_1 and τ_2 are the time constants of the roots.

Substitute $s = j\omega$ into (8.2.11).

$$T(j\omega) = \frac{1/k}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

The magnitude ratio is

$$M(\omega) = |T(j\omega)| = \frac{|1/k|}{|\tau_1 j\omega + 1||\tau_2 j\omega + 1|}$$

Thus

$$m(\omega) = 20 \log M(\omega) = 20 \log \left| \frac{1}{k} \right| - 20 \log |\tau_1 \omega j + 1|$$

$$- 20 \log |\tau_2 \omega j + 1| \tag{8.2.12}$$

The phase angle is

$$\phi(\omega) = \angle \frac{1}{k} - \angle(\tau_1 \omega j + 1) - \angle(\tau_2 \omega j + 1) \tag{8.2.13}$$

The magnitude ratio plot in decibels consists of a constant term, $20 \log |1/k|$, minus the sum of the plots for two first-order lead terms. Assume that $\tau_1 > \tau_2$. Then for $1/\tau_1 < \omega < 1/\tau_2$, the slope is approximately -20 dB/decade. For $\omega > 1/\tau_2$, the contribution of the term $(\tau_2 \omega j + 1)$ becomes significant. This causes the slope to decrease by an additional 20 dB/decade, to produce a net slope of -40 dB/decade for $\omega > 1/\tau_2$. The rest of the plot can be sketched as before. The result is shown in Figure 8.2.2 (upper plot) for $k = 1$. The phase angle plot shown in Figure 8.2.2 (lower plot) is produced in a similar manner by using (8.2.13). Note that if $k > 0$, $\angle(1/k) = 0°$.

TWO COMPLEX ROOTS

We now consider the *underdamped* case of a second-order system that has two complex roots.

Figure 8.2.2 Semilog plots of log magnitude ratio and phase angle of the model $1/(\tau_1 s + 1)(\tau_2 s + 1)$.
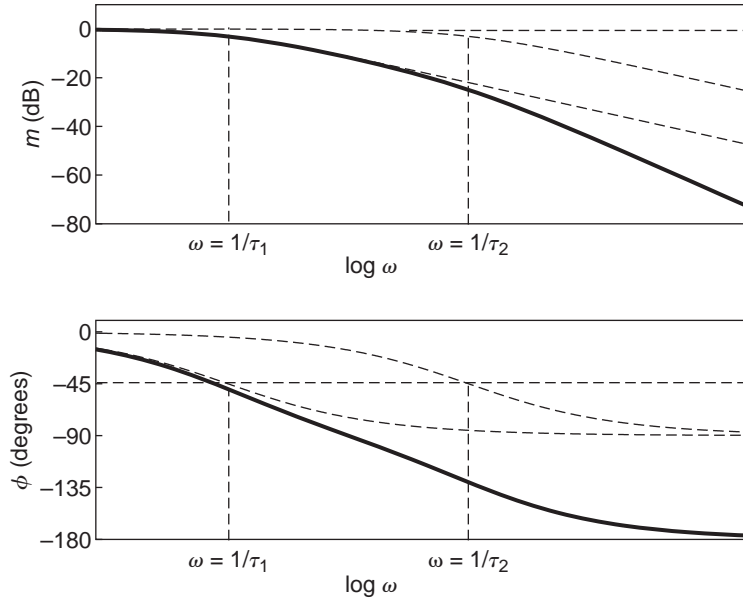
EXAMPLE 8.2.1

Response with Two Complex Roots

■ Problem

The model of a certain system is

$$6\ddot{x} + 12\dot{x} + 174x = 15 f(t)$$

a.   Obtain its steady-state response for $f(t) = 5 \sin 7t$.
b.   Obtain the expressions for $m(\omega)$ and $\phi(\omega)$.

■ Solution

a.   The system's transfer function is

$$T(s) = \frac{X(s)}{F(s)} = \frac{15}{6s^2 + 12s + 174}$$

Substitute $s = 7j$.

$$T(7j) = \frac{15}{-6(7)^2 + 12(7j) + 174} = \frac{15}{-120 + 84j}$$

Then

$$M = |T(7j)| = \frac{15}{\sqrt{(120)^2 + (84)^2}} = 0.1024$$

The phase angle is

$$\phi = \angle T(7j) = \angle(15) - \angle(-120 + 84j) = 0° - \angle(-120 + 84j)$$

Noting that $\angle(-120 + 84j)$ is in the second quadrant, we obtain

$$\phi = -\left[\pi + \tan^{-1}\left(\frac{-84}{120}\right)\right] = -2.531 \text{ rad}$$

Thus the steady-state response is

$$x_{ss}(t) = 5M \sin(7t + \phi) = 0.512 \sin(7t - 2.531)$$
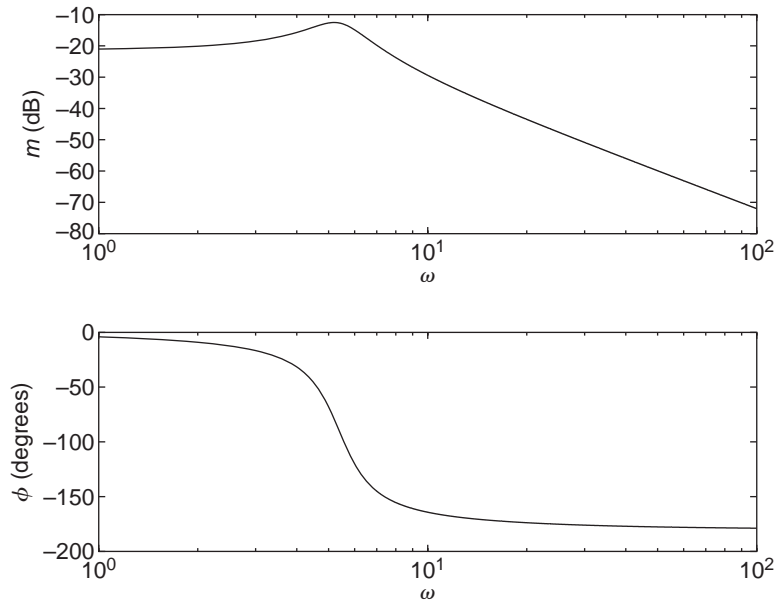
Figure 8.2.3 Semilog plots of log magnitude ratio and phase angle of the model $15/(6s^2 + 12s + 174)$.

b. Replacing s with $j\omega$ gives

$$T(j\omega) = \frac{15}{-6\omega^2 + 12\omega j + 174}$$

Thus,

$$M(\omega) = \frac{15}{\sqrt{(174 - 6\omega^2)^2 + 144\omega^2}}$$

and

$$m(\omega) = 20 \log 15 - 10 \log\left[\left(174 - 6\omega^2\right)^2 + 144\omega^2\right]$$

$$\phi(\omega) = \angle(15) - \angle\left[\left(174 - 6\omega^2\right) + 12\omega j\right]$$

$$= \begin{cases} -\tan^{-1}\dfrac{12\omega}{174 - 6\omega^2} & \text{if } 174 - 6\omega^2 > 0 \\[2mm] \tan^{-1}\dfrac{12\omega}{6\omega^2 - 174} - 180° & \text{if } 174 - 6\omega^2 < 0 \end{cases}$$

The plots are shown in Figure 8.2.3.

If the transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

has complex conjugate roots, it can be expressed as form 4 in Table 8.2.1 as follows:

$$T(s) = \frac{kX(s)}{F(s)} = \frac{1}{(m/k)s^2 + (c/k)s + 1} = \frac{1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

where we have defined $\omega_n = \sqrt{k/m}$ and $\zeta = c/2\sqrt{mk}$, which is called the *damping ratio*. The roots can be expressed in terms of these parameters as

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

The roots are complex if $\zeta < 1$. Thus $T(s)$ becomes

$$T(s) = \frac{kX(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (8.2.14)$$

For most applications of interest here, the quadratic factor given by form 4 in Table 8.2.1 occurs in the denominator; therefore, we will develop the results assuming this will be the case. If a quadratic factor is found in the numerator, its values of $m(\omega)$ and $\phi(\omega)$ are the negative of those to be derived next.

Replacing s with $j\omega$ gives

$$T(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + (2\zeta/\omega_n)j\omega + 1} = \frac{1}{1 - (\omega/\omega_n)^2 + (2\zeta\omega/\omega_n)j} \qquad (8.2.15)$$

To simplify the following expressions, define the following frequency ratio:

$$r = \frac{\omega}{\omega_n} \qquad (8.2.16)$$

Thus the transfer function can be expressed as

$$T(r) = \frac{1}{1 - r^2 + 2\zeta rj} \qquad (8.2.17)$$

The magnitude ratio is

$$M = \frac{1}{|1 - r^2 + 2\zeta rj|} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad (8.2.18)$$

The log magnitude ratio is

$$m = 20\log\left|\frac{1}{1 - r^2 + 2\zeta rj}\right|$$
$$= -20\log\sqrt{(1 - r^2)^2 + (2\zeta r)^2}$$
$$= -10\log\left[(1 - r^2)^2 + (2\zeta r)^2\right] \qquad (8.2.19)$$

The asymptotic approximations are as follows. For $r \ll 1$ (that is, for $\omega \ll \omega_n$), $m \approx -20\log 1 = 0$. For $r \gg 1$ (that is, for $\omega \gg \omega_n$),

$$m \approx -20\log\sqrt{r^4 + 4\zeta^2 r^2}$$
$$\approx -20\log\sqrt{r^4}$$
$$= -40\log r$$

Thus, at low frequencies where $\omega \ll \omega_n$, the curve is horizontal at $m = 0$, while for high frequencies $\omega \gg \omega_n$ where $r \gg 1$, the curve has a slope of -40 dB/decade, just as in the overdamped case (Figure 8.2.4a). The high-frequency and low-frequency asymptotes intersect at the corner frequency $\omega = \omega_n$.

The phase angle plot can be obtained in a similar manner (Figure 8.4.2b). From the additive property for angles, we see that for (8.2.17),

$$\phi = -\angle(1 - r^2 + 2\zeta rj)$$

Thus

$$\phi = -\tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$
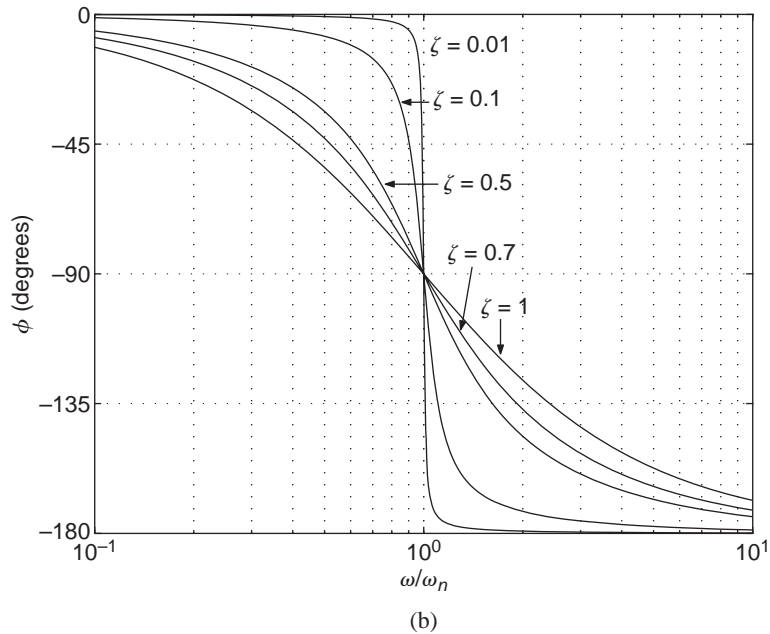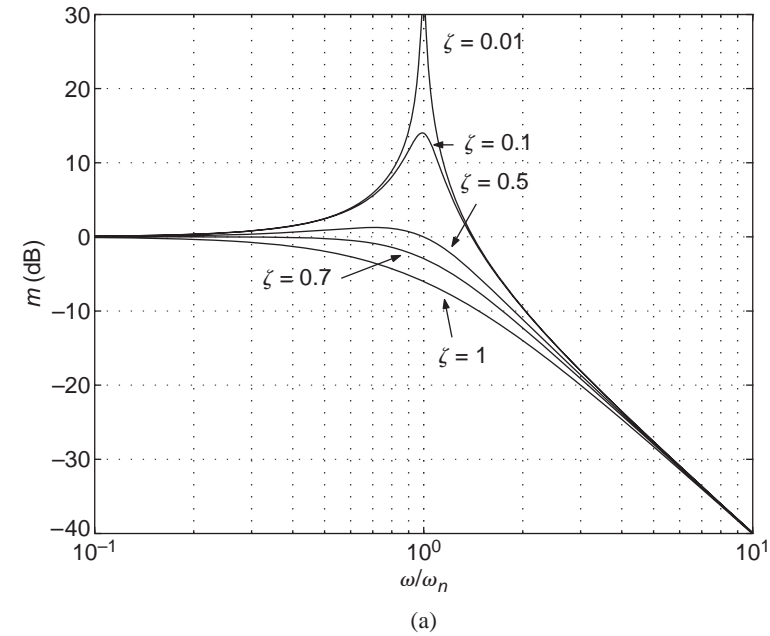
where ϕ is in the third or fourth quadrant. For $r \ll 1$, $\phi \approx 0°$. For $r \gg 1$, $\phi \approx -180°$. At $\omega = \omega_n$, $\phi = -90°$ independently of ζ. The curve is skew-symmetric about the inflection point at $\phi = -90°$ for all values of ζ.

RESONANCE

The complex roots case differs from the real roots case in the vicinity of the corner frequency. To see this, note that M given by (8.2.18) has a maximum value when the

denominator has a minimum. Setting the derivative of the denominator with respect to r equal to zero shows that the maximum M occurs at $r = \sqrt{1 - 2\zeta^2}$, which corresponds to the frequency $\omega = \omega_n \sqrt{1 - 2\zeta^2}$. This frequency is the *resonant frequency* ω_r. The peak of M exists only when the term under the radical is positive; that is, when $\zeta \leq 0.707$. Thus, the resonant frequency is given by

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad 0 \leq \zeta \leq 0.707 \tag{8.2.20}$$

The peak, or resonant, value of M, denoted by M_r, is found by substituting $r = \sqrt{1 - 2\zeta^2}$ into the expression (8.2.18) for M. This gives

$$M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \qquad 0 \leq \zeta \leq 0.707 \tag{8.2.21}$$

If $\zeta > 0.707$, no peak exists, and the maximum value of M occurs at $\omega = 0$ where $M = 1$. Note that as $\zeta \to 0$, $\omega_r \to \omega_n$, and $M_r \to \infty$. For an undamped system, the roots are purely imaginary, $\zeta = 0$, and the resonant frequency is the natural frequency ω_n. These formulas are summarized in Table 8.2.2.

Resonance occurs when the input frequency is close to the resonant frequency of the system. If the damping is small, the output amplitude will continue to increase until either the linear model is no longer accurate or the system fails. When ϕ is near $-90°$; the *velocity $\dot{x}$ is in phase* with the input. This causes the large amplitude. Circuit designers take advantage of resonance by designing amplification circuits whose natural frequency is close to the frequency of a signal they want to amplify (such as the signal from a radio station). Designers of structural systems and suspensions try to avoid resonance because of the damage or discomfort that large motions can produce.

In Figure 8.2.4a, the correction to the asymptotic approximations in the vicinity of the corner frequency depends on the value of ζ. The peak value in decibels is

$$m_r = 20 \log M_r = -20 \log \left(2\zeta \sqrt{1 - \zeta^2}\right) \tag{8.2.22}$$

At the resonant frequency ω_r, the phase angle is

$$\phi|_{r=\sqrt{1-2\zeta^2}} = -\tan^{-1} \frac{\sqrt{1 - 2\zeta^2}}{\zeta} \tag{8.2.23}$$

When $r = 1$ (at $\omega = \omega_n$),

$$m|_{r=1} = -20 \log 2\zeta \tag{8.2.24}$$

Once you understand how each of the forms shown in Table 8.2.1 contributes to the magnitude and phase plots, you can quickly determine the effects of each term in a more complicated transfer function.

Table 8.2.2 Frequency response of a second-order system.

Model:	$T(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
Resonant frequency:	$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \quad 0 \leq \zeta \leq 0.707$
Resonant response:	$M_r = \dfrac{1}{2\zeta \sqrt{1 - \zeta^2}} \quad 0 \leq \zeta \leq 0.707$
	$m_r = -20 \log \left(2\zeta \sqrt{1 - \zeta^2}\right) \quad 0 \leq \zeta \leq 0.707$
	$\phi_r = -\tan^{-1} \dfrac{\sqrt{1 - 2\zeta^2}}{\zeta} \quad 0 \leq \zeta \leq 0.707$

| A Fourth-Order Model | **EXAMPLE 8.2.2** |

■ Problem

Determine the effect of the parameter τ of the magnitude plot of the following transfer function. Assume that time is measured in seconds.

$$T(s) = \frac{\tau s + 1}{s^4 + 40.8s^3 + 8337s^2 + 4.1184 \times 10^4 + 5.184 \times 10^5}$$

Is it possible to choose a value for τ to increase the gain at low frequencies?

■ Solution

First determine the roots of the denominator. They are

$$s = -2.4 \pm 7.632j \qquad -18 \pm 88.10j$$

For the first root pair,

$$\zeta = \cos\left[\tan^{-1}\left(\frac{7.632}{2.4}\right)\right] = 0.3 \qquad \omega_n = \sqrt{2.4^2 + 7.632^2} = 8$$

Similarly, for the second pair,

$$\zeta = \cos\left[\tan^{-1}\left(\frac{88.1}{18}\right)\right] = 0.2 \qquad \omega_n = \sqrt{18^2 + 88.1^2} = 90$$

Thus, $T(s)$ can be expressed in factored form as

$$T(s) = \frac{\tau s + 1}{(s^2 + 4.8s + 64)(s^2 + 36s + 8100)}$$

The model has two resonant frequencies corresponding to the two root pairs. These frequencies are

$$\omega_{r_1} = \omega_n\sqrt{1 - 2\zeta^2} = 8\sqrt{1 - 0.18} = 7.24 \text{ rad/s}$$

$$\omega_{r_2} = \omega_n\sqrt{1 - 2\zeta^2} = 90\sqrt{1 - 0.08} = 86.3 \text{ rad/s}$$

Because ζ is small for each factor, we can expect a resonant peak at these frequencies. However, we cannot use the formula (8.2.22) to calculate m_r at each peak because it applies only to a second-order system. Each quadratic term contributes -40 dB/decade to the composite slope at high frequencies. The numerator term contributes a slope of $+20$ dB/decade at frequencies above $1/\tau$. So the composite curve will have a slope of $20 - 2(40) = -60$ dB/decade at high frequencies.

The numerator term causes the m curve to break upward at $\omega = 1/\tau$. If $1/\tau$ is less than ω_{r_1}, the m curve will break upward before the -40 dB/decade slope of the first quadratic term takes effect. Figure 8.2.5 shows the m plot for three cases, including the case where $\tau = 1.25$, which corresponds to a corner frequency of $\omega = 1/\tau = 0.8$. As compared with the case having no numerator dynamics ($\tau = 0$), the choice of $\tau = 1.25$ can be seen to raise the magnitude. For example, at $\omega = 7.24$ rad/s, the resonant peaks for the two cases are -109 dB ($M = 3.55 \times 10^{-6}$) for $\tau = 0$, and -89.6 ($M = 3.31 \times 10^{-5}$) for $\tau = 1.25$. The amplitude ratio is $3.31 \times 10^{-5}/3.55 \times 10^{-6} = 9.34$ times larger for $\tau = 1.25$.

Using a value of $1/\tau$ that is larger than the smallest resonant frequency will not increase the amplitude ratio because the -40 dB/decade slope from the first quadratic term will take effect before the $+20$ dB/decade slope of the numerator term makes its contribution. An example of this is shown in Figure 8.2.5 for $\tau = 0.0125$ s, which corresponds to a corner frequency of $1/\tau = 80$ rad/s.
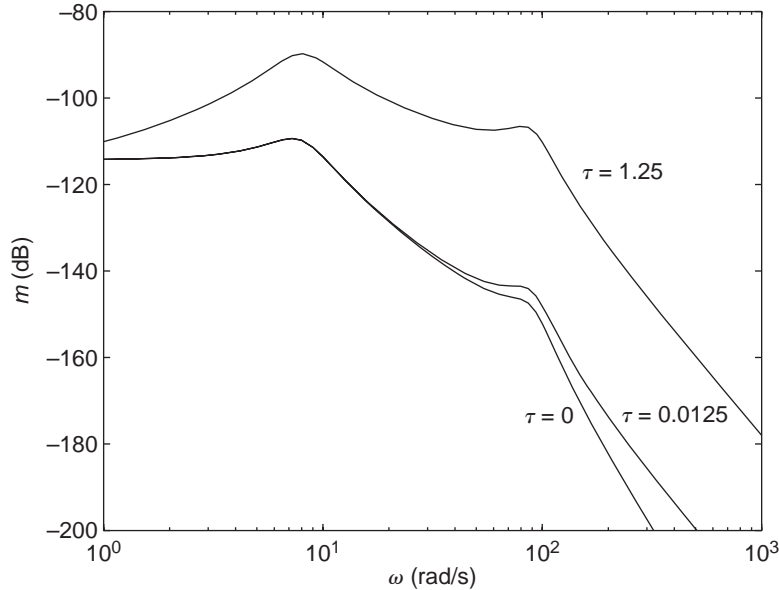
8.3 FREQUENCY RESPONSE EXAMPLES

In this section we present more examples illustrating the concepts and applications of frequency response.

THE NEUTRALLY STABLE CASE

While all real systems will have some damping, it is instructive and useful to obtain some results for the undamped case, where $c = \zeta = 0$. This is because the mathematical results for the undamped case are more easily derived and analyzed, and they give insight into the behavior of many real systems that have a small amount of damping.

   If the model is stable, the free response term disappears in time. The results derived in Section 8.2 therefore must be reexamined for the case where there is no damping because this is not a stable case (it is neutrally stable). In this case, the magnitude and phase angle of the transfer function do not give the entire steady-state response. The free response for the undamped equation $m\ddot{x} + kx = F \sin \omega t$ is found with the Laplace transform as follows:

$$(ms^2 + k)X_{\text{free}}(s) - m\dot{x}(0) - msx(0) = 0$$

$$X_{\text{free}}(s) = \frac{m\dot{x}(0) + msx(0)}{ms^2 + k} = \frac{\dot{x}(0) + sx(0)}{s^2 + \omega_n^2}$$

Thus,

$$x_{\text{free}}(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t$$

   The forced response is found as follows:

$$(ms^2 + k)X_{\text{forced}}(s) = \frac{F\omega}{s^2 + \omega^2}$$

$$X_{\text{forced}}(s) = \frac{1}{m} \frac{F\omega}{(s^2 + \omega_n^2)(s^2 + \omega^2)}$$

Assuming that $\omega \neq \omega_n$, the partial fraction expansion is

$$X_{\text{forced}}(s) = \frac{F\omega}{m\left(\omega^2 - \omega_n^2\right)} \left(\frac{\omega_n}{s^2 + \omega_n^2} - \frac{\omega_n}{\omega} \frac{\omega}{s^2 + \omega^2} \right)$$

Thus the forced response is

$$x_{\text{forced}}(t) = \frac{F\omega}{m\left(\omega^2 - \omega_n^2\right)} \left(\sin \omega_n t - \frac{\omega_n}{\omega} \sin \omega t \right)$$

or, with $r = \omega/\omega_n$,

$$x_{\text{forced}}(t) = -\frac{Fr}{k(1 - r^2)} \sin \omega_n t + \frac{F}{k(1 - r^2)} \sin \omega t$$

Thus the total response is

$$x(t) = \left[\frac{\dot{x}(0)}{\omega_n} - \frac{F}{k} \frac{r}{1 - r^2} \right] \sin \omega_n t + x(0) \cos \omega_n t + \frac{F}{k} \frac{1}{1 - r^2} \sin \omega t \qquad (8.3.1)$$

There is no transient response here; the entire solution is the steady-state response.

BEATING

We may examine the effects of the forcing function independently of the effects of the initial conditions by setting $x(0) = \dot{x}(0) = 0$ in (8.3.1). The result is the forced response:

$$x(t) = \frac{F}{k} \frac{1}{1 - r^2} (\sin \omega t - r \sin \omega_n t) \qquad (8.3.2)$$

When the forcing frequency ω is substantially different than the natural frequency ω_n, the forced response looks somewhat like Figure 8.3.1 and consists of a higher-frequency oscillation superimposed on a lower-frequency oscillation.
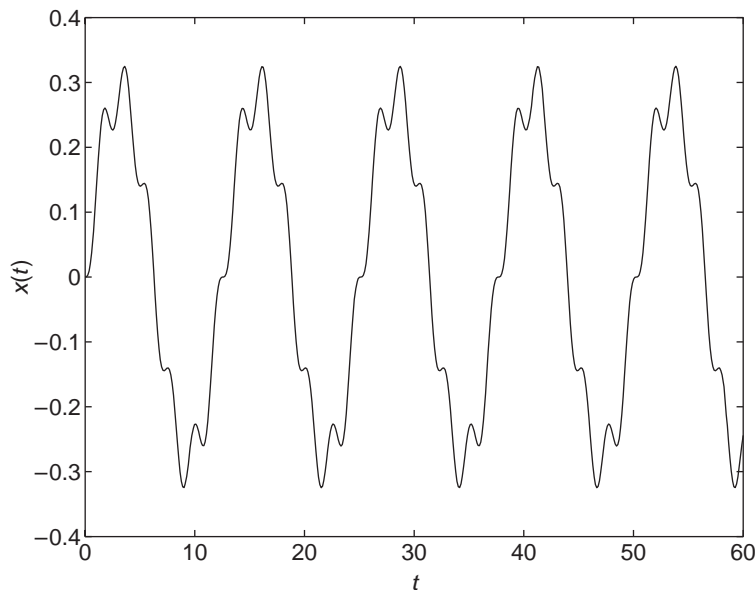
Figure 8.3.1 Response when the forcing frequency is substantially different from the natural frequency.

If the forcing frequency ω is close to the natural frequency ω_n, then $r \approx 1$ and the forced response (8.3.2) can be expressed as follows:

$$x(t) = \frac{F}{k} \frac{1}{1 - r^2} (\sin \omega t - \sin \omega_n t) = \frac{F}{m} \frac{1}{\omega_n^2 - \omega^2} (\sin \omega t - \sin \omega_n t)$$

Using the identity

$$2 \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B) = \sin A - \sin B$$

we see that

$$2 \sin \frac{\omega - \omega_n}{2} t \cos \frac{\omega + \omega_n}{2} t = \sin \omega t - \sin \omega_n t$$

Thus the forced response is given by

$$x(t) = \left(\frac{2F}{m} \frac{1}{\omega_n^2 - \omega^2} \sin \frac{\omega - \omega_n}{2} t \right) \cos \frac{\omega + \omega_n}{2} t \qquad (8.3.3)$$

This can be interpreted as a cosine with a frequency $(\omega + \omega_n)/2$ and a time-varying amplitude of

$$\frac{2F}{m} \frac{1}{\omega_n^2 - \omega^2} \sin \frac{\omega - \omega_n}{2} t$$

The amplitude varies sinusoidally with the frequency $(\omega - \omega_n)/2$, which is lower than the frequency of the cosine. The response thus looks like Figure 8.3.2. This behavior, in which the amplitude increases and decreases periodically, is called *beating*. The *beat period* is the time between the occurrence of zeros in $x(t)$ and thus is given by the half-period of the sine wave, which is $2\pi/|\omega - \omega_n|$. The *vibration period* is the period of the cosine wave, $4\pi/(\omega + \omega_n)$.

Figure 8.3.2 Beating response when the forcing frequency is close to the natural frequency.
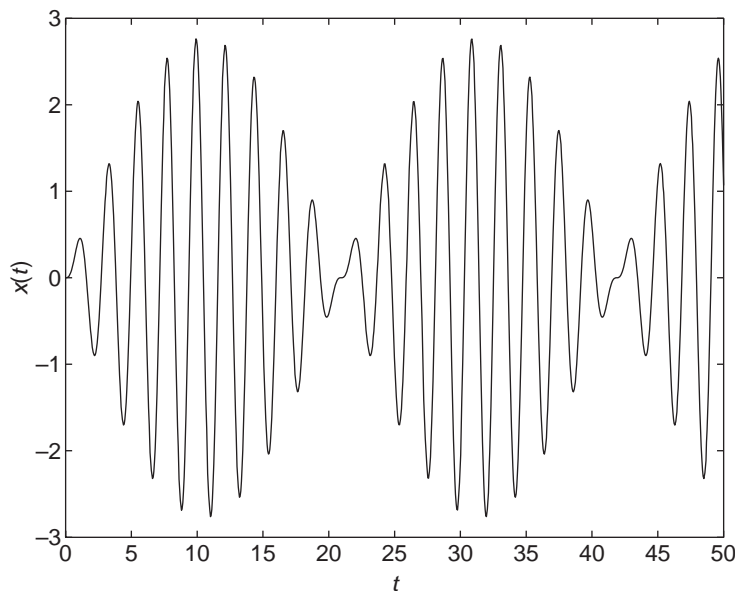
RESPONSE AT RESONANCE

Equation (8.2.18) shows that when $\zeta = 0$ the amplitude of the steady-state response becomes infinite when $r = 1$; that is, when the forcing frequency ω equals the natural frequency ω_n. The phase shift ϕ is exactly -90 degrees at this frequency.

To obtain the expression for $x(t)$ at resonance, we use (8.3.2) to compute the limit of $x(t)$ as $r \to 1$ after replacing ω in $x(t)$ using the relation $\omega = \omega_n r$. We must use l'Hôpital's rule to compute the limit.

$$
\begin{aligned}
x(t) &= \lim_{r \to 1} \frac{F}{k} \frac{1}{1 - r^2} (\sin \omega_n r t - r \sin \omega_n t) \\
&= \frac{F}{k} \lim_{r \to 1} \frac{d(\sin \omega_n r t - r \sin \omega_n t)/dr}{d(1 - r^2)/dr} \\
&= \frac{F}{k} \lim_{r \to 1} \frac{\omega_n t \cos \omega_n r t - \sin \omega_n t}{-2r} \\
&= \frac{F \omega_n}{2k} \left(\frac{\sin \omega_n t}{\omega_n} - t \cos \omega_n t \right)
\end{aligned}
\tag{8.3.4}
$$

The plot is shown in Figure 8.3.3 for the case $m = 4$, $c = 0$, $k = 36$, and $F = 10$. The amplitude increases linearly with time. The behavior for lightly damped systems is similar except that the amplitude does not become infinite. Figure 8.3.4 shows the damped case: $m = 4$, $c = 4$, $k = 36$, and $F = 10$.

For large amplitudes, the linear model on which this analysis is based will no longer be accurate. In addition, all physical systems have some damping, so c will never be exactly zero, and the response amplitude will never be infinite. The important point, however, is that the amplitude might be large enough to damage the system or cause some other undesirable result. At resonance the output amplitude will continue to increase until either the linear model is no longer accurate or the system fails.
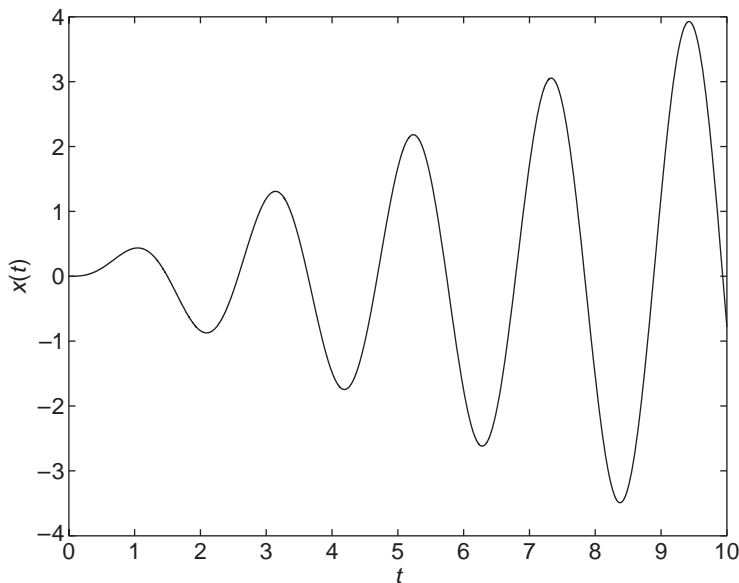
Figure 8.3.3 Response near resonance for an undamped system.

RESONANCE AND TRANSIENT RESPONSE

Although the analysis in Section 8.2 was a steady-state analysis and assumed that the forcing frequency ω is constant, resonance can still occur in transient processes if the input varies slowly enough to allow the steady-state response to begin to appear. For example, resonance can be a problem even if a machine's operating speed is well above its resonant frequency, because the machine's speed must pass through the resonant frequency at startup. If the machine's speed does not pass through the resonant frequency quickly enough, high amplitude oscillations will result. Figure 8.3.5 shows the transient response of the model

$$4\ddot{x} + 3\dot{x} + 100x = 295 \sin[\omega(t)t]$$

Figure 8.3.4 Response near resonance for a damped system.
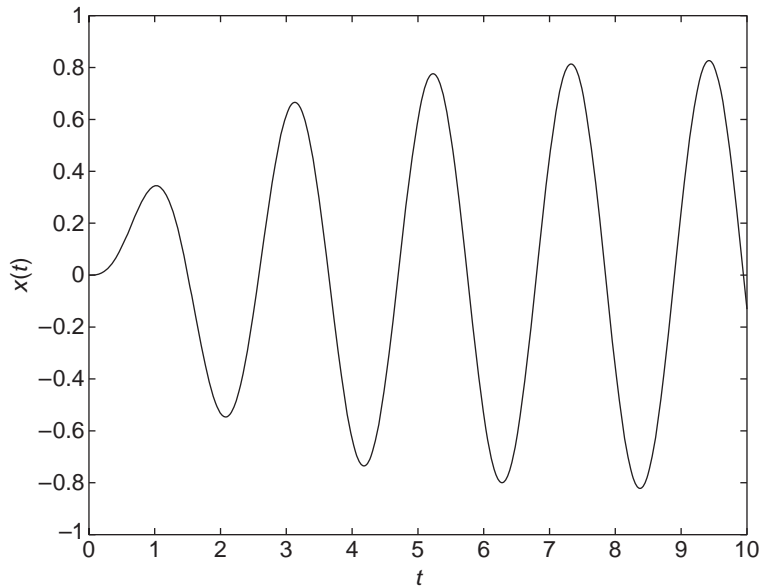
Figure 8.3.5 Transient response when an increasing forcing frequency passes through the resonant frequency.
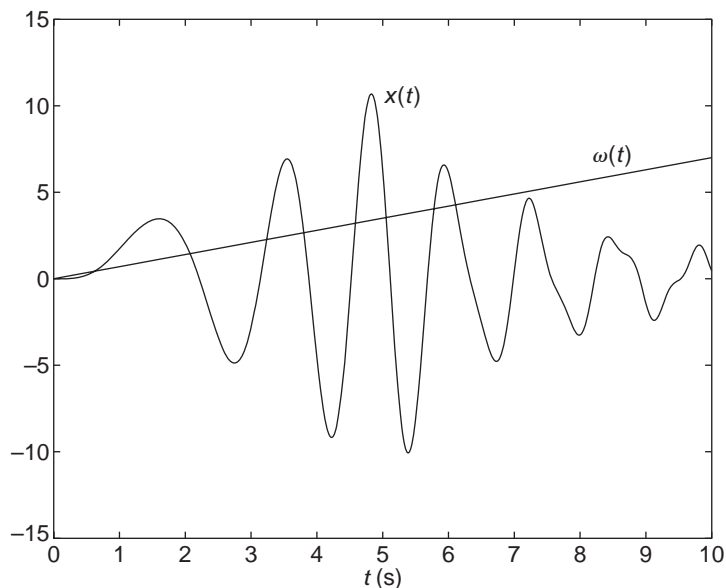
where the frequency $\omega(t)$ increases linearly with time, as $\omega(t) = 0.7t$. Thus the frequency passes through the resonant frequency of this model, which is $\omega_r = 4.97$, at $t = 7.1$ s. The plot shows the large oscillations that occur as the input frequency passes close to the resonant frequency.

BASE MOTION AND TRANSMISSIBILITY

The motion of the mass shown in Figure 8.3.6 is produced by the motion $y(t)$ of the base. This system is a model of many common displacement isolation systems. Assuming that the mass displacement x is measured from the rest position of the mass when $y = 0$, the weight mg is canceled by the static spring force. The force transmitted to the mass by the spring and damper is denoted f_t and is given by

Figure 8.3.6 Base excitation.

$$f_t = c(\dot{y} - \dot{x}) + k(y - x) \tag{8.3.5}$$

This gives the following equation of motion:

$$m\ddot{x} = f_t = c(\dot{y} - \dot{x}) + k(y - x)$$

or

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{8.3.6}$$

The transfer function is

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k} \tag{8.3.7}$$

This transfer function is called the *displacement transmissibility* and can be used to analyze the effects of the base motion $y(t)$ on $x(t)$, the motion of the mass. Notice that this transfer function has numerator dynamics, so its frequency response plots will be different than those of the transfer function $1/(ms^2 + cs + k)$, which was studied in Section 8.2.

From (8.3.5)

$$F_t(s) = (cs + k)[Y(s) - X(s)] \tag{8.3.8}$$

Substituting for $X(s)$ from (8.3.7) gives

$$F_t(s) = (cs + k)\left[Y(s) - \frac{cs + k}{ms^2 + cs + k}Y(s)\right] = (cs + k)\frac{ms^2}{ms^2 + cs + k}Y(s)$$

Thus, the second desired ratio is

$$\frac{F_t(s)}{Y(s)} = (cs + k)\frac{ms^2}{ms^2 + cs + k} \tag{8.3.9}$$

This is the ratio of the transmitted force to the base motion. It is customary to use instead the ratio $F_t(s)/kY(s)$, which is a dimensionless quantity representing how the base displacement y affects the force transmitted to the mass. Thus,

Figure 8.3.7 Single-mass suspension model.

$$\frac{F_t(s)}{kY(s)} = \frac{cs + k}{k}\frac{ms^2}{ms^2 + cs + k} \tag{8.3.10}$$

The ratio $F_t(s)/kY(s)$ is called the *force transmissibility*. It can be used to compute the transmitted force $f_t(t)$ that results from a specified base motion $y(t)$.

An example of base excitation occurs when a car drives over a rough road. Figure 8.3.7 shows a quarter-car representation, where the stiffness k is the series combination of the tire and suspension stiffnesses. The equation of motion is given by (8.3.6). Although road surfaces are not truly sinusoidal in shape, we can nevertheless

use a sinusoidal profile to obtain an approximate evaluation of the performance of the suspension at various speeds.

EXAMPLE 8.3.1

Vehicle Suspension Response

■ Problem

Suppose the road profile is given (in feet) by $y(t) = 0.05 \sin \omega t$, where the amplitude of variation of the road surface is 0.05 ft and the frequency ω depends on the vehicle's speed and the road profile's period. Suppose the period of the road surface is 30 ft. Compute the steady-state motion amplitude and the force transmitted to the chassis, for a car traveling at a speed of 30 mi/hr. The car weighs 3200 lb. The effective stiffness, which is a series combination of the tire stiffness and the suspension stiffness, is $k = 3000$ lb/ft. The damping is $c = 300$ lb-sec/ft.

■ Solution

For a period of 30 ft and a vehicle speed of 30 mi/hr, the frequency ω is

$$\omega = \left(\frac{5280}{30}\right)\left(\frac{1}{3600}\right)(2\pi)30 = 9.215 \text{ rad/sec}$$

For the car weighing 3200 lb, the quarter-car mass is $m = 800/32.2$ slug. From (8.3.7) with $s = j\omega = 9.215j$,

$$\frac{|X|}{|Y|} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = 1.405$$

Thus the displacement amplitude is $|X| = 0.05(1.405) = 0.07$ ft.

The transmitted force is obtained from (8.3.9).

$$\frac{|F_t|}{|Y|} = \sqrt{k^2 + (c\omega)^2}\,\frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = 2960$$

Thus the magnitude of the transmitted force is $|F_t| = 0.05(2960) = 148$ lb.

ROTATING UNBALANCE

A common cause of sinusoidal forcing in machines is the unbalance that exists to some extent in every rotating machine. The unbalance is caused by the fact that the center of mass of the rotating part does not coincide with the center of rotation. Let m be the total mass of the machine and let m_u be the rotating mass causing the unbalance. Consider the entire unbalanced mass m_u to be lumped at its center of mass, a distance ϵ from the center of rotation. This distance is the *eccentricity*. Figure 8.3.8a shows this situation. The main mass is thus $(m - m_u)$ and is assumed to be constrained to allow only vertical motion. The motion of the unbalanced mass m_u will consist of the vector combination of its motion relative to the main mass $(m - m_u)$ and the motion of the main mass. For a constant speed of rotation ω, the rotation produces a radial acceleration of m_u equal to $\epsilon\omega^2$. This causes a force to be exerted on the bearings at the center of rotation. This force has a magnitude $m_u\epsilon\omega^2$ and is directed radially outward. The vertical component of this rotating unbalance force is, from Figure 8.3.8b,

$$f_r = m_u\epsilon\omega^2 \sin \omega t \tag{8.3.11}$$
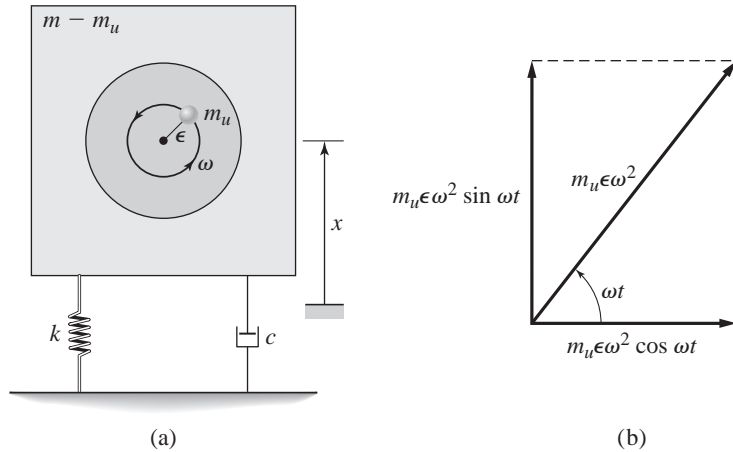
Figure 8.3.8 (a) A machine having rotating unbalance. (b) Vector components of the rotating unbalance force.

(a)   (b)

In many situations involving an unbalanced machine, we are interested in the force that is transmitted to the base or foundation. The equation of motion of a mass-spring-damper system, like that shown in Figure 8.3.8a, with an applied force $f_r(t)$ is

$$m\ddot{x} + c\dot{x} + kx = f_r(t) \tag{8.3.12}$$

where x is the displacement of the mass from its rest position. The force transmitted to the foundation is the sum of the spring and damper forces, and is given by

$$f_t = kx + c\dot{x} \tag{8.3.13}$$

The *force transmissibility* of this system is the ratio $F_t(s)/F_r(s)$, which represents the ratio of the force f_t transmitted to the foundation by the applied force f_r. The most common case of such an applied force is the rotating unbalance force. From (8.3.11), we see that the amplitude F_r of the rotating unbalance force is

$$F_r = m_u \epsilon \omega^2 \tag{8.3.14}$$

The transfer function of (8.3.12) is

$$\frac{X(s)}{F_r(s)} = \frac{1}{ms^2 + cs + k} \tag{8.3.15}$$

From (8.3.13),

$$F_t(s) = (k + cs)X(s) \tag{8.3.16}$$

Substituting $X(s)$ from (8.3.15) into (8.3.16) gives the force transmissibility:

$$\frac{F_t(s)}{F_r(s)} = \frac{k + cs}{ms^2 + cs + k} \tag{8.3.17}$$

Foundation Force Due to Rotating Unbalance | **EXAMPLE 8.3.2**

■ **Problem**

A system having a rotating unbalance, like that shown in Figure 8.3.8, has a total mass of $m = 20$ kg, an unbalanced mass of $m_u = 0.05$ kg, and an eccentricity of $\epsilon = 0.01$ m. The machine rotates at 1150 rpm. Its vibration isolator has a stiffness of $k = 2 \times 10^4$ N/m. Compute the force transmitted to the foundation if the isolator's damping ratio is $\zeta = 0.5$.

■ Solution

First convert the machine's speed to radians per second.

$$\omega = 1150 \text{ rpm} = \frac{1150(2\pi)}{60} = 120 \text{ rad/s}$$

Then,

$$m_u \epsilon \omega^2 = 0.05(0.01)(120)^2 = 7.25 \text{ N}$$

Since the damping ratio is defined as

$$\zeta = \frac{c}{2\sqrt{mk}}$$

the damping constant can be obtained from

$$c = 2\zeta\sqrt{mk} = 632 \text{ N} \cdot \text{s/m}$$

We can calculate the amplitude of the steady-state transmitted force from (8.3.17).

$$|F_t| = |F_r| \left| \frac{k + c\omega j}{k - m\omega^2 + c\omega j} \right| = m_u \epsilon \omega^2 \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = 2 \text{ N}$$

INSTRUMENT DESIGN

When no s term occurs in the numerator of a transfer function, its magnitude ratio is small at high frequencies. On the other hand, introducing numerator dynamics can produce a large magnitude ratio at high frequencies. This effect can be used to advantage in instrument design, for example. The instrument shown in Figure 8.3.9 illustrates this point. With proper selection of the natural frequency of the device, it can be used either as a *vibrometer* to measure the amplitude of a sinusoidal input displacement $z = A_z \sin \omega t$, or an *accelerometer* to measure the amplitude of the acceleration $\ddot{z} = -A_z \omega^2 \sin \omega t$. When used to measure ground motion from an earthquake, for example, the instrument is commonly referred to as a *seismograph*.

The model was obtained in Section 6.6 and is

$$T(s) = \frac{Y(s)}{Z(s)} = \frac{-ms^2}{ms^2 + cs + k} = \frac{-s^2/\omega_n^2}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} \tag{8.3.18}$$

Figure 8.3.9
An accelerometer.

Structure
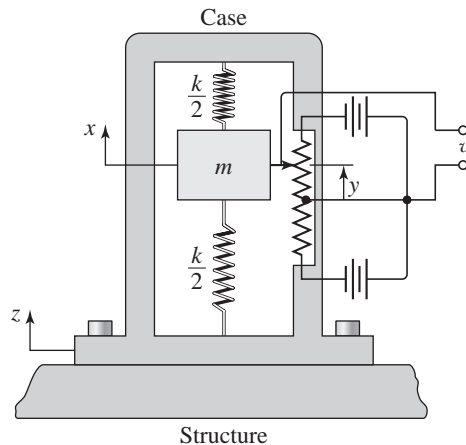
where $y(t)$ is the output displacement, which is measured from the voltage $v(t)$. At steady state, $y(t) = A_y \sin(\omega t + \phi)$.

The numerator $N(s) = -s^2/\omega_n^2$ gives the following contribution to the log magnitude ratio:

$$20 \log |N(j\omega)| = 20 \log \left| \left(\frac{j\omega}{\omega_n} \right)^2 \right| = 40 \log \frac{\omega}{\omega_n} \qquad (8.3.19)$$

This term contributes 0 dB to the net curve at the corner frequency $\omega = \omega_n$, and it increases the slope by 40 dB/decade over all frequencies. Thus, at low frequencies, the slope of $m(\omega)$ corresponding to (8.3.18) is 40 dB/decade, and at high frequencies, the slope is zero. The plot is sketched in Figure 8.3.10.

For the device to act like a vibrometer, this plot shows that the device's natural frequency ω_n must be selected so that $\omega \gg \omega_n$, where ω is the oscillation frequency of the displacement to be measured. For $\omega \gg \omega_n$,

$$|T(j\omega)| \approx 40 \log \frac{\omega}{\omega_n} - 40 \log \frac{\omega}{\omega_n} = 0 \text{ dB}$$

and thus $A_y \approx A_z$ as desired. This is because the mass m cannot respond to high-frequency input displacements. Its displacement x therefore remains approximately constant, and the motion z directly indicates the motion y. To design a vibrometer having specific characteristics, we must know the lower bound of the input displacement frequency ω. The frequency $\omega_n = \sqrt{k/m}$ is then made much smaller than this bound by selecting a large mass and a soft spring (small k). However, these choices are governed by constraints on the allowable deflection. For example, a very soft spring will have a large distance between the free length and the equilibrium positions.

An accelerometer can be obtained by using the lower end of the frequency range; that is, selecting $\omega_n \gg \omega$, or equivalently, for s near zero, (8.3.18) gives

$$T(s) \approx -\frac{s^2}{\omega_n^2} = \frac{Y(s)}{Z(s)}$$
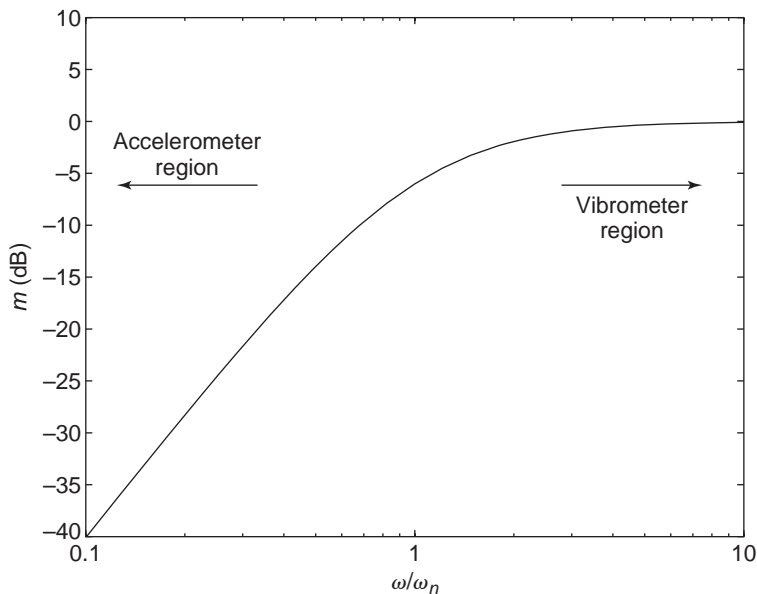
Figure 8.3.10 Frequency response of an accelerometer.

or

$$Y(s) \approx -\frac{1}{\omega_n^2}s^2 Z(s)$$

The term $s^2 Z(s)$ represents the transform of $\ddot{z}$ so the output of the accelerometer is

$$A_y \approx \frac{1}{\omega_n^2}|\ddot{z}| = \frac{\omega^2}{\omega_n^2}A_z$$

With ω_n chosen large (using a small mass and a stiff spring), the input acceleration amplitude $\omega^2 A_z$ can be determined from A_y.

PHYSICAL INTERPRETATION OF THE PHASE PLOT

Figure 8.3.11 Two-mass suspension model.

The phase angle plot can give useful information about the motion of the system relative to the input, or the motion of the constituent masses relative to each other. Consider the following two-mass suspension model analyzed in Example 4.5.9 and shown again in Figure 8.3.11. The equations of motion are

$$m_1\ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) + k_2(y - x_2)$$

We will use the following numerical values: $m_1 = 250$ kg, $m_2 = 40$ kg, $k_1 = 1.5 \times 10^4$ N/m, $k_2 = 1.5 \times 10^5$ N/m, and $c_1 = 1917$ N·s/m. The transfer functions for these values are

$$\frac{X_1(s)}{Y(s)} = \frac{(0.2876s + 2.25)10^5}{s^4 + 55.6s^3 + 4185s^2 + 43.14 \times 10^4 s + 2.75 \times 10^5}$$

$$\frac{X_2(s)}{Y(s)} = \frac{(0.0375s^2 + 0.2876s + 2.25)10^5}{s^4 + 55.6s^3 + 4185s^2 + 4.314 \times 10^4 s + 2.75 \times 10^5}$$

The frequency response plots (obtained with MATLAB, as described in Section 8.6) are shown in Figure 8.3.12. Recall that the input frequency ω in rad/s depends on the period P (in meters) of the road surface and on the vehicle speed v in m/s, as $\omega = 2\pi v/P$. The amplitude plot shows that the chassis amplitude is at most 10% greater than the road motion at low frequencies, where $M \approx 1.1$. At higher frequencies the chassis motion is well isolated from the road motion because $M < 0.5$ at those frequencies. On the other hand, the wheel amplitude is 50% greater than the road motion when $\omega \approx 50$ rad/s.

The relative motion of the chassis, wheel, and road can be determined from the phase plot. For example, at $\omega = 60$, the wheel motion and the chassis motion lag behind the road's sinusoidal motion by 90° and 180°, respectively. Thus at this frequency, the chassis motion is in the opposite direction of the ground motion and it lags the wheel motion by 90°. Such information is useful to designers for understanding the suspension behavior.

Figure 8.3.12 Frequency response of a two-mass suspension model.

8.4 FILTERING PROPERTIES OF DYNAMIC SYSTEMS

We have seen how a *low-pass filter* responds more to sinusoidal inputs having low frequencies. Similarly, a *high-pass filter* responds more to high-frequency inputs, while a *band-pass filter* responds to inputs having a frequency in the midrange. While the term "filter" is normally used to refer to electrical circuits, we may think of other system types as filters. For example, a mass-spring-damper system having a large damping ratio will act like a low-pass filter. If such a system represents a vehicle suspension, the passenger compartment will undergo negligible motion when the vehicle moves with high speed over closely spaced road surface variations.

It is useful to have a specific measure of the range of forcing frequencies to which a system is especially responsive. This single measure may be used as a design specification. The most common such measure is the *bandwidth*. It is commonly defined as the range of frequencies over which the power transmitted or dissipated by the system is no less than one-half of the peak power. For many systems the power transmitted or dissipated is proportional to M^2, where M is the magnitude of the frequency transfer function. For example, the power dissipated by a damper is proportional to the square of the amplitude of the velocity difference across the damper endpoints.

Thus, the bandwidth is commonly defined as the range of frequencies (ω_1, ω_2) over which

$$M^2(\omega_1) \leq \frac{M_{\text{peak}}^2}{2} \geq M^2(\omega_2)$$

or, equivalently,

$$M(\omega_1) \leq \frac{M_{\text{peak}}}{\sqrt{2}} \geq M(\omega_2) \tag{8.4.1}$$

For this reason, the lower and upper bandwidth points are called the "half-power" points.

EXAMPLE 8.4.1

Bandwidth of a First-Order Model

■ **Problem**

Consider the model $\tau \dot{v} + v = f(t)$, for which

$$M = \frac{V}{F} = \frac{1}{\sqrt{1 + \omega^2 \tau^2}}$$

Obtain an expression for the bandwidth in terms of τ and interpret its significance.

■ **Solution**

The peak in M occurs at $\omega = 0$ and is $M_{\text{peak}} = 1$. Thus $\omega_1 = 0$ and ω_2 is found from (8.4.1) as follows:

$$\frac{M_{\text{peak}}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = M(\omega_2) = \frac{1}{\sqrt{1 + \omega_2^2 \tau^2}}$$

This gives $\omega_2 = 1/\tau$. Thus the bandwidth of the model $\tau \dot{v} + v = f(t)$ is the range of frequencies $(0, 1/\tau)$.

Because a small time constant indicates a fast system, the bandwidth is another measure of the speed of response. So the faster the system, the larger the bandwidth, and some engineers describe a system's response speed in terms of its bandwidth rather than in terms of its time constant. However, for other models, the relation between the time constant and the bandwidth is not always so simple, as we will see.

When expressed in decibel units, $m = 20 \log M$, the bandwidth points are those points on the m curve that lie 3 dB below the peak value of m (because $20 \log(1/\sqrt{2}) = -3$ dB). For this reason, the bandwidth points are sometimes called the "3 dB" points.

EXAMPLE 8.4.2

Bandwidth of a Second-Order Model

■ **Problem**

Determine the expression for the bandwidth of the second-order model $m\ddot{x} + c\dot{x} + kx = f(t)$.

■ **Solution**

For this model, the amplitude ratio $M = X/F$ is

$$M = \frac{1}{k} \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{1}$$

where $r = \omega/\omega_n = \omega\sqrt{m/k}$. The peak value of M, denoted M_r, occurs when $0 \leq \zeta \leq 0.707$ and $r = \sqrt{1 - 2\zeta^2}$, and is

$$M_r = \frac{1}{k} \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

The value of r that makes $M(\omega) = M_r/\sqrt{2}$ is found from

$$\frac{1}{k} \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{1}{\sqrt{2}} \left(\frac{1}{k} \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \right)$$

This can be solved for r by squaring both sides and rearranging to obtain

$$r^4 + \left(4\zeta^2 - 2\right)r^2 + 1 - 8\zeta^2 + 8\zeta^4 = 0$$

The solution is

$$r = \sqrt{1 - 2\zeta^2 \pm 2\zeta \sqrt{1 - \zeta^2}} \tag{2}$$

If two positive, real solutions r_1 and r_2 of this equation exist, we can obtain the lower and upper bandwidth frequencies from $\omega_1 = r_1\omega_n$ and $\omega_2 = r_2\omega_n$. If one solution for r is imaginary and the other solution is positive, the lower bandwidth frequency is $\omega_1 = 0$ and the upper frequency ω_2 is obtained from the positive solution. This occurs when $\zeta > 0.382$. So, two half-power points exist only if $0 \leq \zeta \leq 0.382$.

Figure 8.4.1 shows several cases that can occur with various other models. In part (a) of the figure, the peak M_r is large enough so that $M_r/\sqrt{2} > M(0)$, and thus the lower bandwidth frequency ω_1 exists and is greater than zero. Such a system is called a *band-pass* system. Part (b) shows the plot of a *low-pass* system, for which $M(0) > M_r/\sqrt{2}$, so that $\omega_1 = 0$. This occurs for the model $m\ddot{x} + c\dot{x} + kx = f(t)$ when $\zeta > 0.382$. Part (c) shows a case where the upper bandwidth frequency is infinite. Such a system is called a *high-pass* system because it responds more to high-frequency inputs. Part (d) shows a magnitude ratio plot with two peaks. This can occur only with a model of fourth or higher order. Depending on the exact shape of the plot, such a system can have two bandwidths, one for each peak.

ALTERNATIVE DEFINITION OF BANDWIDTH

In the definition of bandwidth specified by (8.4.1), the power transmitted by an input having a frequency outside the bandwidth is less than one-half the power transmitted by an input whose frequency corresponds to M_{peak}, *assuming* that both inputs have the *same* amplitude. However, often the amplitudes of low-frequency forcing functions are larger than those of high-frequency inputs, so the low-frequency inputs might account for more power. Thus inputs whose frequencies lie below the lower bandwidth frequency ω_1
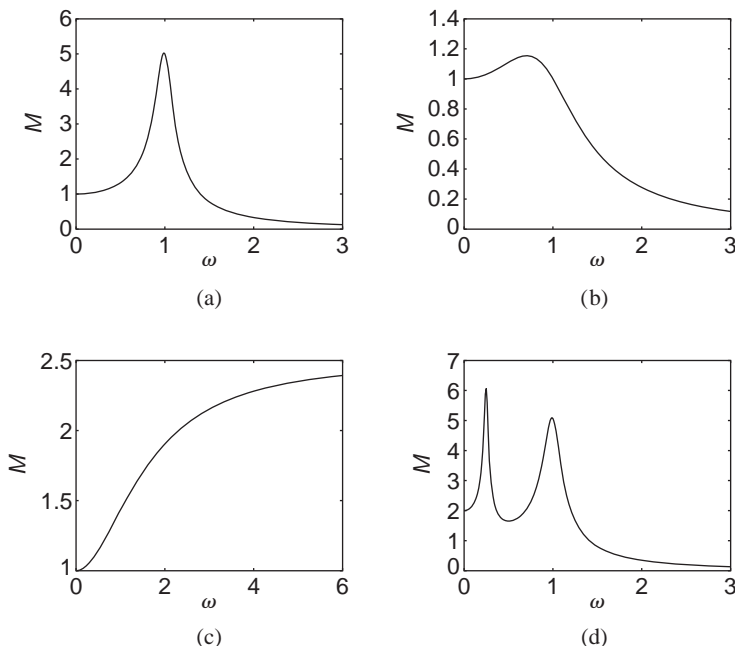
Figure 8.4.1 Some possible frequency response plots.

Figure 8.4.2 Two definitions of bandwidth.
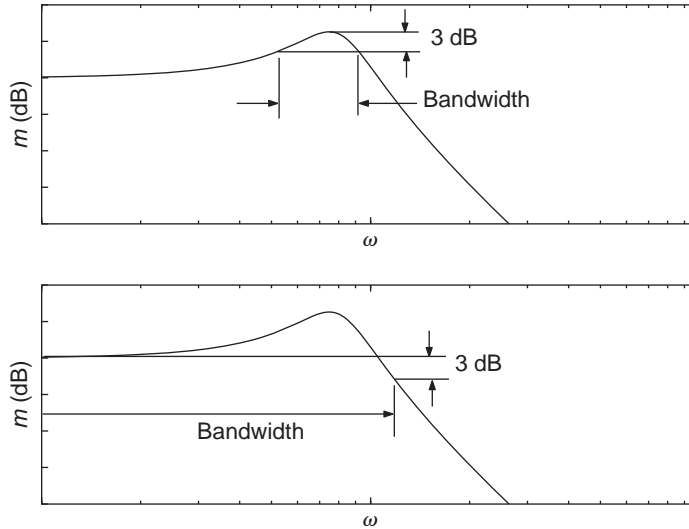
may contribute significantly to the response, and cannot be neglected. Because of this, a modified definition of bandwidth is sometimes used. With this alternative definition of bandwidth, the lower bandwidth frequency ω_1 is assumed to be zero and the upper bandwidth frequency ω_2 is defined to be that frequency at which the power is one-half of the power at *zero* frequency (see Figure 8.4.2).

The two definitions give the same bandwidth for those systems whose peak value of M is $M(0)$. For some systems, however, this alternative definition of bandwidth cannot be applied.[1] For example, with the transfer function of a high-pass filter,

$$T(s) = \frac{\tau s}{\tau s + 1}$$

the value of $M(0)$ is zero and thus $M(0)$ is not the peak value of M. In fact, $m(0) = -\infty$ (see Figure 8.1.9). On the other hand, the bandwidth definition (8.4.1) gives the filter bandwidth to be $1/\tau \leq \omega \leq \infty$, which is a useful and physically meaningful result. Therefore, unless explicitly stated otherwise, we will use the bandwidth definition specified by (8.4.1).

EARPHONE FREQUENCY RESPONSE AND HUMAN HEARING

The peak in the frequency response curve is not always used as a measure of system performance. For example, earphones should be designed so that the magnitude ratio is 1 ($m = 0$ dB) over the entire range of frequencies detectable by the human ear. This range is normally from about 20 Hz to 20,000 Hz. Consequently, a measure of earphone performance is the amount of deviation of its m curve from 0 dB over this frequency range.

Often earphone response is considered acceptable if its m curve deviates from the 0 db line by no more than ± 3 dB. Figure 8.4.3 shows an experimentally determined frequency response curve of a particular earphone supplied with a handheld music player. We see that the earphone amplifies signals in the 20–400 Hz and the 5,000–8,000 Hz ranges, whereas it attenuates most signals above 9,000 Hz.

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[1]The MATLAB function `bandwidth` uses this alternative definition and thus can give meaningless results.
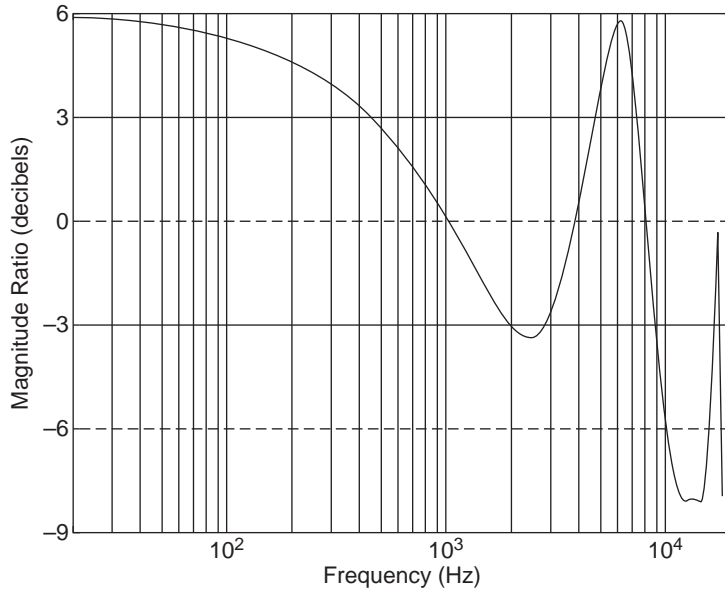
Figure 8.4.3 Frequency response of an earphone.

Thus, this earphone distorts bass and treble tones but reproduces well tones in the mid-range of frequencies.

Superposition and Forced Response | **EXAMPLE 8.4.3**

■ **Problem**

The model of a series RC circuit is

$$RC\dot{v} + v = v_s$$

Suppose the input voltage is the following sum of a constant and two sine waves.

$$v_s(t) = 10 + 5 \sin t + 3 \sin 6t$$

Obtain the expression for the steady-state response if the time constant is $\tau = RC = 0.5$ s.

■ **Solution**

The total forced response at steady state is the sum of the steady-state forced responses due to each of the three terms in $v_s(t)$. For the constant term, 10, the steady-state response is $v = 10$. The magnitude ratio and phase angle for this model are

$$M(\omega) = \frac{1}{\sqrt{1 + \tau^2\omega^2}} = \frac{1}{\sqrt{1 + 0.25\omega^2}}$$

$$\phi(\omega) = -\tan^{-1}(\tau\omega) = -\tan^{-1}(0.5\omega)$$

For the input term $5 \sin t$, $\omega = 1$, $M(1) = 1/\sqrt{1.25} = 2/\sqrt{5}$, and $\phi(1) = -\tan^{-1}(0.5) = -0.464$ rad. For the input term $3 \sin 6t$, $\omega = 6$, $M(6) = 1/\sqrt{10}$, and $\phi(6) = -\tan^{-1}(3) = -1.25$ rad. Thus the steady-state response is

$$v_{ss}(t) = 10 + 5M(1) \sin[t + \phi(1)] + 3M(6) \sin[6t + \phi(6)]$$

$$= 10 + \frac{10}{\sqrt{5}} \sin(t - 0.464) + \frac{3}{\sqrt{10}} \sin(6t - 1.25)$$

$$= 10 + 4.47 \sin(t - 0.464) + 0.948 \sin(6t - 1.25)$$

Figure 8.4.4 Steady-state
response of $0.5\dot{v} + v = 10 + 5\sin t + 3\sin 6t$.
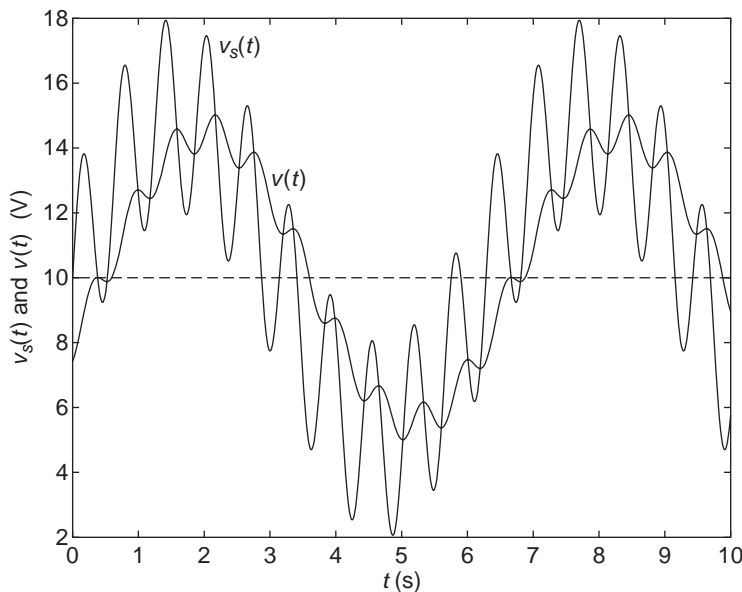
Note that the amplitude of the last sine term is much less than that of the first sine term, whose
amplitude is smaller than the constant term 10. The circuit does not filter the constant term at
all, filters the lower-frequency term somewhat, and heavily filters the higher frequency term.
Figure 8.4.4 shows the input and the response.

RESPONSE TO GENERAL PERIODIC INPUTS

The application of the sinusoidal input response is not limited to cases involving a
single sinusoidal input. A basic theorem of analysis states that under some assumptions,
which are generally satisfied in most practical applications, any periodic function can be
expressed by a constant term plus an infinite series of sines and cosines with increasing
frequencies. This theorem is the *Fourier theorem,* and its associated series is the *Fourier
series.* The series has the form

$$f(t) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi t}{p}\right) + a_2 \cos\left(\frac{2\pi t}{p}\right) + \cdots$$

$$+ \; b_1 \sin\left(\frac{\pi t}{p}\right) + b_2 \sin\left(\frac{2\pi t}{p}\right) + \cdots \tag{8.4.2}$$

where $f(t)$ is the periodic function and p is the half-period of $f(t)$. The constants
a_i and b_i are determined by integration formulas applied to $f(t)$. These formulas are
given in Appendix B.

We have seen in Example 8.4.3 how to determine the steady-state response of a
linear system when subjected to an input that is a constant, a sine, or a cosine. When the
input $f(t)$ is periodic and expressed in the form of (8.4.2), the superposition principle
states that the complete steady-state response is the sum of the steady-state responses
due to each term in (8.4.2). Although this is an infinite series, in practice we have to
deal with only a few of its terms, because those terms whose frequencies lie outside the
system's bandwidth can be neglected as a result of the filtering property of the system.

<div align="right">

EXAMPLE 8.4.4

</div>

Response to Nonsinusoidal Inputs

■ Problem

An engine valve train can be modeled as an equivalent mass, equivalent damping, and two stiffnesses, one due to the valve spring and one due to the elasticity of the push rod (Figure 8.4.5). The equation of motion is

$$\frac{I_o}{a^2}\ddot{x} + c_e\dot{x} + \left(k_1 + k_2\frac{a^2}{b^2}\right)x = \frac{a}{b}k_2 y(t)$$

The input displacement $y(t)$ is determined by the shape and rotational speed of the cam. Suppose the input is as shown in Figure 8.4.6a. Determine the steady-state response of the model for the following parameter values:

$$\ddot{x} + 20\dot{x} + 625x = 600y(t)$$
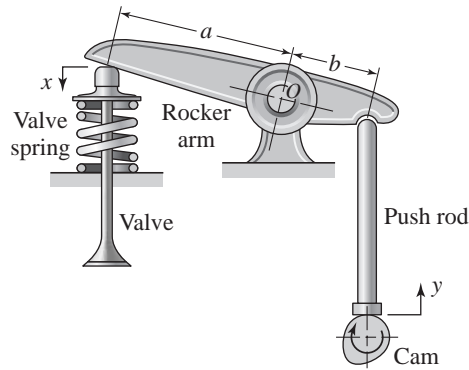
Figure 8.4.5 A valve system.
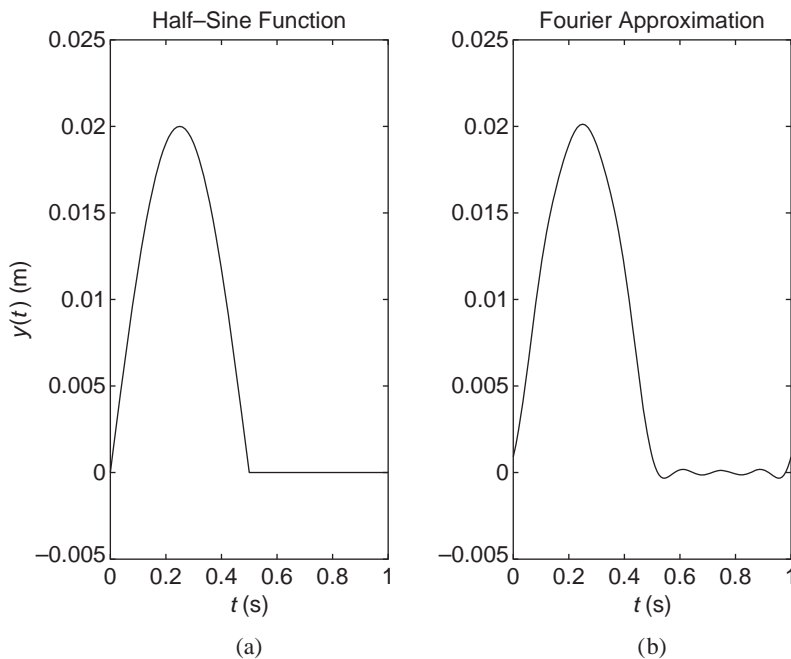
Figure 8.4.6 (a) Half-sine function. (b) Fourier series approximation.

(a)

(b)

■ Solution

The transfer function is

$$T(s) = \frac{X(s)}{Y(s)} = \frac{600}{s^2 + 20s + 625}$$

and

$$T(j\omega) = \frac{600}{625 - \omega^2 + 20\omega j}$$

$$M(\omega) = \frac{600}{\sqrt{\left(625 - \omega^2\right)^2 + 400\omega^2}} \tag{1}$$

$$\phi(\omega) = -\tan^{-1}\frac{20\omega}{625 - \omega^2} \tag{2}$$

The Fourier series' representation of $y(t)$ can be determined from the formulas in Appendix B, and is

$$y(t) = A_0 + A_1 \sin\omega_1 t + A_2 \cos\omega_2 t + A_3 \cos\omega_3 t + A_4 \cos\omega_4 t + \cdots$$

$$= 0.02\left[\frac{1}{\pi} + \frac{1}{2}\sin 2\pi t - \frac{2}{\pi}\left(\frac{\cos 4\pi t}{1(3)} + \frac{\cos 8\pi t}{3(5)} + \frac{\cos 12\pi t}{5(7)} + \cdots\right)\right] \tag{3}$$

Figure 8.4.6b is a plot of equation (3) including only those series terms shown. The plot illustrates how well the Fourier series represents the input function.

The steady-state response will have the form

$$x(t) = B_0 + B_1 \sin(\omega_1 t + \phi_1) + B_2 \cos(\omega_2 t + \phi_2) + B_3 \cos(\omega_3 t + \phi_3) + B_4 \cos(\omega_4 t + \phi_4) + \cdots$$

where

$$B_i = A_i M_i \tag{4}$$

Note that the amplitudes M_i and phases ϕ_i of the response due to cosine inputs are computed just as for a sine input.

Using equation (2) of Example 8.4.2, we obtain one positive solution, $r = 1.187$, and one imaginary solution. Thus, the lower bandwidth frequency is 0, and the upper frequency is $\omega_2 = 1.187\sqrt{625} = 29.7$ rad/sec. Because the bandwidth is from 0 to 29.7 rad/sec, the only series terms in $y(t)$ lying within the bandwidth are the constant term (whose frequency is 0), and the $\sin 2\pi t$, $\cos 4\pi t$, and $\cos 8\pi t$ terms, whose frequencies are 6.28, 12.6, and 25.1 rad/sec, respectively. To demonstrate the filtering property of the system, however, we also compute the effect of the first term lying outside the bandwidth. This is the $\cos 12\pi t$ term, whose frequency is 37.7 rad/sec. The following table was computed using equations (1), (2), (3), and (4).

i	ω_i	A_i	M_i	B_i	ϕ_i
0	0	0.006366	0.96	0.006112	0
1	2π	0.01	1.001913	0.010019	−0.211411
2	4π	−0.004244	1.1312	−0.004801	−0.493642
3	8π	−0.000849	1.193557	−0.001013	−1.584035
4	12π	−0.000364	0.547162	−0.000199	−2.383436

Using this table we can express the steady-state response as follows:

$$x(t) = 0.006112 + 0.010019\sin(2\pi t - 0.211411)$$

$$- 0.004801\cos(4\pi t - 0.493642) - 0.001013\cos(8\pi t - 1.584035)$$

$$- 0.000199\cos(12\pi t - 2.383436)$$
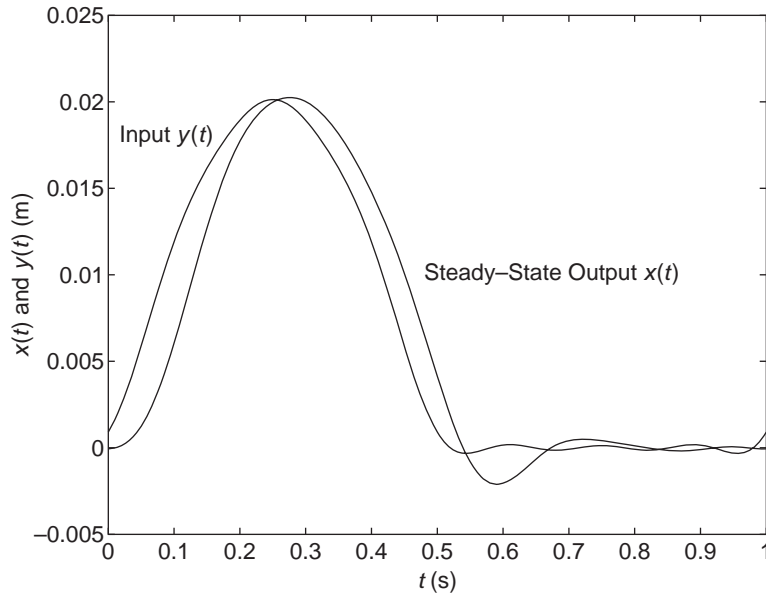
Figure 8.4.7 Steady-state response to a half-sine input.

The input and the response are plotted in Figure 8.4.7. The difference between the input and output results from the resistive or lag effect of the system, not from the omission of the higher-order terms in the series. To see this, note that A_4 is 43% of A_3 but B_4 is only 20% of B_3. The decreasing amplitude of the higher-order terms in the series for $y(t)$, when combined with the filtering property of the system, enables us to truncate the series when the desired accuracy has been achieved.

8.5 SYSTEM IDENTIFICATION FROM FREQUENCY RESPONSE

In cases where a transfer function or differential equation model is difficult to derive from general principles, or where the model's coefficient values are unknown, often an experimentally obtained frequency response plot can be used to determine the form of an appropriate model and the values of the model's coefficients.

TEST PROCEDURES

Often a sinusoidal input is easier to apply to a system than a step input, because many devices, such as ac circuits and rotating machines, naturally produce a sinusoidal signal or motion. If a suitable apparatus can be devised to provide a sinusoidal input of adjustable frequency, then the system's output amplitude and phase shift relative to the input can be measured for various input frequencies. When these data are plotted on the logarithmic plot for a sufficient frequency range, the form of the model can often be determined. This procedure is easiest for systems with electrical inputs and outputs, because for these systems, variable-frequency oscillators and frequency response analyzers are commonly available. Some of these can automatically sweep through a range of frequencies and plot the decibel and phase angle data. For nonelectrical outputs, such as a displacement, a suitable transducer can be used to produce an electrical measurement that can be analyzed.

Another advantage of frequency response tests is that they can often be applied to a system or process without interrupting its normal operations. A small-amplitude sinusoidal test signal is superimposed on the operating inputs, and the sinusoidal component of the output having the input's frequency is subtracted from the output measurements. Specialized computer algorithms and test equipment are available for this purpose.

USE OF ASYMPTOTES AND CORNER FREQUENCIES

When using frequency response data for identification, it is important to understand how to reconstruct a transfer function form from the asymptotes and corner frequencies. The slopes of the asymptotes can be used to determine the orders of the numerator and the denominator. The corner frequencies can be used to estimate time constants and natural frequencies.

All real data will have some "scatter," and thus a perfect model fit will not be practical. However, to illustrate the methods clearly, in the following examples we use data that have very little scatter, and thus the derived models are unambiguous.

EXAMPLE 8.5.1

Identifying a First-Order System

■ Problem

An input $v_s(t) = 5 \sin \omega t$ V was applied to a certain electrical system for various values of the frequency ω, and the amplitude $|v_o|$ of the steady-state output was recorded. The data are shown in the first two columns of the following table. Determine the transfer function.

ω (rad/s)	$|v_o|$ (V)	$|v_o|/5$	$20 \log (|v_o|/5)$
1	10.95	2.19	6.81
2	10.67	2.13	6.57
3	10.3	2.06	6.28
4	9.84	1.97	5.89
5	9.33	1.87	5.44
6	8.80	1.76	4.91
7	8.28	1.66	4.40
8	7.782	1.56	3.86
9	7.31	1.46	3.29
10	6.87	1.37	2.73
15	5.18	1.04	0.34
20	4.09	0.82	−1.72
30	2.83	0.57	−4.88
40	2.16	0.43	−7.33
50	1.74	0.35	−9.12
60	1.45	0.29	−10.75
70	1.25	0.25	−12.04
80	1.10	0.22	−13.15
90	0.97	0.19	−14.43
100	0.88	0.18	−14.89

■ Solution

First divide the output amplitude $|v_o|$ by the amplitude of the input. The result is given in the third column. This is the amplitude ratio M. Next convert this data to decibels using the conversion
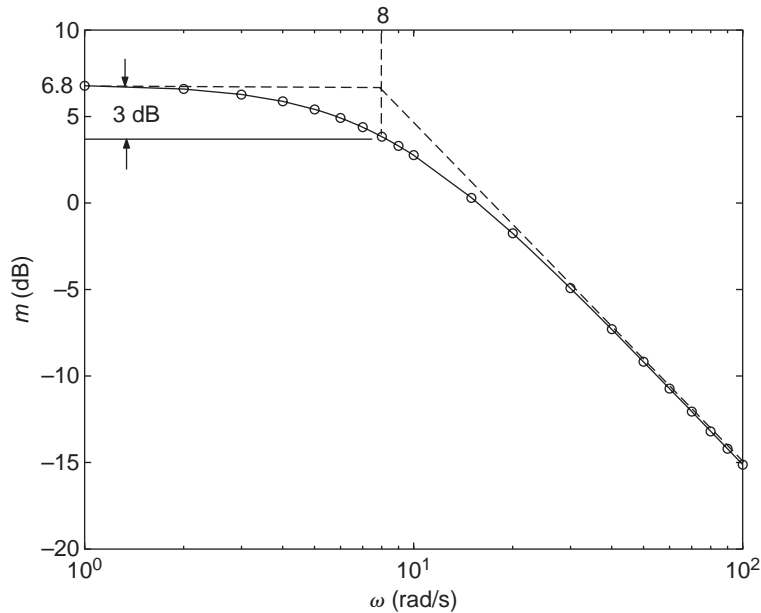
Figure 8.5.1 Frequency response data for Example 8.5.1.

$m = 20 \log M$. This is the fourth column. The plot of m versus ω is shown by the small circles in Figure 8.5.1.

After drawing the asymptotes shown by the dashed lines, we first note that the data has a low-frequency asymptote of zero slope, and a high-frequency asymptote of slope -20 dB/decade.

This suggests a model of the form

$$T(s) = \frac{K}{\tau s + 1}$$

In decibel units,

$$m = 20 \log K - 10 \log(\tau^2 \omega^2 + 1)$$

The corner frequency $\omega = 1/\tau$ occurs where m is 3 dB below the peak value of 6.81. From the plot or the data we can see that the corner frequency is $\omega = 8$ rad/s. Thus $\tau = 1/8$ s.

At low frequencies, $\omega \ll 1/\tau$ and $m \approx 20 \log K$. From the plot, at low frequency, $m = 6.81$ dB. Thus $6.81 = 20 \log K$, which gives

$$K = 10^{6.81/20} = 2.19$$

Thus the estimated model is

$$\frac{V_o(s)}{V_s(s)} = \frac{2.19}{\frac{1}{8}s + 1}$$

Another first-order model form is

$$\frac{V_o(s)}{V_s(s)} = K \frac{\tau_1 s + 1}{\tau_2 s + 1}$$

Because the high-frequency asymptote of this model has zero slope, it cannot describe the given data.

EXAMPLE 8.5.2

Identifying a Second-Order System

■ Problem

Measured response data are shown by the small circles in Figure 8.5.2. Determine the transfer function.

■ Solution

After drawing the asymptotes shown by the dashed lines, we first note that the data have a low-frequency asymptote of zero slope, and a high-frequency asymptote of slope -40 dB/decade. This suggests a second-order model without numerator dynamics, either of the form having real roots:

$$T(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

or the form having complex roots:

$$T(s) = \frac{K}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

However, the peak in the data eliminates the form having real roots.

At low frequencies, $m \approx 20 \log K$. From the plot, at low frequencies, $m = 75$ dB. Thus $75 = 20 \log K$, which gives

$$K = 10^{75/20} = 5623$$

The peak is estimated to be 83 dB. From Table 8.2.1 the peak when $K = 1$ is given by $m_r = -20 \log(2\zeta \sqrt{1 - \zeta^2})$. Thus with $K = 5623$ the formula for the peak becomes

$$m_r = 20 \log 5623 - 20 \log \left(2\zeta \sqrt{1 - \zeta^2} \right)$$

or

$$83 = 75 - 20 \log \left(2\zeta \sqrt{1 - \zeta^2} \right)$$

Figure 8.5.2 Frequency response data for Example 8.5.2.
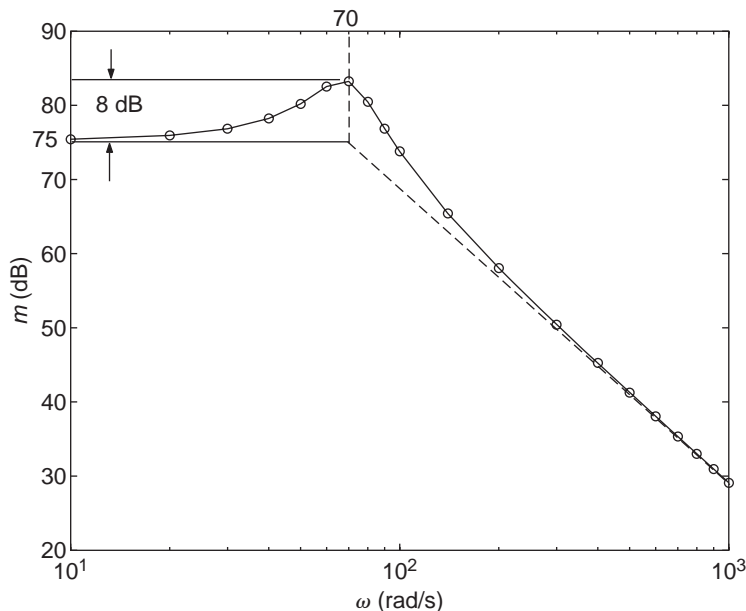
Thus,

$$\log\left(2\zeta\sqrt{1-\zeta^2}\right) = \frac{75-83}{20} = -0.4$$

and

$$2\zeta\sqrt{1-\zeta^2} = 10^{-0.4}$$

Solve for ζ by squaring both sides.

$$4\zeta^2\left(1-\zeta^2\right) = 10^{-0.8}$$

$$4\zeta^4 - 4\zeta^2 + 10^{-0.8} = 0$$

This gives $\zeta^2 = 0.9587$ and 0.0413. The positive solutions are $\zeta = 0.98$ and 0.2. Because there is a resonance peak in the data, the first solution is not valid, and we obtain $\zeta = 0.2$.

Knowing ζ, we can now estimate ω_n from the peak frequency, which is estimated to be $\omega_r = 70$ rad/s. Thus from Table 8.2.1, $\omega_r = \omega_n\sqrt{1-2\zeta^2}$, or

$$70 = \omega_n\sqrt{1-2(0.2)^2}$$

This gives $\omega_n = 73$ rad/s.

Thus, the estimated model is

$$T(s) = \frac{5623}{s^2 + 29.2s + 5329}$$

Application of the Phase Plot | **EXAMPLE 8.5.3**

■ Problem

Consider the experimentally determined plots shown in Figure 8.5.3. Determine the forms of the transfer function.
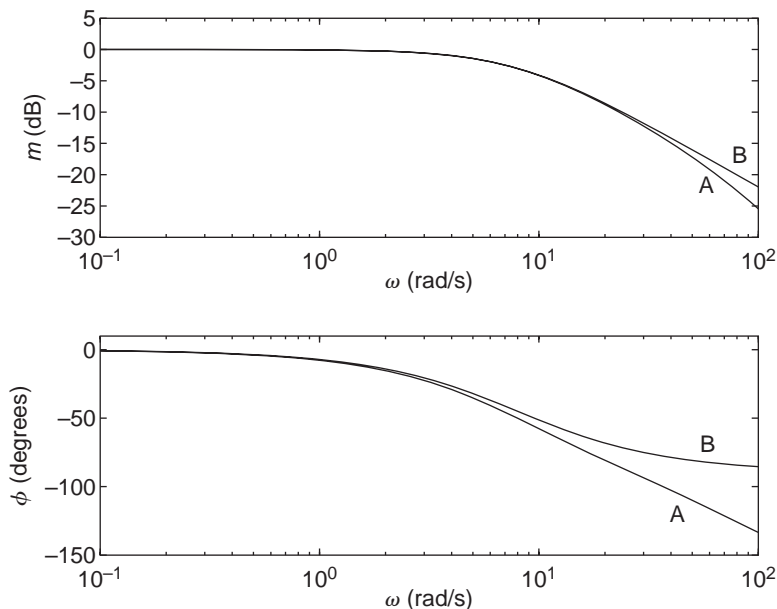
Figure 8.5.3 Frequency response data for Example 8.5.3.

■ Solution

At low frequencies both magnitude curves start at 0 dB and have zero slope. Therefore, the numerator of both transfer functions is 1 at low frequencies. Both magnitudes drop by 3 dB near $\omega = 8$ rad/s, and both phase curves pass near $-45°$ near $\omega = 8$ rad/s. Thus, we conclude that each transfer function has a denominator term $\tau s + 1$ where $\tau = 1/8$ s, approximately.

The high-frequency slope of curve B is -20 dB/decade, and thus we conclude that its transfer function is

$$T_B(s) = \frac{1}{\frac{1}{8}s + 1}$$

The high-frequency slope of curve A is steeper, and thus we conclude that its transfer function has another denominator term that contributes to the slope at high frequencies. This is more apparent in the phase plot, where ϕ_B becomes horizontal but ϕ_A appears to be heading toward $-180°$. Thus we suspect that its transfer function has the form

$$T_A(s) = \frac{1}{\left(\frac{1}{8}s + 1\right)(\tau_2 s + 1)}$$

but we cannot confirm this or determine the value of τ_2 without data at frequencies higher than 10^2 rad/s.

8.6 FREQUENCY RESPONSE ANALYSIS USING MATLAB

The logarithmic frequency response plots are sometimes called "Bode" plots, and are named after H. W. Bode, who developed frequency response methods for electronic circuit design. MATLAB provides the `bode`, `bodemag`, `evalfr`, and `freresp` functions to compute and plot frequency response. These functions are available in the Control Systems Toolbox.

THE `bode` FUNCTION

The `bode` function generates frequency response plots. The `bodemag(sys)` function plots the magnitude only. The basic syntax of the `bode` function is `bode(sys)`, where `sys` is an LTI system model. Such a model can be created with the `tf` function or the `ss` function. In this basic syntax the `bode` function produces a screen plot of the logarithmic magnitude ratio m and the phase angle ϕ versus ω. Note that the phase angle is plotted in *degrees,* not radians. MATLAB uses the corner frequencies of the numerator and denominator of the transfer function to automatically select an appropriate frequency range for the plots. For example, to obtain the plots for the transfer function

$$T(s) = \frac{70}{3s^2 + 4s + 192} \tag{8.6.1}$$

the session is

```
≫sys = tf(70,[3, 4, 192]);
≫bode(sys)
```

Figure 8.6.1 shows the plots you will see on the screen.

If you created the LTI model from a state variable model that has more than one input or more than one output, then the `bode` function will display a series of plots, one pair for each input-output combination.
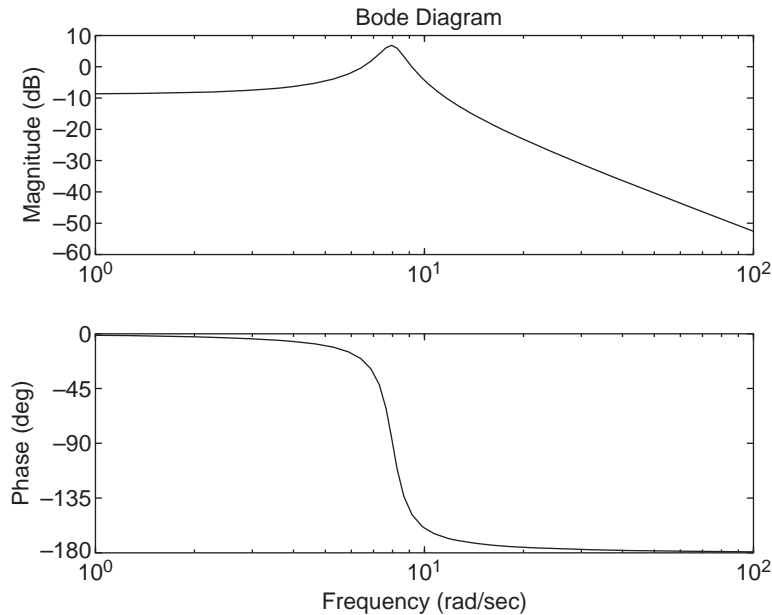
Figure 8.6.1 Plots produced with the MATLAB `bode` function.

SPECIFYING THE FREQUENCY RANGE

To specify a set of frequencies rather than letting MATLAB select them, you can use the syntax `bode(sys,w)`, where w is a vector containing the desired frequencies. The frequencies must be specified in *radians* per unit time (for example, radians/second). For example, you might want to examine more closely the peak in the *m* curve near the resonant frequency of (8.6.1). To see 401 regularly spaced points on the curve for $7 \leq \omega \leq 9$, the session is

```
≫sys = tf(70,[3, 4, 192]);
≫w = (7:0.005:9);
≫bode(sys,w)
```

To obtain 401 points in the range $7 \leq \omega \leq 9$ that are logarithmically spaced, type instead `w = logspace(log10(7),log10(9),401)`.

If you want to specify just the lower and upper frequencies `wmin` and `wmax`, and let MATLAB select the spacing, generate the vector w by typing `w = {7, 9}`.

A variation is the following: `[mag, phase, w] = bode(sys)`. This form returns the magnitude ratio M, the phase angle ϕ, and the frequencies ω in the arrays `mag`, `phase`, and `w`, but does not display a plot. It is very important to note that the array `mag` contains the ratio M, and *not* the logarithmic ratio *m*. Note also that the array `phase` contains the phase angle in *degrees,* not radians. However, the vector w contains the frequencies in *radians* per unit time. The frequencies in the vector w are the frequencies automatically selected by MATLAB. If you want to use your own set of frequencies, use the syntax: `[mag, phase] = bode(sys,w)`. The extended syntax of the `bodemag` function is similar.

These two forms are useful when you want to save the frequency response calculations generated by the `bode` function. You can use the arrays `mag`, `phase`, and w for further analysis, or to generate plots that you can format as you want. If you do this note that the variables `mag` and `phase` are returned as $1 \times 1 \times n$ arrays. To collapse the values into a $1 \times n$ array suitable for plotting, type `mag = mag(:)` and `phase = phase(:)`.

For example, to plot M versus ω for (8.6.1) using 401 regularly spaced points over the range $7 \leq \omega \leq 9$, the session is

```
≫sys = tf(70,[3, 4, 192]);
≫w = (7:0.005:9);
≫[mag] = bode(sys,w); mag = mag(:);
≫plot(w,mag),xlabel('\omega (rad/sec)'),ylabel('M')
```

Note that the `phase` array need not be generated if you are not going to use it.

To plot the logarithmic magnitude ratio m versus $\log \omega$, using the frequencies generated by MATLAB, the session is

```
≫sys = tf(70,[3, 4, 192]);
≫[mag, phase, w] = bode(sys); mag = mag(:);
≫semilogx(w,20*log10(mag)),grid,...
≫xlabel('\omega (rad/sec)'),ylabel('m (dB)')
```

Note that you must use the decibel conversion $m = 20 \log M$.

Both the `bodemag` and the `bode` functions can be used to obtain the curves for several transfer functions on the same plot by using the syntax `bode(sys1, sys2, ...)` or `bodemag(sys1, sys2, ...)`.

OBTAINING INFORMATION FROM THE PLOT

Once the plot is displayed on the screen, right-clicking on it brings up a menu. One choice on the menu is "Characteristics." Left-clicking on this choice brings up a submenu, one of whose choices is "Peak Response." When this is selected the peak of the magnitude curve is identified with a dot, and the corresponding frequency is displayed. Left-clicking on the curve enables you to trace the curve with the cursor while the coordinates are displayed. With this method you can determine the bandwidth. For the transfer function (8.6.1), with the plot covering the frequency range $7 \leq \omega \leq 9$, the peak is found to be $m = 6.83$ dB at $\omega = 7.94$ rad/sec, and the lower and upper bandwidth frequencies are found to be 7.25 and 8.59 rad/sec.

THE `evalfr` FUNCTION

The `evalfr` function stands for "evaluation of frequency response." It computes the complex value of a transfer function at a specified value of s. For example, to compute the value of $T(s)$ given by (8.6.1) at $s = 6j$, and find its magnitude and phase angle, the session is

```
≫fr = evalfr(sys,6j);
≫mag = abs(fr);
≫ang = angle(fr);
```

The results are `fr` $= 0.7704 - 0.2201j$, `mag` $= 0.8013$, and `phase` $= -0.2783$ rad.

THE `freqresp` FUNCTION

The `evalfr` function can be used to evaluate `sys` at one frequency only. For a vector of frequencies, use the `freqresp` function, whose syntax is `fr = freqresp(sys,w)`, where `w` is a vector of frequencies, which must be in radians/unit time.

The following MATLAB script file calculates the coefficients and phase angles of the steady-state response described in Example 8.4.4.

```
sys1 = tf(600, [1, 20, 625]);
w = [0, 2*pi, 4*pi, 8*pi, 12*pi];
A = 0.02*[1/pi, 1/2, -2/(3*pi), -2/(3*5*pi), -2/(5*7*pi)];
fr = freqresp(sys1,w);
M = abs(fr);
B = A.*M(:)'
ph = angle(fr(:))
```

The results are shown in the table in Example 8.4.4.

8.7 CHAPTER REVIEW

The frequency response of a system describes its steady-state behavior resulting from periodic inputs. This chapter demonstrates the usefulness of the transfer function for understanding and analyzing a system's frequency response.

A sinusoidal input applied to a stable linear system produces a steady-state sinusoidal output of the same frequency, but with a different amplitude and a phase shift. The frequency transfer function is the transfer function with the Laplace variable s replaced by $j\omega$, where ω is the input frequency. The magnitude M of the frequency transfer function is the amplitude ratio between the input and output, and its argument is the phase shift ϕ. Both are functions of ω and usually plotted against $\log \omega$, with M expressed in decibels as $m = 20 \log M$. The logarithmic plots can be sketched by using low- and high-frequency asymptotes intersecting at the corner frequencies.

Most transfer functions consist of a combination of the following terms: K, s, $\tau s + 1$, and $s^2 + 2\zeta \omega_n s + \omega_n^2$, so we analyzed the frequency response of each term. The composite frequency response plot consists of the sum of the plots of the numerator terms minus the sum of the plots of the denominator terms.

Important engineering applications of frequency response covered in this chapter are an understanding of resonance and the design of isolators to minimize the transmission of motion between a base and a machine or to minimize the effects of rotating unbalance on a supporting structure.

The bandwidth measures the filtering property of the system. This concept can be used with the Fourier series representation of a periodic function, which is an infinite series, by eliminating those terms that lie outside the bandwidth. The truncated series can be used to compute the response to a general periodic input.

The form of the transfer function and an estimate of the numerical values of the parameters can often be obtained from experiments involving frequency response.

The MATLAB Control Systems toolbox provides the `bode`, `bodemag`, `evalfr`, and `freresp` functions, which are useful for frequency response analysis.

Now that you have finished this chapter, you should be able to

1. Sketch frequency response plots using asymptotes, and use the plots or the frequency transfer function to determine the steady-state output amplitude and phase that results from a sinusoidal input.
2. Compute resonance frequencies and bandwidth.
3. Analyze vibration isolation systems and the effects of base motion and rotating unbalance.

4. Determine the steady-state response to a periodic input, given the Fourier series description of the input.
5. Estimate the form of a transfer function and its parameter values, given frequency response data.
6. Use MATLAB as an aid in the preceding tasks.

PROBLEMS

Section 8.1 Frequency Response of First-Order Systems

8.1   Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a.
$$T(s) = \frac{Y(s)}{F(s)} = \frac{25}{14s + 18}, \quad f(t) = 15 \sin 1.5t$$

b.
$$T(s) = \frac{Y(s)}{F(s)} = \frac{15s}{3s + 4}, \quad f(t) = 5 \sin 2t$$

c.
$$T(s) = \frac{Y(s)}{F(s)} = \frac{s + 50}{s + 150}, \quad f(t) = 3 \sin 100t$$

d.
$$T(s) = \frac{Y(s)}{F(s)} = \frac{33}{200} \frac{s + 100}{s + 33}, \quad f(t) = 8 \sin 50t$$

8.2   Use asymptotic approximations to sketch the frequency response plots for the following transfer functions.

a.
$$T(s) = \frac{15}{6s + 2}$$

b.
$$T(s) = \frac{9s}{8s + 4}$$

c.
$$T(s) = 6\frac{14s + 7}{10s + 2}$$

8.3   Figure P8.3 is a representation of the effects of the tide on a small body of water connected to the ocean by a channel. Assume that the ocean height h_i varies sinusoidally with a period of 12 hr with an amplitude of 3 ft about a mean height of 10 ft. If the observed amplitude of variation of h is 2 ft, determine the time constant of the system and the time lag between a peak in h_i and a peak in h.

Figure P8.3
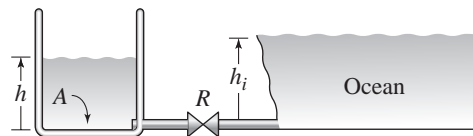
8.4   A single-room building has four identical exterior walls, 5 m wide by 3 m high, with a perfectly insulated roof and floor. The thermal resistance of the walls is $R = 4.5 \times 10^{-3}$ K/W · m². Taking the only significant thermal capacitance to be the room air, obtain the expression for the steady-state room air temperature if the outside air temperature varies sinusoidally about 15°C with an amplitude of 5° and a period of 24 h. The specific heat and density of air at these conditions are $c_p = 1004$ J/kg · K and $\rho = 1.289$ kg/m³.

8.5   For the rotational system shown in Figure P8.5, $I = 2 \text{ kg} \cdot \text{m}^2$ and $c = 4$ $\text{N} \cdot \text{m} \cdot \text{s/rad}$. Obtain the transfer function $\Omega(s)/T(s)$, and derive the expression for the steady-state speed $\omega_{ss}(t)$ if the applied torque in $\text{N} \cdot \text{m}$ is given by

$$T(t) = 30 + 5 \sin 3t + 2 \cos 5t$$

8.6   For the system shown in Figure P8.6, the bottom area is $A = 4\pi \text{ ft}^2$ and the linear resistance is $R = 1500 \text{ sec}^{-1}\text{ft}^{-1}$. Suppose the volume inflow rate is

$$q_{vi}(t) = 0.2 + 0.1 \sin 0.002t \text{ ft}^3/\text{sec}$$

Obtain the expression for the steady-state height $h_{ss}(t)$. Compute the lag in seconds between a peak in $q_{vi}(t)$ and a peak in $h_{ss}(t)$.

Figure P8.5
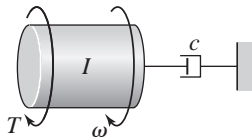
Figure P8.6
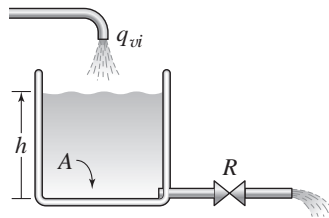
8.7   For the circuit shown in Figure 8.1.11a, can values be found for R_1, R_2, and C to make a low-pass filter? Prove your answer mathematically.

Section 8.2 Frequency Response of Higher-Order Systems

8.8   Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a. $$T(s) = \frac{Y(s)}{F(s)} = \frac{10}{(10s+1)(4s+1)}, \qquad f(t) = 10 \sin 0.2t$$

b. $$T(s) = \frac{Y(s)}{F(s)} = \frac{1}{2s^2 + 20s + 200}, \qquad f(t) = 16 \sin 5t$$

8.9   Use the following transfer functions to find the steady-state response $y_{ss}(t)$ to the given input function $f(t)$.

a. $$T(s) = \frac{Y(s)}{F(s)} = \frac{8}{s(s^2 + 10s + 100)}, \qquad f(t) = 6 \sin 9t$$

b. $$T(s) = \frac{Y(s)}{F(s)} = \frac{10}{s^2(s+1)}, \qquad f(t) = 9 \sin 2t$$

c. $$T(s) = \frac{Y(s)}{F(s)} = \frac{s}{(2s+1)(5s+1)}, \qquad f(t) = 9 \sin 0.7t$$

d. $$T(s) = \frac{Y(s)}{F(s)} = \frac{s^2}{(2s+1)(5s+1)}, \qquad f(t) = 9 \sin 0.7t$$

8.10 The model of a certain mass-spring-damper system is

$$10\ddot{x} + c\dot{x} + 20x = f(t)$$

Determine its resonant frequency ω_r and its peak magnitude M_r if (a) $\zeta = 0.1$ and (b) $\zeta = 0.3$.

8.11 The model of a certain mass-spring-damper system is

$$10\ddot{x} + c\dot{x} + 20x = f(t)$$

How large must the damping constant c be so that the maximum steady-state amplitude of x is no greater than 3, if the input is $f(t) = 11\sin\omega t$, for an arbitrary value of ω?

8.12 The model of a certain mass-spring-damper system is

$$13\ddot{x} + 2\dot{x} + kx = 10\sin\omega t$$

Determine the value of k required so that the maximum response occurs at $\omega = 4$ rad/sec. Obtain the steady-state response at that frequency.

8.13 Determine the resonant frequencies of the following models.

a.
$$T(s) = \frac{7}{s(s^2 + 6s + 58)}$$

b.
$$T(s) = \frac{7}{(3s^2 + 18s + 174)(2s^2 + 8s + 58)}$$

Figure P8.14
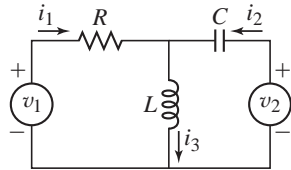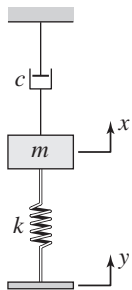
8.14 For the circuit shown in Figure P8.14, $L = 0.1$ H, $C = 10^{-6}$ F, and $R = 100\,\Omega$. Obtain the transfer functions $I_3(s)/V_1(s)$ and $I_3(s)/V_2(s)$. Using asymptotic approximations, sketch the m curves for each transfer function and discuss how the circuit acts on each input voltage (does it act like a low-pass filter, a high-pass filter, or other?).

8.15 (a) For the system shown in Figure P8.15, $m = 1$ kg and $k = 600$ N/m. Derive the expression for the peak amplitude ratio M_r and resonant frequency ω_r, and discuss the effect of the damping c on M_r and on ω_r. (b) Extend the derivation of the expressions for M_r and ω_r to the case where the values of m, c, and k are arbitrary.

Figure P8.15

8.16 For the *RLC* circuit shown in Figure P8.16, $C = 10^{-5}$ F and $L = 5 \times 10^{-3}$ H. Consider two cases: (a) $R = 10\,\Omega$ and (b) $R = 1000\,\Omega$. Obtain the transfer function $V_o(s)/V_s(s)$ and the log magnitude plot for each case. Discuss how the value of R affects the filtering characteristics of the system.

8.17 A model of a fluid clutch is shown in Figure P8.17. Using the values $I_1 = I_2 = 0.02$ kg $\cdot$ m^2, $c_1 = 0.04$ N $\cdot$ m $\cdot$ s/rad, and $c_2 = 0.02$ N $\cdot$ m $\cdot$ s/rad, obtain the transfer function $\Omega_2(s)/T_1(s)$, and derive the expression for the

Figure P8.16
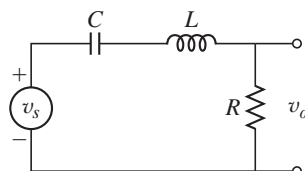
Figure P8.17
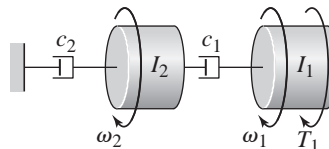
steady-state speed $\omega_2(t)$ if the applied torque in $N \cdot m$ is given by

$$T_1(t) = 4 + 2 \sin 1.5t + 0.9 \sin 2t$$

Section 8.3 Frequency Response Examples

8.18 Determine the beat period and the vibration period of the model

$$3\ddot{x} + 75x = 7 \sin 5.2t$$

8.19 Resonance will produce large vibration amplitudes, which can lead to system failure. For a system described by the model

$$\ddot{x} + 64x = 0.2 \sin \omega t$$

where x is in feet, how long will it take before $|x|$ exceeds 0.1 ft, if the forcing frequency ω is close to the natural frequency?

8.20 A certain system having a rotating unbalance, like that shown in Figure 8.3.8, has a total mass of $m = 15$ kg, an unbalanced mass of $m_u = 0.1$ kg, and an eccentricity of $\epsilon = 0.02$ m. The machine rotates at 3400 rpm. Its vibration isolator has a stiffness of $k = 10^4$ N/m. Compute the force transmitted to the foundation if the isolator's damping ratio is $\zeta = 0.707$.

8.21 A device of mass $m = 500$ kg used for machining must not have an amplitude of vibration greater than 2 mm due to the floor vibration caused by nearby machinery. The isolation system has the values $k = 5 \times 10^5$ N/m and $c = 1600$ N $\cdot$ s/m (see Figure P8.21). What is the maximum amplitude of the floor vibration that can be tolerated?

Figure P8.21

8.22 The quarter-car weight of a certain vehicle is 625 lb and the weight of the associated wheel and axle is 190 lb. The suspension stiffness is 8000 lb/ft and the tire stiffness is 10,000 lb/ft. If the amplitude of variation of the road surface is 0.25 ft with a period of 20 ft, determine the critical (resonant) speeds of this vehicle.

8.23 A motor of mass 50 kg is supported at the end of a steel cantilever beam of length 2 m, width 0.3 m, and thickness 0.03 m. The rotating unbalance of the motor is $m_u \epsilon = 0.02$ kg $\cdot$ m. Including the effect of the beam mass, determine the steady-state amplitude of motion of the motor if the motor speed is 3400 rpm.

8.24 A motor of mass 50 kg is supported in the middle of the doubly clamped steel beam of length 2 m. The rotating unbalance of the motor is $m_u \epsilon = 0.02$ kg $\cdot$ m and the motor speed is 3400 rpm. Including the effect of the beam mass, determine the required width and thickness of the beam so that the steady-state amplitude of motion of the motor is no greater than 0.01 m.

8.25 A certain factory contains a heavy rotating machine that causes the factory floor to vibrate. We want to operate another piece of equipment nearby and we measure the amplitude of the floor's motion at that point to be 0.01 m. The mass of the equipment is 1500 kg and its support has a stiffness of $k = 2 \times 10^4$ N/m and a damping ratio of $\zeta = 0.04$. Calculate the maximum force that will be transmitted to the equipment at resonance.

8.26 An electronics module inside an aircraft must be mounted on an elastic pad to protect it from vibration of the airframe. The largest amplitude vibration produced by the airframe's motion has a frequency of 40 cycles per second. The module weighs 200 N, and its amplitude of motion is limited to 0.003 m because of space. Neglect damping and calculate the percent of the airframe's motion transmitted to the module.
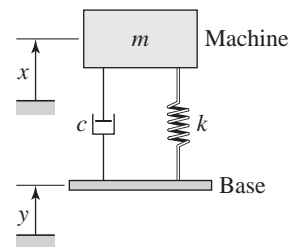
8.27 An electronics module used to control a large crane must be isolated from the crane's motion. The module weighs 2 lb. (a) Design an isolator so that no more than 10% of the crane's motion amplitude is transmitted to the module. The crane's vibration frequency is 3000 rpm. (b) What percentage of the crane's motion will be transmitted to the module if the crane's frequency can be anywhere between 2500 and 3500 rpm?

8.28 A certain pump weighs 50 lb and has a rotating unbalance. The unbalanced weight is 0.05 lb and has an eccentricity of 0.1 in. The pump rotates at 1000 rpm. Its vibration isolator has a stiffness of $k = 500$ lb/ft. Compute the force transmitted to the foundation if the isolator's damping ratio is (a) $\zeta = 0.05$ and (b) $\zeta = 0.7$.

8.29 To calculate the effects of rotating unbalance, we need to know the value of the product $m_u \epsilon$, where m is the unbalanced mass and ϵ is the eccentricity. These two quantities are sometimes difficult to calculate separately, but sometimes an experiment can be performed to estimate the product $m_u \epsilon$. An experiment was performed on a particular rotating machine whose mass is 75 kg. The machine's support has negligible damping and a stiffness of $k = 2500$ N/m. When the machine operates at 200 rpm, the measured force transmitted to the foundation was 15 N. Estimate the value of $m_u \epsilon$.

8.30 Design a vibrometer having a mass of 0.1 kg, to measure displacements having a frequency near 200 Hz.

Section 8.4 Filtering Properties of Dynamic Systems

Figure P8.31

8.31 A certain mass is driven by base excitation through a spring (see Figure P8.31). Its parameter values are $m = 200$ kg, $c = 2000$ N · s/m, and $k = 2 \times 10^4$ N/m. Determine its resonant frequency ω_r, its resonance peak M_r, and its bandwidth.

8.32 A certain series *RLC* circuit has the following transfer function.

$$T(s) = \frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}$$

Suppose that $L = 300$ H, $R = 10^4$ Ω, and $C = 10^{-6}$ F. Find the bandwidth of this system.

8.33 Obtain the expressions for the bandwidths of the two circuits shown in Figure P8.33.

8.34 For the circuit shown in Figure P8.34, $L = 0.1$ H and $C = 10^{-6}$ F. Investigate the effect of the resistance R on the bandwidth, resonant frequency, and resonant peak over the range $100 \le R \le 1000$ Ω.
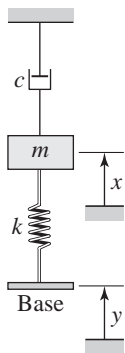
Figure P8.33

Figure P8.34

(a)

(b)
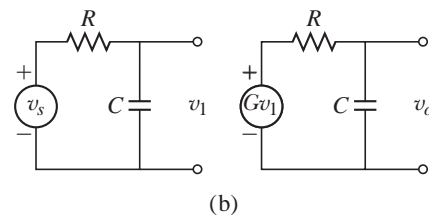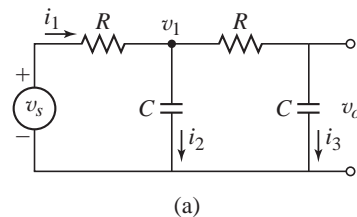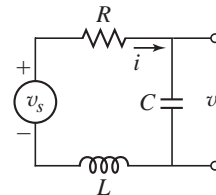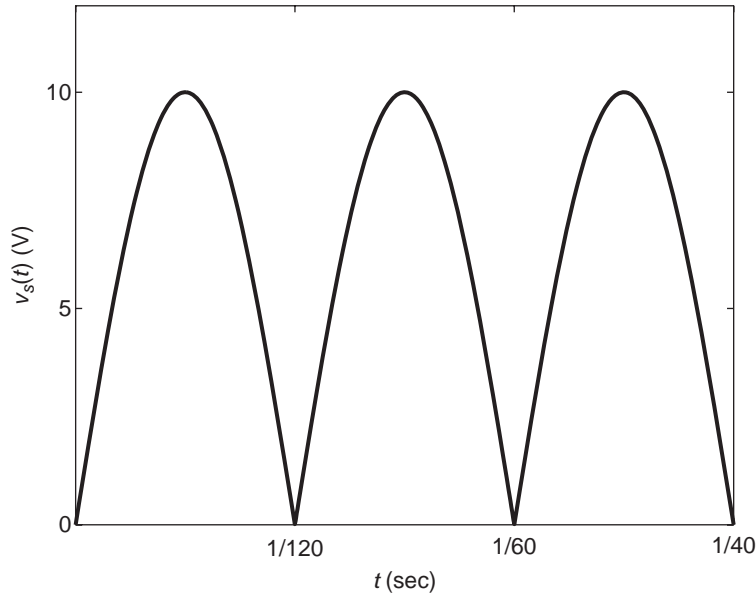
8.35 The voltage shown in Figure P8.35 is produced by applying a sinusoidal voltage to a *full wave rectifier*. The Fourier series approximation to this function is

$$v_s(t) = \frac{20}{\pi} - \frac{40}{\pi} \left[\frac{\cos 240\pi t}{1(3)} + \frac{\cos 480\pi t}{3(5)} + \frac{\cos 720\pi t}{5(7)} + \cdots \right]$$

Suppose this voltage is applied to a series RC circuit whose transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

where $R = 600 \ \Omega$ and $C = 10^{-6}$ F. Keeping only those terms in the Fourier series whose frequencies lie within the circuit's bandwidth, obtain the expression for the steady-state voltage $v_o(t)$.

8.36 The voltage shown in Figure P8.36 is called a square wave. The Fourier series approximation to this function is

$$v_s(t) = 5 \left[1 + \frac{4}{\pi} \left(\frac{\sin 120\pi t}{1} + \frac{\sin 360\pi t}{3} + \frac{\sin 600\pi t}{5} + \cdots \right) \right]$$

Suppose this voltage is applied to a series RC circuit whose transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

where $R = 10^3 \ \Omega$ and $C = 10^{-6}$ F. Keeping only those terms in the Fourier series whose frequencies lie within the circuit's bandwidth, obtain the expression for the steady-state voltage $v_o(t)$.

8.37 The displacement shown in Figure P8.37a is produced by the cam shown in part (b) of the figure. The Fourier series approximation to this function is

$$y(t) = \frac{1}{20\pi} \left[\pi - 2 \left(\frac{\sin 10\pi t}{1} + \frac{\sin 20\pi t}{2} + \frac{\sin 30\pi t}{3} + \cdots \right) \right]$$

Figure P8.36

Figure P8.37

(a)   (b)
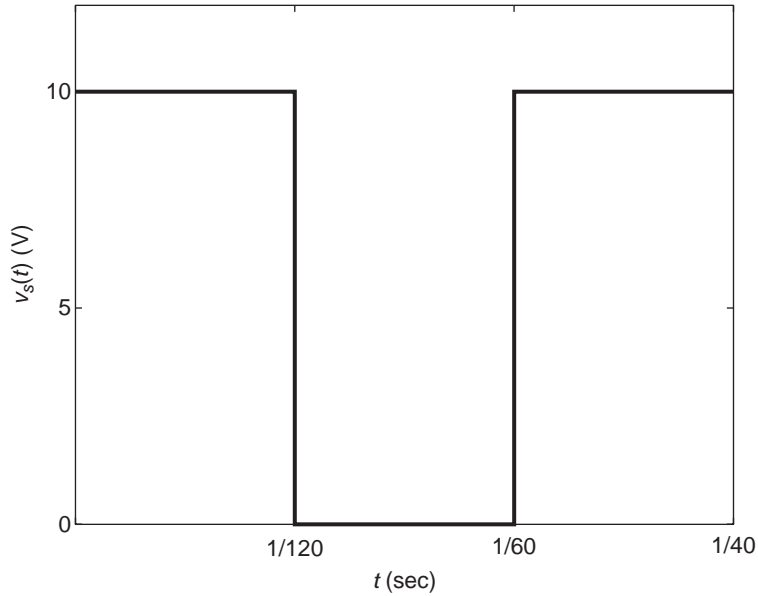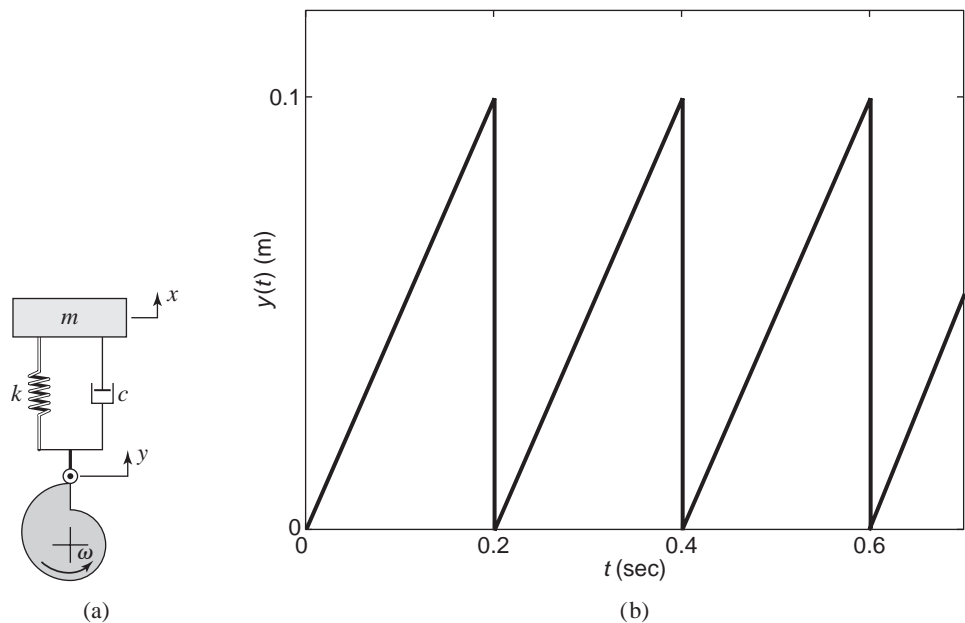
For the values $m = 1$ kg, $c = 98$ N $\cdot$ s/m, and $k = 4900$ N/m, keeping only those terms in the Fourier series whose frequencies lie within the system's bandwidth, obtain the expression for the steady-state displacement $x(t)$.

8.38 Given the model

$$0.5\dot{y} + 5y = f(t)$$

with the following Fourier series representation of the input

$$f(t) = \sin 4t + 4 \sin 8t + 0.04 \sin 12t + 0.06 \sin 16t + \cdots$$

Find the steady-state response $y_{ss}(t)$ by considering only those components of the $f(t)$ expansion that lie within the bandwidth of the system.

8.39 A mass-spring-damper system is described by the model

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

where $m = 0.25$ slug, $c = 2$ lb-sec/ft, $k = 25$ lb/ft, and $f(t)$ (lb) is the externally applied force shown in Figure P8.39. The forcing function can be expanded in a Fourier series as follows:

$$f(t) = -\left(\sin 3t + \frac{1}{3} \sin 9t + \frac{1}{5} \sin 15t + \frac{1}{7} \sin 21t + \cdots + \frac{1}{n} \sin 3nt \pm \cdots \right)$$

$$n \text{ odd}$$

Find an approximate description of the output $x_{ss}(t)$ at steady state, using only those input components that lie within the bandwidth.
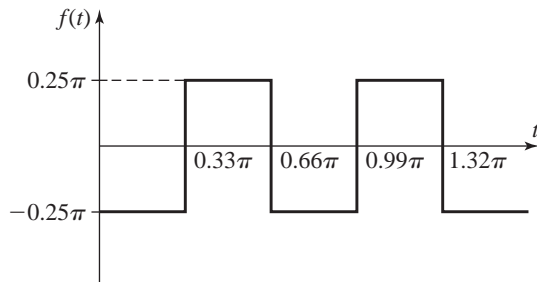
Figure P8.39

Section 8.5 System Identification from Frequency Response

8.40 An input $v_s(t) = 20 \sin \omega t$ V was applied to a certain electrical system for various values of the frequency ω, and the amplitude $|v_o|$ of the steady-state output was recorded. The data are shown in the following table. Determine the transfer function $V_o(s)/V_s(s)$.

ω (rad/s)	$|v_o|$ (V)
0.1	5.48
0.2	5.34
0.3	5.15
0.4	4.92
0.5	4.67
0.6	4.40
0.7	4.14
0.8	3.89
0.9	3.67
1	3.20
1.5	2.59
2	2.05
3	1.42
4	1.08
5	0.87
6	0.73
7	0.63

8.41 An input $f(t) = 15 \sin \omega t$ N was applied to a certain mechanical system for various values of the frequency ω, and the amplitude $|x|$ of the steady-state

output was recorded. The data are shown in the following table. Determine the transfer function $X(s)/F(s)$.

ω (rad/s)	$|x|$ (mm)
0.1	209
0.4	52
0.7	28
1	19
2	7
4	2
6	1

8.42 The following data were taken by driving a machine on its support with a rotating unbalance force at various frequencies. The machine's mass is 50 kg, but the stiffness and damping in the support are unknown. The frequency of the driving force is f Hz. The measured steady-state displacement of the machine is $|x|$ mm. Estimate the stiffness and damping in the support.

f (Hz)	$|x|$ (mm)	f (Hz)	$|x|$ (mm)
0.2	2	3.8	26
1	4	4	22
2	8	5	16
2.6	24	6	14
2.8	36	7	12
3	50	8	12
3.4	36	9	12
3.6	30	10	10

Section 8.6 Frequency Response Analysis Using MATLAB

8.43 For the system shown in Figure P8.43, $m_1 = m_2$, $k_1 = k_2$, and $k_1/m_1 = 64$ N/(m · kg). Obtain the transfer function $X_1(s)/Y(s)$ and its Bode plots. Identify the resonant frequencies and bandwidth.

8.44 For the system shown in Figure P8.44, $I_1 = I_2$, $c_{T_1} = c_{T_2}$, $c_{T_1}/I_1 = 0.1$ rad^{-1}· s^{-1}, and $k_T/I_1 = 1$ s^{-2}. Obtain the transfer function $\Theta_1(s)/\Phi(s)$ and its Bode plots. Identify the resonant frequencies and bandwidth.

Figure P8.43
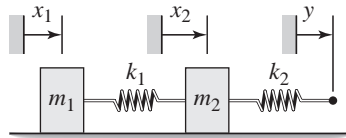
Figure P8.44
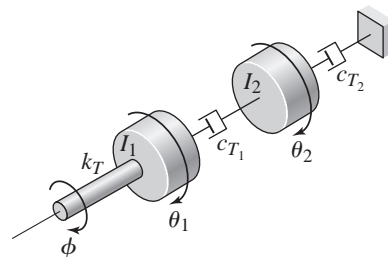
8.45 A certain mass is driven by base excitation through a spring (see Figure P8.31). Its parameter values are $m = 50$ kg, $c = 200$ N · s/m, and $k = 5000$ N/m.

Determine its resonant frequency ω_r, its resonance peak M_r, and the lower and upper bandwidth frequencies.

8.46 The transfer functions for an armature-controlled dc motor with the speed as the output are

$$\frac{\Omega(s)}{V(s)} = \frac{K_T}{(Is + c)(Ls + R) + K_b K_T}$$

$$\frac{\Omega(s)}{T_d(s)} = -\frac{Ls + R}{(Is + c)(Ls + R) + K_b K_T}$$

A certain motor has the following parameter values:

$$K_T = 0.04 \text{ N} \cdot \text{m/A} \qquad K_b = 0.04 \text{ V} \cdot \text{s/rad}$$
$$c = 7 \times 10^{-5} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad R = 0.6 \text{ }\Omega$$
$$L = 0.1 \text{ H} \qquad I_m = 2 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$
$$I_L = 4 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

where I_m is the motor's inertia and I_L is the load inertia. Thus $I = I_m + I_L$. Obtain the frequency response plots for both transfer functions. Determine the bandwidth and resonance frequency, if any.

8.47 The transfer function of the speaker model derived in Chapter 6 is, for $c = 0$,

$$\frac{X(s)}{V(s)} = \frac{K_f}{mLs^3 + mRs^2 + (kL + K_f K_b)s + kR}$$

where x is the diaphragm's displacement and v is the applied voltage. A certain speaker has the following parameter values:

$$m = 0.002 \text{ kg} \qquad k = 10^6 \text{ N/m}$$
$$K_f = 20 \text{ N/A} \qquad K_b = 15 \text{ V} \cdot \text{s/m}$$
$$R = 10 \text{ }\Omega \qquad L = 10^{-3} \text{ H}$$

Obtain the speaker's frequency response plots. Determine the speaker's bandwidth and resonant frequency, if any.

8.48 The following is a two-mass model of a vehicle suspension.

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - c_1 \dot{x}_2 - k_1 x_2 = 0$$
$$m_2 \ddot{x}_2 + c_1 \dot{x}_2 + (k_1 + k_2)x_2 - c_1 \dot{x}_1 - k_1 x_1 = k_2 y$$

Mass m_1 is one-fourth the mass of the car body, and m_2 is the mass of the wheel-tire-axle assembly. Spring constant k_1 represents the suspension's elasticity, and k_2 represents the tire's elasticity. The input is the road surface displacement y. A certain vehicle has the following parameter values:

$$m_1 = 250 \text{ kg} \qquad k_1 = 1.5 \times 10^4 \text{ N/m}$$
$$m_2 = 40 \text{ kg} \qquad k_2 = 1.5 \times 10^5 \text{ N/m}$$
$$c_1 = 1000 \text{ N} \cdot \text{m/s} \qquad c_2 = 0$$

(a) Obtain the suspension's frequency response plots. Determine the bandwidth and resonant frequencies, if any. (b) If the road surface is approximately sinusoidal with a period of 10 m, at what speeds will the car mass experience the greatest oscillation amplitude?

Transient Response and Block Diagram Models

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Obtain and interpret the free, step, ramp, and impulse response of linear models.

2. Compute and use the time constant τ, the undamped natural frequency ω_n, and other parameters to describe and assess system response.

3. Use time-domain response data to estimate coefficient values in dynamic models.

4. Interpret and draw block diagrams, given a system's equations or transfer functions.

5. Use Simulink to simulate nonlinear systems and systems with inputs more complicated than the impulse, step, and ramp functions.

Now that we have seen how to model the various types of physical systems (mechanical, electrical, fluid, and thermal), it is appropriate at this point to pull together our modeling knowledge and our analytical and computer solution methods. The purpose of this chapter is to integrate this knowledge with emphasis on understanding system behavior in the *time domain*, in particular, the transient response, as opposed to the steady-state frequency response.

The forcing functions commonly used to model real inputs or to test a system's response in the time domain are the *impulse,* the *step,* and the *ramp* functions. The impulse models a suddenly applied and suddenly removed input. The step function models a suddenly applied input that remains constant. The ramp models an input that is changing at a constant rate. In this chapter we will summarize the response of systems subjected to these inputs.

The transfer function and the characteristic polynomial are the principal tools for analyzing linear system response, and so it is now appropriate to review these concepts. Consider the following general form of a single, constant-coefficient, linear ordinary differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$

$$= b_m \frac{d^m f}{dt^m} + b_{m-1} \frac{d^{m-1} f}{dt^{m-1}} + \cdots + b_1 \frac{df}{dt} + b_0 f$$

where $a_n \neq 0$ and $m \leq n$. The response (the output) is $y(t)$, and $f(t)$ is the forcing function (the input). From the initial value properties of the Laplace transform, we have $\mathcal{L}(\dot{y}) = sY(s) - y(0)$, $\mathcal{L}(\ddot{y}) = s^2 Y(s) - sy(0) - \dot{y}(0)$, etc. In our applications, the initial values (at $t = 0$) of the input $f(t)$ and its derivatives are zero; thus $\mathcal{L}(\dot{f}) = sF(s)$, $\mathcal{L}(\ddot{f}) = s^2 F(s)$, etc. Applying the Laplace transform to the differential equation gives

$$Y(s) = \frac{I(s)}{D(s)} + \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) F(s)}{D(s)}$$

$$= Y_{\text{free}}(s) + Y_{\text{forced}}(s)$$

where $D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$. The *characteristic polynomial* is the polynomial $D(s)$. The term $I(s)$ contains the initial response values $y(0), \dot{y}(0), \ldots$.

We see that the response is the sum of the *free response,* which is independent of the input, and the *forced response,* which is independent of the initial values of y and its derivatives. The system transfer function $T(s)$ is given by $T(s) = Y_{\text{forced}}(s)/F(s)$; that is,

$$T(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Sections 9.1 and 9.2 summarize the response of first- and second-order systems. We review the concepts of the *time constant τ* and the *undamped natural frequency ω_n*. These parameters help to describe the speed and the oscillation characteristics of the response. In Section 9.3 we will introduce additional parameters that are useful for describing and for specifying the response.

Response prediction cannot be precisely made unless we have numerical values for the model's coefficients, and Section 9.4 shows how measurements of the response as a function of time can be used to obtain such values.

Section 9.5 introduces block diagrams, which are based on the transfer function concept. The block diagram is a way of representing the dynamics of a system in graphical form. These diagrams will be used often in subsequent chapters to describe dynamic systems, and they also assist in developing Simulink models. Section 9.6 gives several examples of the application of block diagram models.

Section 9.7 shows how to use MATLAB to compute transient response characteristics and to do block-diagram algebra. Finally, Section 9.8 introduces several Simulink blocks that are useful for simulating nonlinear systems or systems with complicated inputs. ∎

9.1 RESPONSE OF FIRST-ORDER SYSTEMS

We now summarize the response of first-order systems to step and ramp inputs. The free response of a model is its solution in the absence of an input. We have seen that the free response of the model

$$m\dot{v} + cv = f(t) \tag{9.1.1}$$

is

$$v(t) = v(0)e^{-ct/m} \tag{9.1.2}$$

where $v(0)$ is the initial value of the response $v(t)$, and m and c are constants. When $c/m < 0$ the solution grows exponentially; this is the *unstable* case. If $c/m = 0$, the model is *neutrally stable,* and $v(t) = v(0)$. If c/m is positive, the model is *stable,* and the solution decays exponentially.

THE TIME CONSTANT

For the stable case, we introduced a new parameter τ with the definition

$$\tau = \frac{m}{c} \tag{9.1.3}$$

The free response can be written as

$$v(t) = v(0)e^{-t/\tau} \tag{9.1.4}$$

and is illustrated in Figure 9.1.1.

The new parameter τ has units of time and is the model's *time constant*. It gives a convenient measure of the exponential decay curve. After a time equal to one time constant has elapsed, v has decayed to 37% of its initial value. We can also say that v has decayed *by* 63%. At $t = 4\tau$, $v(t)$ has decayed to 2% of its initial value. At $t = 5\tau$, $v(t)$ has decayed to 1% of its initial value.

Figure 9.1.1 The free response $v(t) = v(0)e^{-t/\tau}$.

(a)

$$m\frac{dv}{dt} + cv = f$$

$$\tau = \frac{m}{c}$$

(b)

$$I\frac{d\omega}{dt} + c\omega = T$$

$$\tau = \frac{I}{c}$$

(c)

$$RC\frac{dv}{dt} + v = v_s$$

$$\tau = RC$$

(d)

$$AR\frac{dh}{dt} + gh = Rq_v$$

$$\tau = \frac{AR}{g}$$

(e)

$$mc_p R\frac{dT}{dt} + T = T_b$$

$$\tau = mc_p R$$
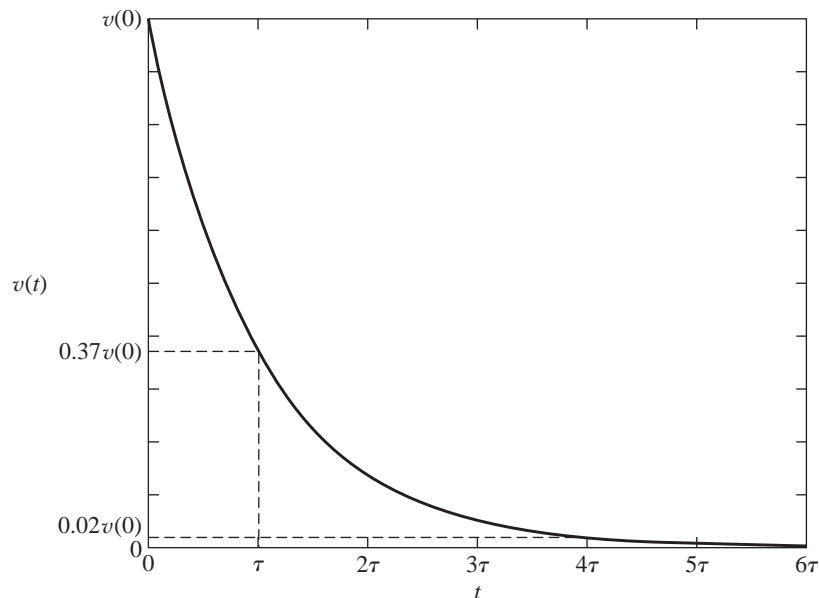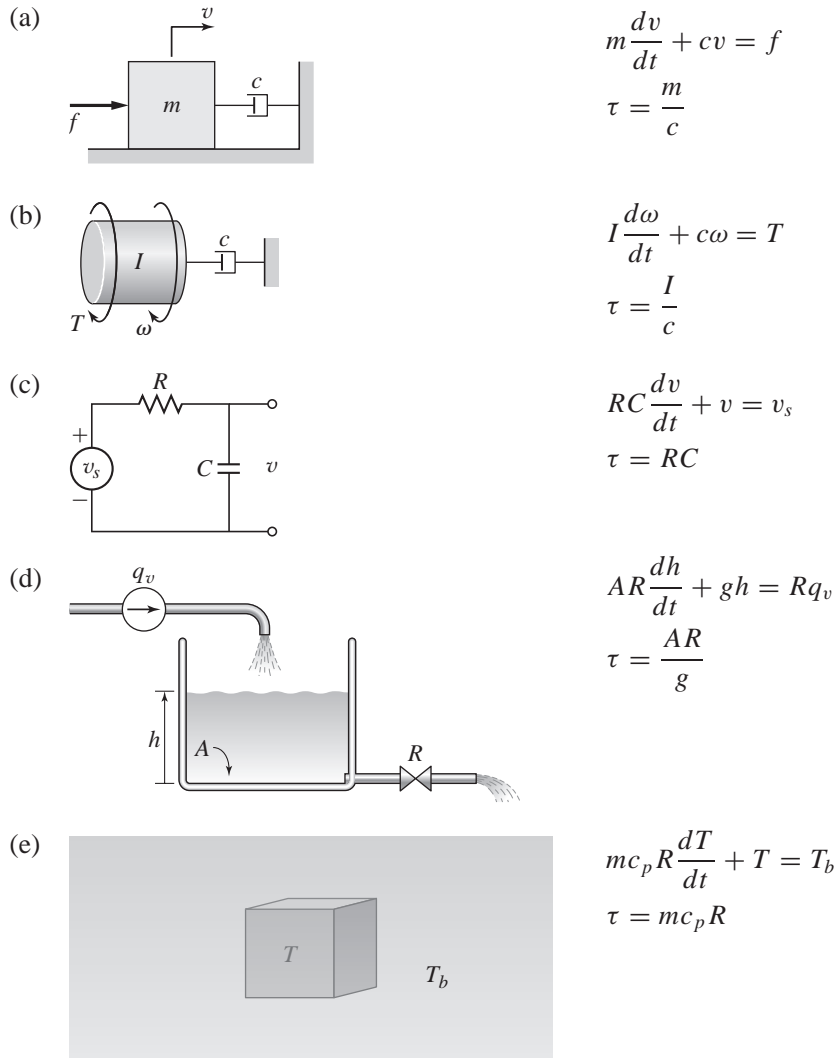
Figure 9.1.2 shows some systems whose models have the same form as that of (9.1.1); namely, $a\dot{y} + by = f(t)$. This can be rewritten as

$$\frac{a}{b}\dot{y} + y = \frac{1}{b}f(t)$$

In this form, the time constant may be identified as $\tau = a/b$. This form can be used to determine the time constant expressions for each of the systems shown in Figure 9.1.2.

STEP RESPONSE OF A FIRST-ORDER MODEL

The step response of the model $m\dot{v} + cv = f$ is

$$v(t) = \underbrace{v(0)e^{-ct/m}}_{\text{Free response}} + \underbrace{\frac{F}{c}\left(1 - e^{-ct/m}\right)}_{\text{Forced response}} \tag{9.1.5}$$

where F is the magnitude of the step input. Note that the response is the sum of the free and the forced responses.
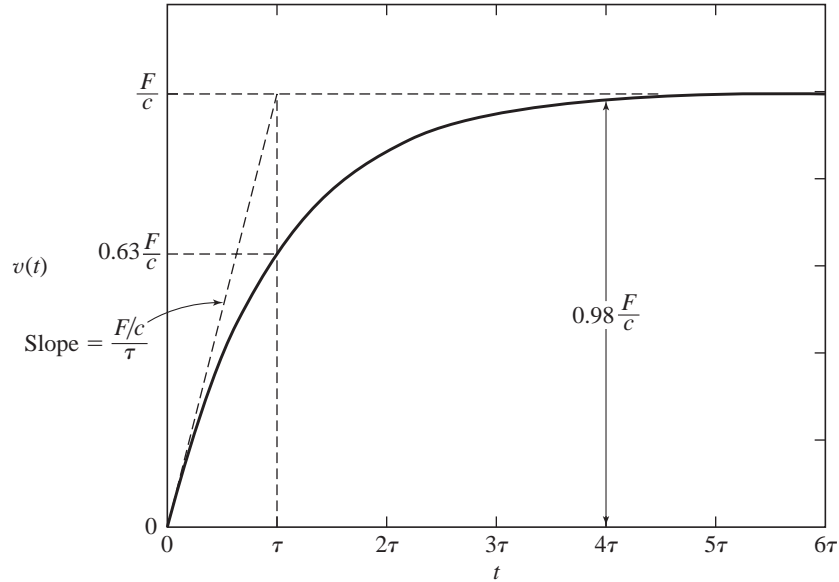
The time constant is useful also for analyzing the response when the forcing function is a step. We can express the solution in terms of the time constant τ by substituting $c/m = 1/\tau$ to obtain

$$v(t) = v(0)e^{-t/\tau} + \frac{F}{c}\left(1 - e^{-t/\tau}\right) = v(0)e^{-t/\tau} + v_{ss}\left(1 - e^{-t/\tau}\right) \qquad (9.1.6)$$

where $v_{ss} = F/c$. The solution approaches the constant value F/c as $t \to \infty$. This is called the *steady-state* response, denoted v_{ss}.

The response is plotted in Figure 9.1.3 for $v(0) = 0$. At $t = \tau$, the response is 63% of the steady-state value. At $t = 4\tau$, the response is 98% of the steady-state value, and at $t = 5\tau$, it is 99% of steady state. Thus, because the difference between 98% and 99% is so small, for most engineering purposes we can say that $v(t)$ reaches steady state at $t = 4\tau$, although mathematically, steady state is not reached until $t = \infty$.

If $v(0) \neq 0$, the response is shifted by $v(0)e^{-t/\tau}$. At $t = \tau$, 37% of the difference between the initial value $v(0)$ and the steady-state value remains. At $t = 4\tau$, only 2% of this difference remains.

Table 9.1.1 summarizes the response of the model $m\dot{y} + cy = r(t)$.

Table 9.1.1 Free, step, and ramp
response of $\tau\dot{y} + y = r(t)$.

Free response $[r(t) = 0]$
  $y(t) = y(0)e^{-t/\tau}$
  $y(\tau) \approx 0.37y(0)$
  $y(4\tau) \approx 0.02y(0)$
Step response $[r(t) = Ru_s(t), y(0) = 0]$
  $y(t) = R(1 - e^{-t/\tau})$
  $y(\infty) = y_{ss} = R$
  $y(\tau) \approx 0.63y_{ss}$
  $y(4\tau) \approx 0.98y_{ss}$
Ramp response $[r(t) = mt, y(0) = 0]$
  $y(t) = m(t - \tau + \tau e^{-t/\tau})$

Figure 9.1.4 Step response for two values of c.

RESPONSE TIME AND THE TIME CONSTANT

When using the time constant to compare the response times of two models subjected to a step input, the comparison should be done only for systems whose steady-state responses are identical. Note that because $\tau = m/c$, a small value of the damping constant c produces a large time constant, which indicates a sluggish system. The result is counter to our intuition, which tells us that a small amount of damping should correspond to a system with a fast response. This counterintuitive result is explained by noting that a larger steady-state response F/c results from a smaller damping constant. Thus, for a large value of c, even though the time constant is smaller, the steady-state response is smaller, and thus it takes less time to reach a specific steady-state value. This is illustrated in Figure 9.1.4 for $v(0) = 0$ with two values of c.

THE STEP FUNCTION APPROXIMATION

The step function is an approximate description of an input that can be switched on in a time interval that is very short compared to the time constant of the system. A good example of a step input is the voltage applied to a circuit due to the sudden closure of a switch.

In some applications, however, it may not be clear whether a step function is a good model of the input. For example, we might model rocket thrust as a step function if it reaches a constant value quickly compared to the vehicle's time constant. Solid-rocket thrust as a function of time depends on the cross-sectional shape of the solid fuel. The propellant grain shape called rod-and-tube was designed to give a constant thrust by keeping the total burning area approximately constant. The step function is a good model for this case if the time to reach constant thrust is short compared to the vehicle time constant (Figure 9.1.5a). For a propellant grain having a tubular cross-section, the fuel burns from the inside to the outside (Figure 9.1.5b). The increasing burning area produces a thrust that increases with time. A step function usually would not be used to model the thrust, unless we needed a quick, very approximate answer, in which case we would take the average thrust as the step magnitude.

Figure 9.1.5 Solid fuel rocket thrust as a function of time.

(a)

Rod-and-Tube

(b)

Tubular
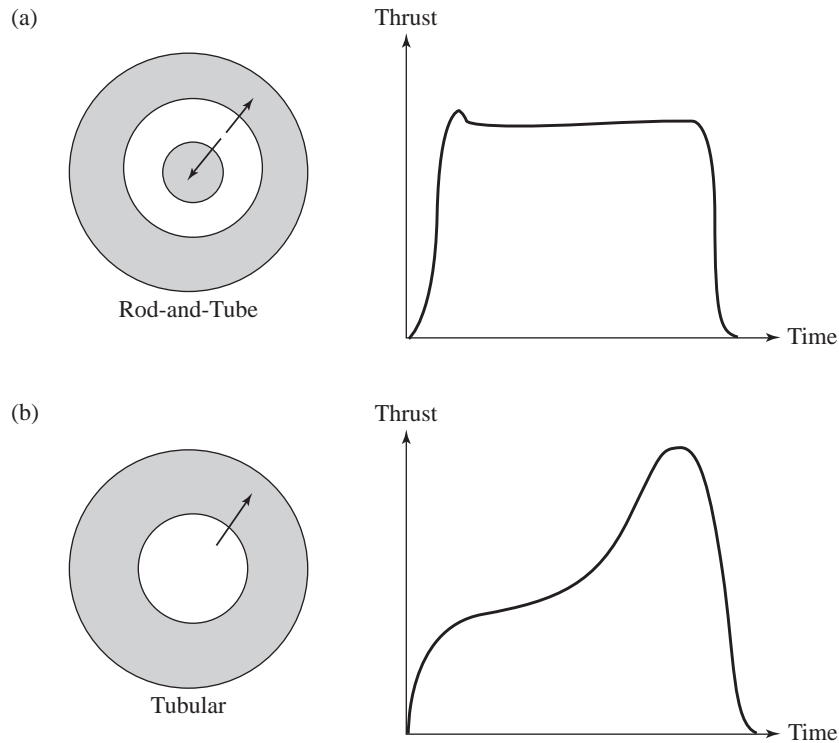
EXAMPLE 9.1.1

Speed Response of a Rotational System

■ Problem

A certain rotational system has an inertia $I = 50 \text{ kg} \cdot \text{m}^2$ and a viscous damping constant $c = 10 \text{ N} \cdot \text{m} \cdot \text{s/rad}$. The torque $T(t)$ is applied by an electric motor (Figure 9.1.6a). From the free body diagram shown in part (b) of the figure, the equation of motion is

$$50\frac{d\omega}{dt} + 10\omega = T(t) = K_f i_f \tag{1}$$

The model of the motor's field current i_f in amperes is

$$0.001\frac{di_f}{dt} + 5i_f = v(t) \tag{2}$$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 25 \text{ N} \cdot \text{m/A}$.

Suppose the applied voltage is 10 V. Determine the steady-state speed of the inertia and estimate the time required to reach that speed.

■ Solution

From equation (2) we see that the time constant of the motor circuit is $0.001/5 = 2 \times 10^{-4}$ s. Thus the current will reach a steady-state value of $10/5 = 2$ A in approximately $4(2 \times 10^{-4}) = 8 \times 10^{-4}$ s. The resulting steady-state torque is $K_T(2) = 25(2) = 50 \text{ N} \cdot \text{m}$.

From equation (1) we find the time constant of the rotational system to be $50/10 = 5$ s. Since this is much larger than the time constant of the circuit (2×10^{-4} s), we conclude that the motor torque may be modeled as a step function. The magnitude of the step function is $50 \text{ N} \cdot \text{m}$. The steady-state speed is $\omega = 50/10 = 5$ rad/s, and therefore it will take approximately $4(5) = 20$ s to reach this speed.
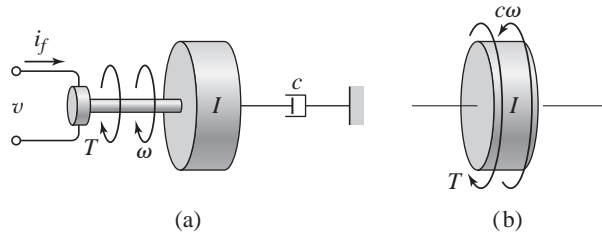
Figure 9.1.6 A rotational system.

(a)   (b)

STEP INPUTS VERSUS CONSTANT INPUTS

Consider the two input functions shown in Figure 9.1.7. The one shown in part (a) is constant, whereas in part (b) the input is a step function. For the constant input, the derivative is always zero, but for the step input the derivative is an impulse function, which is infinite at $t = 0$ and zero elsewhere. So if we are interested in finding the response of a model only for $t \geq 0$, then we can conclude that it does not matter whether we model the input as a constant or as a step, *as long as the model does not contain any time derivatives of the input*.

If the model contains input derivatives, then we must be careful in how we model the input and in how we solve for the response. For example, if the input is the thrust of a rocket engine that is turned on at $t = 0$, a step function model of the thrust is appropriate. If the input, however, is a gravity force, which is always present, then a step function model is not appropriate. This distinction needs to be made only if the model has input derivatives.

The source of the initial value of the response depends on the form of the input. If the input is a constant, then we may take the initial value to be due to the effect of the input prior to $t = 0$. If the input is a step whose value is zero for $t < 0$, then the initial value must have been caused by some other agent or process not described by the model.

Although we have made a distinction between constant inputs and step inputs, we will follow common practice and call the response due to either type of input the "step" response.

STEP RESPONSE WITH AN INPUT DERIVATIVE

The general form of the first-order linear model is

$$m\dot{v} + cv = b\dot{f}(t) + f(t) \tag{9.1.7}$$

whose transfer function is of the form

$$\frac{V(s)}{F(s)} = \frac{bs + 1}{ms + c} \tag{9.1.8}$$
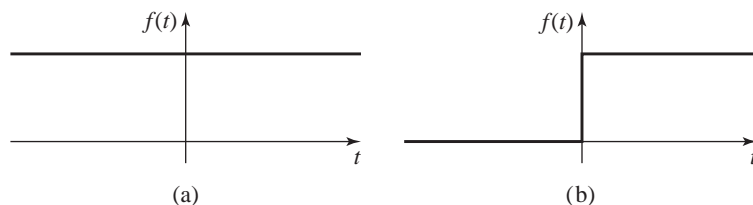
Figure 9.1.7 (a) Constant input. (b) Step input.

(a)   (b)

The presence of an input derivative is indicated by an s in the numerator of the transfer function. Such a transfer function is said to have *numerator dynamics*.

If $f(t)$ is a step function of magnitude F, then $F(s) = F/s$ and $f(0) = 0$, and (9.1.8) gives

$$V(s) = \frac{mv(0) + bF}{ms + c} + \frac{F}{s(ms + c)}$$

which gives

$$v(t) = \left[v(0) + \frac{bF}{m} \right] e^{-ct/m} + \frac{F}{c} \left(1 - e^{-ct/m} \right) \tag{9.1.9}$$

Comparing this with (9.1.6), we see that the effect of the $b\dot{f}$ term is to increase the effective initial value of $v(t)$ by the amount bF/m.

IMPULSE RESPONSE

The impulse response of $m\dot{v} + cv = f(t)$ is found as follows, where A is the impulse *strength*, or the area under the impulse versus time curve.

$$ms V(s) - mv(0) + cV(s) = A$$

$$V(s) = \frac{mv(0) + A}{ms + c} = \frac{v(0) + A/m}{s + c/m}$$

$$v(t) = \left[v(0) + \frac{A}{m} \right] e^{-ct/m} \tag{9.1.10}$$

We can see that the effect of the impulse is to increase the effective initial condition by A/m.

Because $\dot{f}(t)$ is undefined if $f(t)$ is an impulse, we do not consider the impulse response of the equation $m\dot{v} + cv = f(t) + b\dot{f}(t)$.

RAMP RESPONSE AND THE TIME CONSTANT

We can obtain the response of the general equation $\tau\dot{v} + v = f(t)$ to the ramp input $f(t) = mt$ as follows. Setting $v(0) = 0$ and transforming the equation with $F(s) = m/s^2$ gives

$$\tau s V(s) + V(s) = F(s) = \frac{m}{s^2}$$

or

$$V(s) = \frac{m}{s^2(\tau s + 1)} = \frac{m}{s^2} - \frac{m\tau}{s} + \frac{m\tau}{s + 1/\tau}$$

The inverse transforms give

$$v(t) = m(t - \tau) + m\tau e^{-t/\tau}$$

The response is in steady state after approximately $t = 4\tau$. At steady state, $v(t) = m(t - \tau)$, so the response is parallel to the input but lags behind it by a time τ (Figure 9.1.8).

Obtaining the ramp response can be tedious for higher-order systems, but sometimes we only need to find the steady-state *difference* between the input and the output. This can be done easily with the final value theorem, as is illustrated by Example 9.1.2.
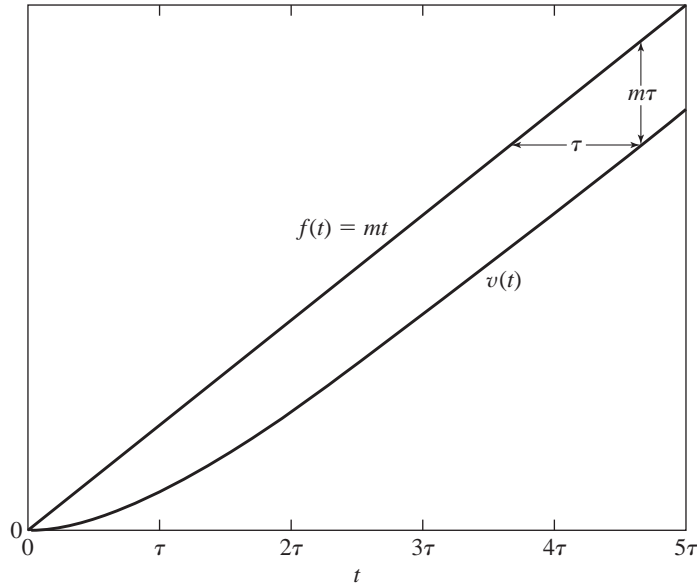
Figure 9.1.8 Ramp input and ramp response of the model $\tau \dot{v} + v = f(t)$.

The Final Value Theorem and Ramp Response | **EXAMPLE 9.1.2**

■ **Problem**

Obtain the steady-state difference $f(\infty) - v(\infty)$ between the input and output of the following model: $\tau \dot{v} + v = bf(t)$, where b is a constant and $f(t) = mt$. Assume that $v(0) = 0$ and that the model is stable ($\tau > 0$).

■ **Solution**

The transform of the response is

$$V(s) = \frac{b}{\tau s + 1} F(s) = \frac{b}{\tau s + 1} \frac{m}{s^2}$$

Use this with the final value theorem to find the steady-state difference:

$$f(\infty) - v(\infty) = \lim_{s \to 0}[s F(s)] - \lim_{s \to 0}[s V(s)] = \lim_{s \to 0} s[F(s) - V(s)]$$

$$= \lim_{s \to 0} s \left(\frac{m}{s^2} - \frac{b}{\tau s + 1} \frac{m}{s^2} \right)$$

$$= \lim_{s \to 0} \frac{m}{s} \left(\frac{\tau s + 1 - b}{\tau s + 1} \right)$$

$$= \begin{cases} \infty & b \neq 1 \\ m\tau & b = 1 \end{cases}$$

Thus, the steady-state difference is infinite unless $b = 1$. Both the input and output approach straight lines at steady state. The preceding result shows that the lines diverge unless $b = 1$. If $b = 1$, the lines are a vertical distance $m\tau$ apart. This is the case shown in Figure 9.1.8.

9.2 RESPONSE OF SECOND-ORDER SYSTEMS

The equations of motion of many systems containing mass, spring, and damping elements have the form

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{9.2.1}$$

where $f(t)$ is the input. Its transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{9.2.2}$$

Figure 9.2.1 shows examples of other types of systems that have the same model form. The solution of this equation, and therefore the form of the free and forced responses, depends on the values of the two characteristic roots, obtained from the characteristic

Figure 9.2.1 Some second-order systems.

(a)

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

(b)

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$

(c)

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

(d)

$$RA\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_v$$

$$RA\frac{dh_2}{dt} + g(h_2 - h_1) + gh_2 = 0$$

(e)

$$R_1 C_1\frac{dT}{dt} + T = T_b$$

$$R_1 R_2 C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b$$

$$= R_2 T + R_1 T_o$$
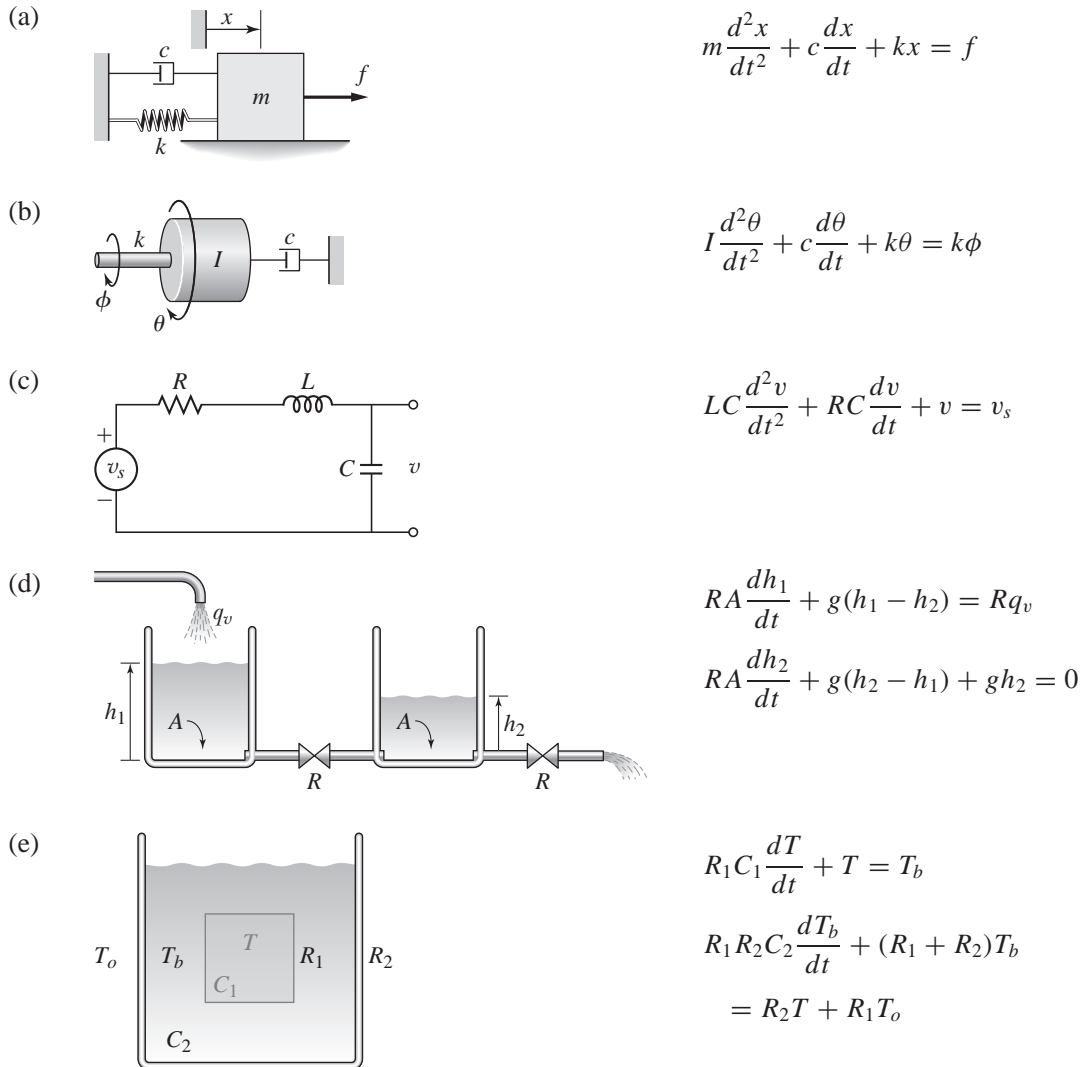
equation $ms^2 + cs + k = 0$. Recall from the discussion in Chapter 3 that this model is stable if both of its roots are real and negative or if the roots are complex with negative real parts. This is true if m, c, and k have the same sign.

A related model form is

$$m\ddot{x} + c\dot{x} + kx = a\dot{g}(t) + bg(t) \tag{9.2.3}$$

where $g(t)$ is the input. Its transfer function is

$$\frac{X(s)}{G(s)} = \frac{as + b}{ms^2 + cs + k} \tag{9.2.4}$$

So this model has numerator dynamics. It is important to understand that the input does not affect the characteristic equation, and therefore does not affect the stability of the model or its free response. Thus (9.2.1) and (9.2.3) have the same stability characteristics and the same free response, because they have the same characteristic equation, $ms^2 + cs + k = 0$.

The formulas to be developed in this section are based on the transfer function model form. Models in state variable form can always be reduced to transfer function form. For example, consider the model of an armature-controlled dc motor, developed in Section 6.4.

$$L_a \frac{di_a}{dt} = v_a - i_a R_a - K_b\omega \tag{9.2.5}$$

$$I\frac{d\omega}{dt} = K_T i_a - c\omega - T_L \tag{9.2.6}$$

This is a second-order linear model and it can be reduced to the forms of (9.2.2) and (9.2.4), as shown in Section 6.4. The results are

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a I s^2 + (R_a I + cL_a)s + cR_a + K_b K_T} \tag{9.2.7}$$

which has the form of (9.2.2), and

$$\frac{I_a(s)}{V_a(s)} = \frac{I s + c}{L_a I s^2 + (R_a I + cL_a)s + cR_a + K_b K_T} \tag{9.2.8}$$

which has the form of (9.2.4).

UNDAMPED RESPONSE

Consider the undamped systems shown in Figure 9.2.2. They all have the same model form: $m\ddot{x} + kx = f(t)$. In the first system, suppose $f(t) = 0$ and we set the mass in motion at time $t = 0$ by pulling it to a position $x(0)$ and releasing it with an initial velocity $\dot{x}(0)$. From Case 3 in Table 9.2.1, the response has the form $x(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t$, where we have defined

$$\omega_n = \sqrt{\frac{k}{m}} \tag{9.2.9}$$

Using the initial conditions we find that the constants are $C_1 = \dot{x}(0)/\omega_n$ and $C_2 = x(0)$. Thus the solution is

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t \tag{9.2.10}$$

This solution shows that the mass oscillates about the rest position $x = 0$ with a frequency of $\omega_n = \sqrt{k/m}$ radians per unit time. The period of the oscillation is $2\pi/\omega_n$.

Figure 9.2.2 Examples of undamped systems.

(a)

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

(b)

$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{L}}$$

(c)

$$LC\frac{d^2v}{dt^2} + v = v_s$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$
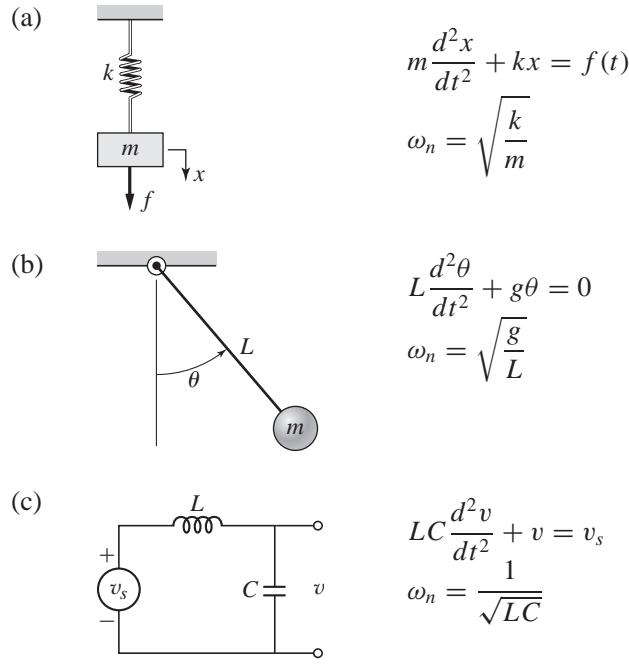
Table 9.2.1 Solution forms for the free response of a second-order model.

Equation	Solution form
$\ddot{x} + a\dot{x} + bx = 0 \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	**1.** $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$
2. $(a^2 = 4b)$ repeated, real roots: s_1, s_1	**2.** $x(t) = (C_1 + tC_2)e^{s_1 t}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$	**3.** $x(t) = C_1 \sin \omega t + C_2 \cos \omega t$
$\quad \omega = \sqrt{b}$	
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = r \pm j\omega$	**4.** $x(t) = e^{rt}(C_1 \sin \omega t + C_2 \cos \omega t)$
$\quad r = -a/2, \omega = \sqrt{4b - a^2}/2$	

The frequency of oscillation ω_n is called the *natural frequency*. The natural frequency is greater for stiffer springs (larger k values). The amplitude of the oscillation depends on the initial conditions $x(0)$ and $\dot{x}(0)$.

Damping is present in the model $m\ddot{x} + c\dot{x} + kx = f(t)$ if $c \neq 0$. If so, Cases 1, 2, and 4 in Table 9.2.1 summarize the free response forms. See Example 4.4.4 for specific examples of the damped free response. Figure 9.2.3 shows the free response for four values of c, with $m = 1$, $k = 16$, $x(0) = 1$, and $\dot{x}(0) = 0$. For no damping, the system is neutrally stable and the mass oscillates with a constant amplitude and a radian frequency of $\sqrt{k/m} = 4$, which is the natural frequency ω_n. As the damping is increased slightly to $c = 4$, the system becomes stable and the mass still oscillates but with a smaller radian frequency ($\sqrt{12} = 3.464$). The oscillations die out eventually as the mass returns to rest at the equilibrium position $x = 0$. As the damping is increased further to $c = 8$, the mass no longer oscillates because the damping force is large enough to limit the velocity and thus the momentum of the mass to a value that prevents the mass from overshooting the equilibrium position. For a larger value of c ($c = 10$), the mass takes longer to return to equilibrium because the damping force greatly slows down the mass.

Figure 9.2.3 Responses for four values of c.

EFFECT OF ROOT LOCATION

Figure 9.2.4 shows how the location of the characteristic roots in the complex plane affects the free response. The real part of the root is plotted on the horizontal axis, and the imaginary part is plotted on the vertical axis. Because the roots are conjugate, we show only the upper root. Using the results we have found in earlier chapters, we see that

1. Unstable behavior occurs when the root lies to the right of the imaginary axis.
2. Neutrally stable behavior occurs when the root lies on the imaginary axis.
3. The response oscillates only when the root has a nonzero imaginary part.
4. The greater the imaginary part, the higher the frequency of the oscillation.
5. The farther to the left the root lies, the faster the response decays.

The characteristic equation of the model (9.2.1) is

$$ms^2 + cs + k = 0 \qquad (9.2.11)$$

Its roots are

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = r \pm j\omega \qquad (9.2.12)$$

where r and ω denote the real and imaginary parts of the roots.

THE DAMPING RATIO

A second-order system's free response for the stable case can be conveniently characterized by the *damping ratio* ζ (sometimes called the *damping factor*). For the characteristic equation (9.2.11), the damping ratio is defined as

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad (9.2.13)$$

Figure 9.2.4 Effect of root location on the free response.
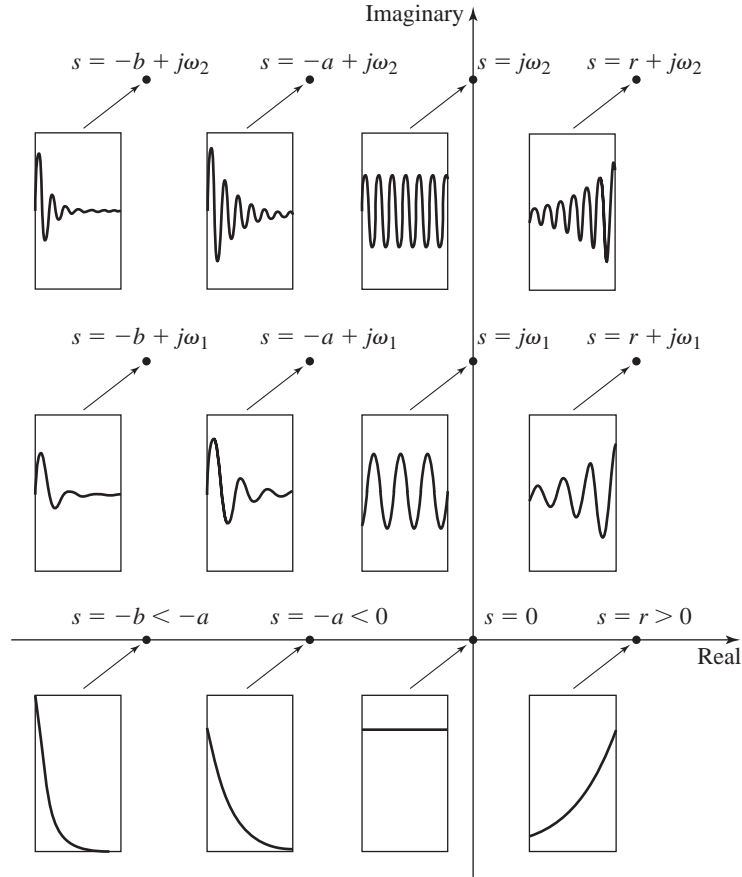
This definition is not arbitrary but is based on the way the roots change from real to complex as the value of c is changed; that is, from (9.2.12) we see that three cases can occur:

1. *The Critically Damped Case:* Repeated roots occur if $c^2 - 4mk = 0$; that is, if $c = 2\sqrt{mk}$. This value of the damping constant is the *critical damping constant* c_c, and when c has this value the system is said to be *critically damped*.
2. *The Overdamped Case:* If $c > c_c = 2\sqrt{mk}$, two real distinct roots exist, and the system is *overdamped*.
3. *The Underdamped Case:* If $c < c_c = 2\sqrt{mk}$, complex roots occur, and the system is *underdamped*.

The damping ratio is thus seen to be the ratio of the actual damping constant c to the critical value c_c. Note that

1. For a critically damped system, $\zeta = 1$.
2. Exponential behavior occurs if $\zeta > 1$ (the overdamped case).
3. Oscillations exist when $\zeta < 1$ (the underdamped case).

For an unstable system the damping ratio is meaningless and therefore not defined. For example, the equation $3s^2 - 5s + 4 = 0$ is unstable, and we do not compute its damping ratio, which would be negative if you used (9.2.13).

The damping ratio can be used as a quick check for oscillatory behavior. For example, the model whose characteristic equation is $s^2 + 5ds + 4d^2 = 0$ is stable if $d > 0$ and it has the following damping ratio.

$$\zeta = \frac{5d}{2\sqrt{4d^2}} = \frac{5}{4} > 1$$

Because $\zeta > 1$, no oscillations can occur in the system's free response regardless of the value of d and regardless of the initial conditions.

The motor transfer function (9.2.7) is repeated here.

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T}$$

The denominator of this model has the standard form $ms^2 + cs + k$, and thus the damping ratio of the motor model is, from (9.2.13),

$$\zeta = \frac{R_a I + c L_a}{2\sqrt{L_a I (R_a c + K_b K_T)}}$$

Even if the damping constant c is zero, the damping ratio is still nonzero because of the term $R_a I$.

NATURAL AND DAMPED FREQUENCIES OF OSCILLATION

The roots of (9.2.11) are purely imaginary when there is no damping. The imaginary part and the frequency of oscillation for this case is the *undamped natural frequency* $\omega_n = \sqrt{k/m}$.

We can write the characteristic equation in terms of the parameters ζ and ω_n. First divide (9.2.11) by m and use the fact that $\omega_n^2 = k/m$ and that

$$2\zeta \omega_n = 2 \left(\frac{c}{2\sqrt{mk}} \right) \left(\sqrt{\frac{k}{m}} \right) = \frac{c}{m}$$

The characteristic equation becomes

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \tag{9.2.14}$$

and the roots are

$$s = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \tag{9.2.15}$$

The frequency of oscillation is $\omega_n \sqrt{1 - \zeta^2}$ and is called the *damped natural frequency,* or simply the damped frequency, to distinguish it from the undamped natural frequency ω_n. We will denote the damped frequency by ω_d.

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{9.2.16}$$

The frequencies ω_n and ω_d have meaning only for the underdamped case ($\zeta < 1$). For this case (9.2.16) shows that $\omega_d < \omega_n$. Thus the damped frequency is always less than the undamped frequency.

Table 9.2.2 Free response of $m\ddot{x} + c\dot{x} + kx = f(t)$ for the stable case.

Characteristic roots	$s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
Undamped natural frequency	$\omega_n = \sqrt{\dfrac{k}{m}}$
Damping ratio	$\zeta = \dfrac{c}{2\sqrt{km}}$
Damped natural frequency	$\omega_d = \omega_n\sqrt{1 - \zeta^2}$
Overdamped case ($\zeta > 1$)	Distinct, real roots: $s = -r_1, \ s = -r_2 \quad (r_1 \neq r_2)$
	$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t}$
	$A_1 = \dfrac{\dot{x}(0) + r_2 x(0)}{r_2 - r_1}$
	$A_2 = \dfrac{-r_1 x(0) - \dot{x}(0)}{r_2 - r_1} = x(0) - A_1$
Critically damped case ($\zeta = 1$)	Repeated roots: $s = -r_1, \ s = -r_1$
	$x(t) = (A_1 + A_2 t)\, e^{-r_1 t}$
	$A_1 = x(0)$
	$A_2 = \dot{x}(0) + r_1 x(0)$
Underdamped case ($0 \leq \zeta < 1$)	Complex conjugate roots: $s = -a \pm jb, \ b > 0$
	$x(t) = D e^{-at} \sin(bt + \phi)$
	$D = +\dfrac{1}{b}\sqrt{[bx(0)]^2 + [\dot{x}(0) + ax(0)]^2}$
	$\sin\phi = \dfrac{x(0)}{D} \quad \cos\phi = \dfrac{\dot{x}(0) + ax(0)}{bD}$
Alternative form for $0 \leq \zeta < 1$	$x(t) = e^{-\zeta\omega_n t}\,[A \sin\omega_d t + x(0)\cos\omega_d t]$
	$A = \dfrac{\zeta}{\sqrt{1 - \zeta^2}}x(0) + \dfrac{1}{\omega_d}\dot{x}(0)$

TIME CONSTANT

Comparison of (9.2.12) with (9.2.15) shows that $r = -\zeta\omega_n$ and $\omega = \omega_n\sqrt{1 - \zeta^2}$. Because the time constant τ is $-1/r$, we have

$$\tau = \frac{1}{\zeta\omega_n} \tag{9.2.17}$$

Remember that this formula applies only if $\zeta \leq 1$ (otherwise, $\sqrt{1 - \zeta^2}$ is imaginary).

   Table 9.2.2 summarizes the free response of the stable, linear, second-order model in terms of the parameters ζ, ω_n, and ω_d. Table 9.2.3 summarizes the formulas for these parameters.

GRAPHICAL INTERPRETATION

The preceding relations can be represented graphically by plotting the location of the roots (9.2.15) in the complex plane (Figure 9.2.5). The parameters ζ, ω_n, ω_d, and τ are normally used to describe stable systems only, and so we will assume for now that all the roots lie to the left of the imaginary axis (in the "left half-plane").

Table 9.2.3 Response parameters for second-order systems.

Model	$m\ddot{x} + c\dot{x} + kx = f(t)$
	m, c, k constant
Characteristic Equation	$ms^2 + cs + k = 0$
1. Roots	$s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
2. Stability Property	Stable if and only if both roots have negative real parts. This occurs if and only if m, c, and k have the same sign.
3. Alternative forms for underdamped systems	
Characteristic Equation:	$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
Roots:	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
4. Damping ratio or damping factor	$\zeta = \dfrac{c}{2\sqrt{mk}}$
5. Undamped natural frequency	$\omega_n = \sqrt{\dfrac{k}{m}}$
6. Damped natural frequency	$\omega_d = \omega_n\sqrt{1 - \zeta^2}$
7. Time constant	$\tau = 2m/c = 1/\zeta\omega_n \quad \text{if } \zeta \leq 1$
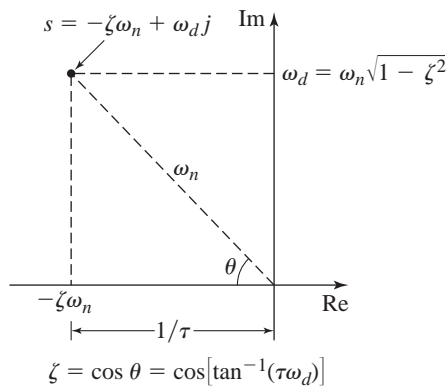
Figure 9.2.5 Graphical interpretation of the parameters ζ, τ, ω_n, and ω_d.

The lengths of two sides of the right triangle shown in Figure 9.2.5 are $\zeta\omega_n$ and $\omega_n\sqrt{1 - \zeta^2}$. Thus the hypotenuse is of length ω_n. It makes an angle θ with the *negative* real axis, and

$$\cos\theta = \zeta \tag{9.2.18}$$

Therefore all roots lying on the circumference of a given circle centered on the origin are associated with the same undamped natural frequency ω_n. From (9.2.18) we see that all roots lying on the same line passing through the origin are associated with the same damping ratio. The limiting values of ζ correspond to the imaginary axis ($\zeta = 0$) and the negative real axis ($\zeta = 1$). Roots lying on a given line parallel to the real axis all give the same damped natural frequency. All roots lying on a line parallel to the imaginary axis have the same time constant. The farther to the left this line is, the smaller the time constant.

9.3 DESCRIPTION AND SPECIFICATION OF STEP RESPONSE

We can express the free response and the step response of the underdamped second-order model $m\ddot{x} + c\dot{x} + kx = f(t)$ in terms of the parameters ζ and ω_n as follows. The form of the free response is

$$x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \qquad (9.3.1)$$

where B and ϕ depend on the initial conditions $x(0)$ and $\dot{x}(0)$. The unit step response for zero initial conditions is

$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + 1\right] \qquad (9.3.2)$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi \qquad (9.3.3)$$

Because $\zeta > 0$, ϕ lies in the third quadrant. Table 9.3.1 gives the step response in terms of ζ and ω_n for the underdamped, critically damped, and overdamped cases.

The free response for $\dot{x}(0) = 0$ and the step response for $x(0) = \dot{x}(0) = 0$ of the second-order model $m\ddot{x} + c\dot{x} + kx = f(t)$ are illustrated in Figures 9.3.1 and 9.3.2 for several values of the damping ratio ζ. Note that the response axis has been normalized by k and the time axis has been normalized by $\omega_n = \sqrt{k/m}$. Thus a variation of ζ should be interpreted as a variation of c, and not of k or m. When $\zeta > 1$, the response is

Table 9.3.1 Unit step response of a stable second-order model.

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. Overdamped case ($\zeta > 1$): distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2}e^{-r_1 t} - \frac{r_1}{r_1 - r_2}e^{-r_2 t} + 1\right)$$

2. Critically damped case ($\zeta = 1$): repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t)e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1)e^{-r_1 t} + 1]$$

3. Underdamped case ($0 \leq \zeta < 1$): complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$x(t) = Be^{-t/\tau}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + 1\right]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi \quad \text{(third quadrant)}$$

Time constant: $\tau = 1/\zeta\omega_n$

Figure 9.3.1 Free response of second-order systems for various values of ζ.
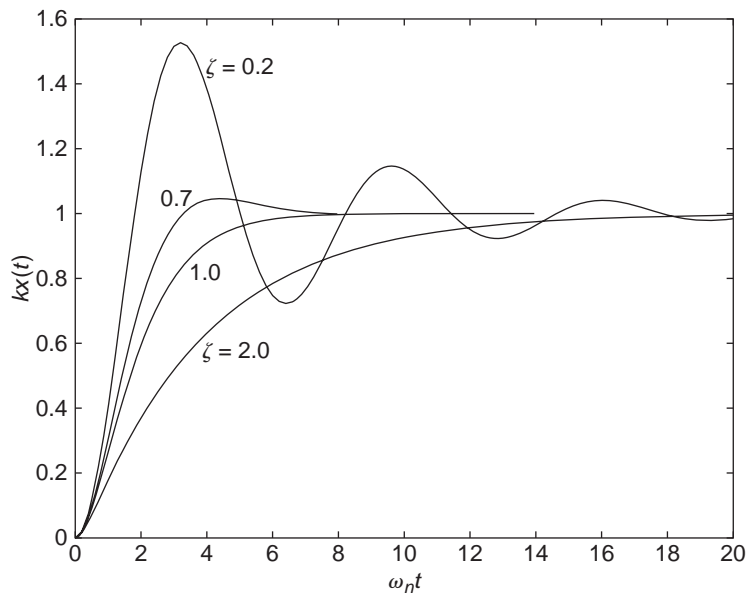
Figure 9.3.2 Step response of second-order systems for various values of ζ.

sluggish and does not overshoot the steady-state value. As ζ is decreased, the speed of response increases. The critically damped case $\zeta = 1$ is the case in which the steady-state value is reached most quickly but without oscillation. As ζ is decreased below 1, the response overshoots and oscillates about the final value. The smaller ζ is, the larger the overshoot, and the longer it takes for the oscillations to die out. There are design applications in which we wish the response to be near its final value as quickly as possible, with some oscillation tolerated. This corresponds to a value of ζ slightly less than 1. The value $\zeta = 0.707$ is a common choice for such applications. As ζ is decreased to zero (the undamped case), the oscillations never die out.

DESCRIPTION OF STEP RESPONSE

Suppose you obtained the response plot shown in Figure 9.3.3 either from a measured response or from a computer simulation. Suppose also that you want to describe the plot to someone over the phone (assuming you cannot send them the plot!). You would say that $x(t)$ starts at 0 and rises to the steady-state value of $x = 100$ mm. It oscillates briefly around the steady-state value (there are about two oscillations with a period of about 6.6 s). The first oscillation has the largest peak, which is 37 mm and occurs at $t = 3.3$ s. You might also note that the response reaches 50% of the steady-state value in 1.2 s, and it first reaches 100% of the steady-state value in 2 s. Note that, depending on the resolution of the plot, you might not be able to determine the time for the response to reach the steady state, so this time estimate might be subject to great error.

   The parameters we have just used to describe the plot not only are the standard ways of describing step response, but also are the standard ways of specifying desired performance. These transient-response specifications are illustrated in Figure 9.3.4. Note that the response need not be that of a second-order system. The *maximum or peak overshoot M_p* is the maximum deviation of the output x above its steady-state value x_{ss}. It is sometimes expressed as a percentage of the final value and denoted $M_\%$. Because the maximum overshoot increases with decreasing ζ, it is sometimes used as an indicator of the relative stability of the system. The *peak time t_p* is the time at which the maximum overshoot occurs. The *settling time t_s* is the time required for the oscillations to stay within some specified small percentage of the final value. The most common values used are 2% and 5%. If the final value of the response differs from some desired value, a steady-state error exists.

   The *rise time t_r* can be defined as the time required for the output to rise from 0% to 100% of its final value. However, no agreement exists on this definition. Sometimes, the rise time is taken to be the time required for the response to rise from 10% to 90% of the final value. Finally, the *delay time t_d* is the time required for the response to reach 50% of its final value.

Figure 9.3.3 Underdamped step response.
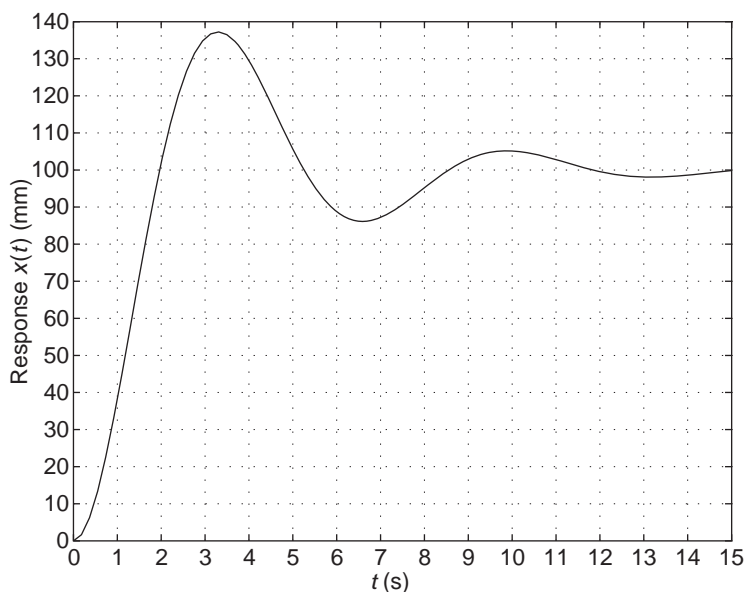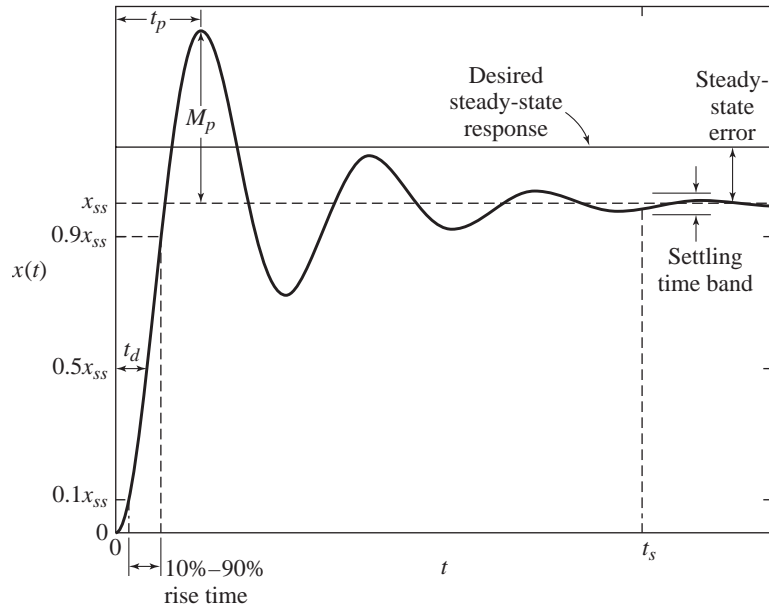
Figure 9.3.4 Transient-response specifications.

Except for the settling time, these parameters are relatively easy to obtain from an experimentally determined step-response plot. They can also be determined in analytical form for a second-order model, whose underdamped step response is given by (9.3.2) and (9.3.3).

MAXIMUM OVERSHOOT

Setting the derivative of (9.3.2) equal to zero gives expressions for both the maximum overshoot and the peak time t_p. After some trigonometric manipulation, the result is

$$\frac{dx}{dt} = \frac{1}{k}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t\right) = 0$$

For $t < \infty$, this gives

$$\omega_n\sqrt{1-\zeta^2}t = n\pi \quad n = 0, 1, 2, \ldots$$

The times at which extreme values of the oscillations occur are thus

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{9.3.4}$$

The odd values of n give the times of overshoots, and the even values correspond to the times of undershoots. The maximum overshoot occurs when $n = 1$. Thus,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{9.3.5}$$

The magnitudes $|x_n|$ of the overshoots and undershoots are found by substituting (9.3.4) into (9.3.2). After some manipulation, the result is

$$|x_n| = \frac{1}{k}\left[1 + (-1)^{n-1}e^{-n\pi\zeta/\sqrt{1-\zeta^2}}\right] \tag{9.3.6}$$

The largest value $|x_n|$ occurs when $n = 1$. Thus the maximum overshoot M_p is found when $n = 1$. It is

$$M_p = x_{\max} - x_{ss} = \frac{1}{k}e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{9.3.7}$$

The preceding expressions show that the maximum overshoot and the peak time are functions of only the damping ratio ζ for a second-order system. The *percent* maximum overshoot $M_\%$ is

$$M_\% = \frac{x_{\max} - x_{ss}}{x_{ss}}100 = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{9.3.8}$$

Frequently we need to compute the damping ratio from a measured value of the maximum overshoot. In this case, we can solve (9.3.8) for ζ as follows. Let $R = \ln 100/M_\%$. Then (9.3.8) gives

$$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} \qquad R = \ln\frac{100}{M_\%} \tag{9.3.9}$$

RISE TIME

To obtain the expression for the 100% rise time t_r, set $x = x_{ss} = 1/k$ in (9.3.2) to obtain

$$e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = 0$$

This implies that for $t < \infty$,

$$\omega_n\sqrt{1-\zeta^2}t + \phi = n\pi \quad n = 0, 1, 2, \ldots \tag{9.3.10}$$

For $t_r > 0$, $n = 2$, because ϕ is in the third quadrant. Thus,

$$t_r = \frac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \tag{9.3.11}$$

where ϕ is given by (9.3.3). The rise time is inversely proportional to the natural frequency ω_n for a given value of ζ.

SETTLING TIME

To express the settling time in terms of the parameters ζ and ω_n, we can use the fact that the exponential term in the solution (9.3.2) provides the envelopes of the oscillations. The time constant of these envelopes is $1/\zeta\omega_n$, and thus the 2% settling time t_s is

$$t_s = \frac{4}{\zeta\omega_n} \tag{9.3.12}$$

DELAY TIME

An exact analytical expression for the delay time is difficult to obtain, but we can obtain an approximate expression as follows. Set $x = 0.5x_{ss} = 0.5/k$ in (9.3.2) to obtain

$$e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) = -0.5\sqrt{1-\zeta^2} \tag{9.3.13}$$

where ϕ is given by (9.3.3). For a given ζ and ω_n, t_d can be obtained by a numerical procedure, using the following straight-line approximation as a starting guess:

$$t_d \approx \frac{1 + 0.7\zeta}{\omega_n} \tag{9.3.14}$$

Table 9.3.2 summarizes these formulas. Figure 9.3.5 shows the plots of the maximum percent overshoot, the peak time, and the 100% rise time as functions of ζ. In Section 9.5 we will see how these formulas can be used with experimentally determined response plots to estimate the parameters m, c, and k. In Chapters 10, 11, and 12, we will use these specifications to describe the desired performance of control systems.

Table 9.3.2 Step response specifications for the underdamped model $m\ddot{x} + c\dot{x} + kx = f$.

Maximum percent overshoot	$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
	$\zeta = \dfrac{R}{\sqrt{\pi^2 + R^2}}, \quad R = \ln\dfrac{100}{M_\%}$
Peak time	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Delay time	$t_d \approx \dfrac{1 + 0.7\zeta}{\omega_n}$
100% rise time	$t_r = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$
	$\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$
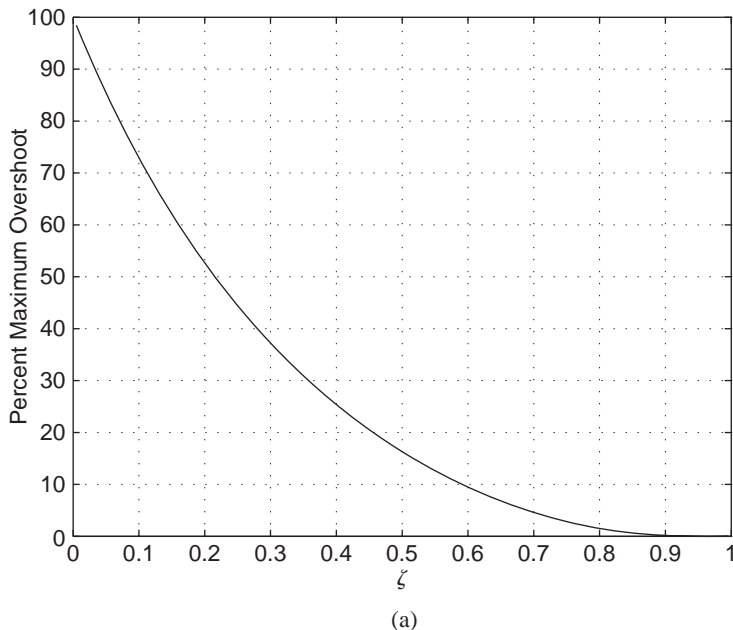
Figure 9.3.5 Percent maximum overshoot, peak time, and rise time as functions of the damping ratio ζ. (*Continued*)

Figure 9.3.5 (*Continued*)

(b)

(c)
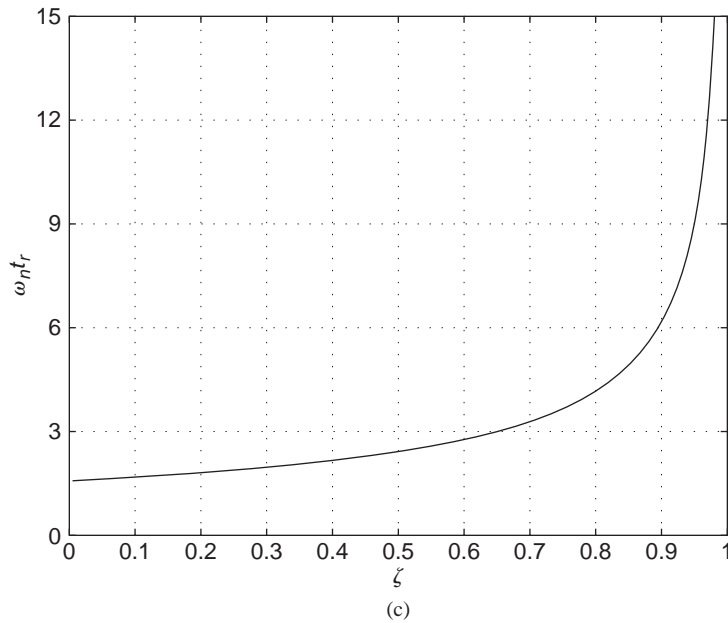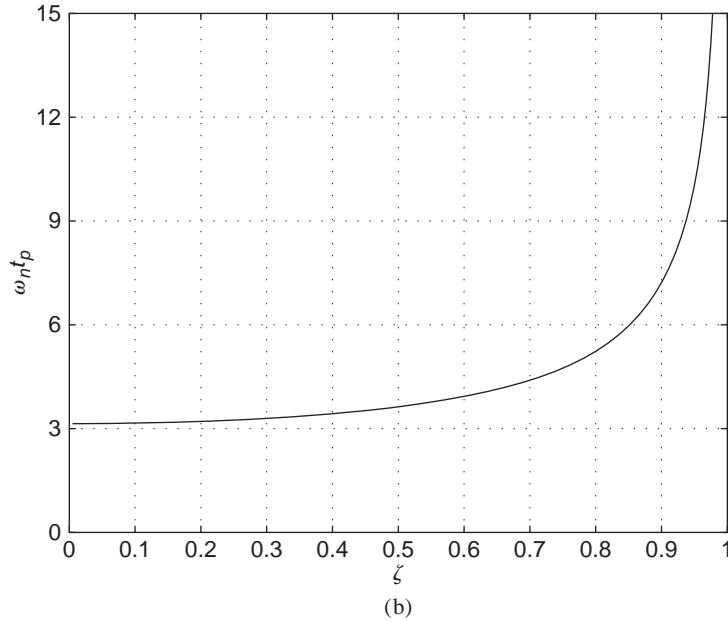
Figure 9.3.6 illustrates the effect of root location on decay rate, peak time, and overshoot. In part (a) of the figure, we see that roots lying on the same vertical line have the same decay rate because they have the same time constant. Part (b) shows that roots lying on the same horizontal line have the same oscillation frequency, period, and peak time. Part (c) of the figure shows that roots lying on the same radial line have the same damping ratio and therefore the same maximum percent overshoot $M_\%$. You can

Models A and B have the same real part, the same time constant, and the same decay time.

(a)

Models A and B have the same imaginary part, the same period, and the same peak time.

(b)
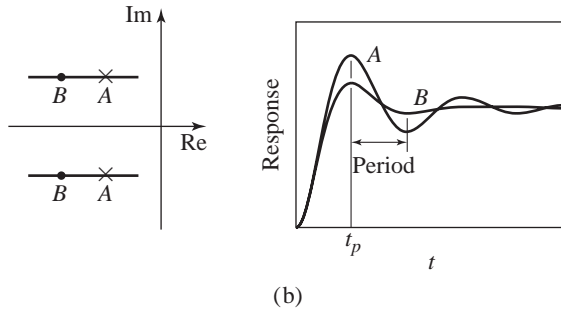
Models A, B, and C have the same damping ratio and the same overshoot.

(c)
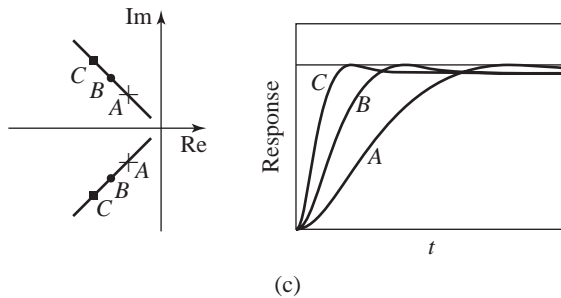
see this from the $M_\%$ formula, (9.3.8), which is a function of ζ only. Roots lying on the same horizontal line have the same damped frequency ω_d, and therefore have the same peak time t_p. This is true because $t_p = \pi/\omega_n\sqrt{1 - \zeta^2}$, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, and therefore, $t_p = \pi/\omega_d$. Thus if ω_d is constant, so is t_p.

NUMERATOR DYNAMICS AND SECOND-ORDER SYSTEM RESPONSE

Be aware that the preceding analysis is based on the model $m\ddot{x} + c\dot{x} + kx = f$, which does not have numerator dynamics. The presence of numerator dynamics can change the system behavior quite a bit, decreasing the response time and producing overshoots or undershoots in the response.

For the model $m\ddot{x} + c\dot{x} + kx = a\dot{g}(t) + bg(t)$, the effect on the step response of the numerator dynamics coefficient a is to change the formulas for the constants A_1 and A_2 in Table 9.3.1 for the real roots cases, and the formulas for the amplitude B and

angle ϕ for the complex roots case. These formulas are too cumbersome to be of use, so we do not list them.

For example, consider the two systems shown in Figure 9.3.7 for $m = 1$, $c = 6$, and $k = 8$. The model for part (a) of the figure is $\ddot{x} + 6\dot{x} + 8x = 6\dot{y}(t) + 8y(t)$, and the unit-step response is obtained from the Laplace transform.

$$X(s) = \frac{6s + 8}{s(s^2 + 6s + 8)} = \frac{1}{s} + \frac{1}{s + 2} - \frac{2}{s + 4}$$

Thus

$$x(t) = 1 + e^{-2t} - 2e^{-4t} \tag{9.3.15}$$

The model for part (b) of the figure is $\ddot{x} + 6\dot{x} + 8x = 8y(t)$, and the unit-step response is given by

$$X(s) = \frac{8}{s(s^2 + 6s + 8)} = \frac{1}{s} - \frac{2}{s + 2} + \frac{1}{s + 4}$$

$$x(t) = 1 - 2e^{-2t} + e^{-4t} \tag{9.3.16}$$

The responses are shown in Figure 9.3.8. Curve (1) corresponds to the case with numerator dynamics. We see that the numerator dynamics produces a smaller 10%–90% rise time, and causes an overshoot here, even though $\zeta > 1$.

Figure 9.3.7 Effect of damper location on numerator dynamics.

(a)   (b)
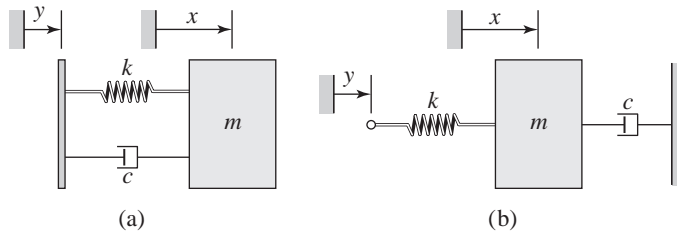
Figure 9.3.8 Response with and without numerator dynamics.
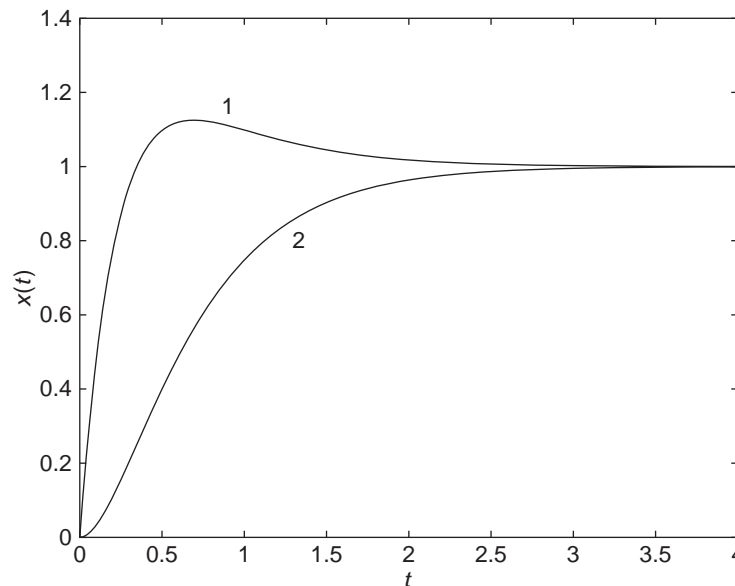
IMPULSE RESPONSE OF SECOND ORDER SYSTEMS

The impulse response of $m\ddot{x} + c\dot{x} + kx = f(t)$ is found as follows, where A is the impulse strength.

$$m\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + c\left[sX(s) - x(0)\right] + kX(s) = A$$

$$X(s) = \frac{msx(0) + A + m\dot{x}(0) + cx(0)}{ms^2 + cs + k}$$

Thus we see that the impulse has the effect of an additional initial velocity A/m. That is, the impulse has changed the momentum $m\dot{x}(0)$ by the amount A.

9.4 PARAMETER ESTIMATION IN THE TIME DOMAIN

The response calculations developed in the preceding sections cannot be used to make predictions about system behavior unless we have numerical values for the parameters in the model, such as values for m, c, and k in the model $m\ddot{x} + c\dot{x} + kx = f(t)$. In Chapter 1 we introduced some methods for estimating parameter values. In this section, we introduce additional methods, which are based on the response solutions of first- and second-order models.

USING THE FREE RESPONSE OF THE FIRST-ORDER MODEL

The free and the step response can be used to estimate one or more of the parameters of a dynamic model. For example, consider the first-order model

$$m\frac{dv}{dt} + cv = f(t) \tag{9.4.1}$$

where $f(t)$ is the input. The time constant is $\tau = m/c$, and the free response is

$$v(t) = v(0)e^{-t/\tau} \tag{9.4.2}$$

If we take the natural logarithm of both sides we obtain

$$\ln v(t) = \ln v(0) - \frac{t}{\tau}$$

By defining $V(t) = \ln v(t)$, we can transform this equation into the equation of a straight line, as follows:

$$V(t) = V(0) - \frac{t}{\tau} \tag{9.4.3}$$

This describes a straight line in terms of $V(t)$ and t. Its slope is $-1/\tau$ and its intercept is $V(0)$. These quantities may be estimated by drawing a straight line through the tranformed data points $[V(t_i), t_i]$ if the scatter in the data is not too large. Otherwise we can use the least-squares method to estimate the parameters.

If the measurement of $v(0)$ is subject to random measurement error, then $V(0)$ is not known precisely, and we can use the least squares method to compute estimates of the coefficients τ and $V(0)$. Using the least-squares equations (1.5.1) and (1.5.2) from

Chapter 1 for a first-order polynomial, we can derive the following equations:

$$-\frac{1}{\tau} \sum_{i=1}^{n} t_i^2 + V(0) \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} V_i t_i \qquad (9.4.4)$$

$$-\frac{1}{\tau} \sum_{i=1}^{n} t_i + V(0)n = \sum_{i=1}^{n} V_i \qquad (9.4.5)$$

These are two linear algebraic equations, which can be solved for τ and $V(0)$.

On the other hand, in many applications the starting value $v(0)$ can be measured without significant error. In this case, we can transform the data by using $z(t) = V(t) - V(0)$ so that the model (9.4.3) becomes $z(t) = -t/\tau$, which is a linear equation constrained to pass through the origin. We can then use (1.5.3) from Chapter 1, expressed here as

$$-\frac{1}{\tau} \sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} t_i z_i \qquad (9.4.6)$$

to find the time constant τ.

EXAMPLE 9.4.1

Estimating Capacitance from the Free Response

■ Problem
Commercially available resistors are marked with a color code that indicates their resistance value. Suppose that the resistance in the circuit of Figure 9.4.1a is 10^5 Ω. A voltage is applied to the circuit for $t < 0$ and then is suddenly removed at time $t = 0$. The voltage across the capacitor as measured by a data acquisition system is plotted in part (b) of the figure and is given in the following table. Use the data to estimate the value of the capacitance C.

Time t (s)	Voltage v_C (V)
0	5
0.25	3.3
0.5	2.2
0.75	1.4
1	0.9
1.25	0.6
1.5	0.4
1.75	0.3
2	0.2

■ Solution
The circuit model may be derived from Kirchhoff's voltage law, which gives

$$v_s = Ri + v_C \qquad \text{or} \qquad i = \frac{v_s - v_C}{R} \qquad (1)$$

For the capacitor we have

$$v_C = \frac{1}{C} \int i \, dt \qquad \text{or} \qquad \frac{dv_C}{dt} = \frac{i}{C}$$
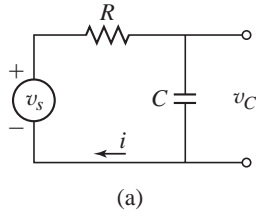
Figure 9.4.1 An RC circuit.
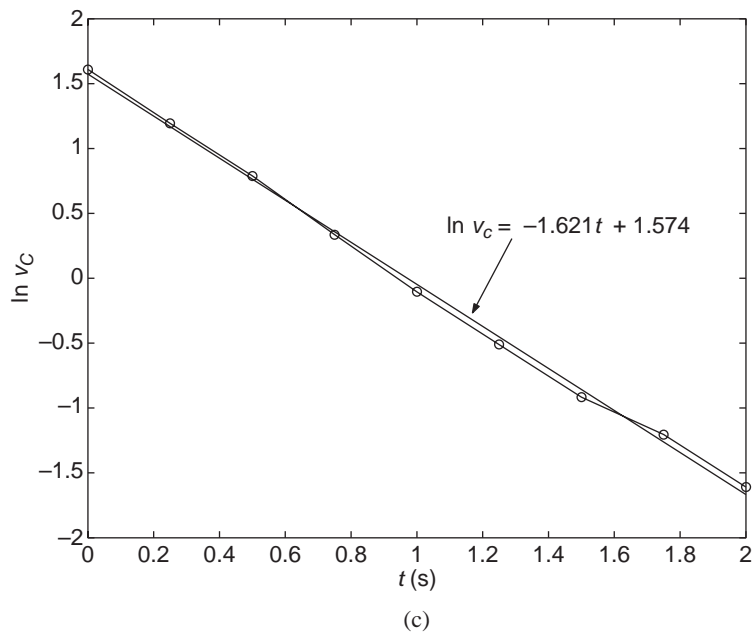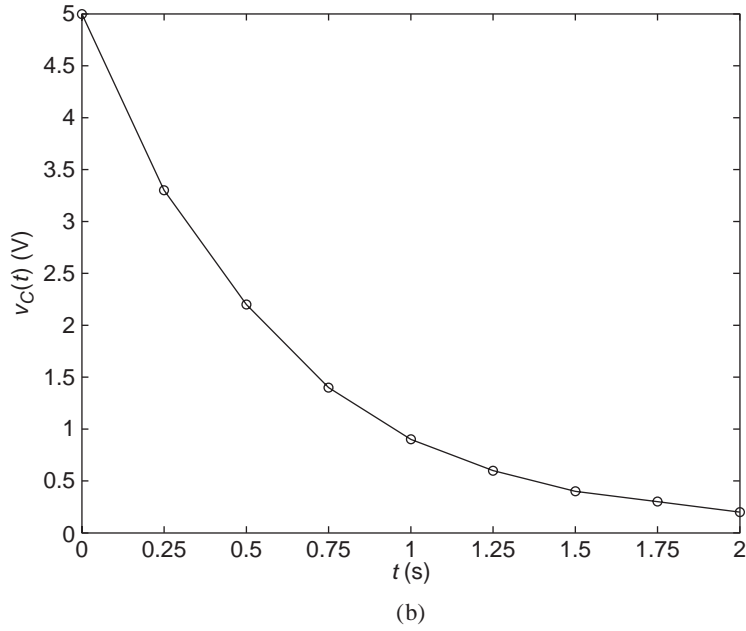
Substituting from equation (1) gives the circuit model.

$$\frac{dv_C}{dt} = \frac{v_s - v_C}{RC} \qquad \text{or} \qquad RC\frac{dv_C}{dt} + v_C = v_s$$

The free response has the form

$$v_C(t) = v_C(0)e^{-t/RC} = v_C(0)e^{-t/\tau}$$

where the time constant is $\tau = RC$. Taking the natural logarithm of both sides gives

$$\ln v_C(t) = \ln v_C(0) - \frac{t}{\tau} \qquad (2)$$

When the logarithmic transformation is applied to the original data, we obtain the following table.

Time t (s)	$\ln v_C$
0	1.609
0.25	1.194
0.5	0.789
0.75	0.337
1	−0.105
1.25	−0.511
1.5	−0.916
1.75	−1.204
2	−1.609

These transformed data are plotted in Figure 9.4.1c. Note that the data lie close to a straight line, which is given by $\ln v_C = -1.621t + 1.574$. This line was found with the least-squares method, but a similar line could have been obtained by using a straightedge to draw a line through the data. The least-squares method is required when there is considerable scatter in the data.

Comparing the equation for the line with equation (2), we obtain $\ln v_C(0) = 1.574$, which gives $v_C(0) = 4.825$, and $1/\tau = 1.621$, which gives $\tau = 0.617$. Because we know that $R = 10^5$ Ω, we obtain $C = \tau/R = 0.617/10^5 = 6.17 \times 10^{-6}$ F.

EXAMPLE 9.4.2

Temperature Dynamics

■ Problem
The temperature of liquid cooling in a porcelain mug at room temperature (68°F) was measured at various times. The data are given below.

Time t (sec)	Temperature T (°F)
0	145
620	130
2266	103
3482	90

Develop a model of the liquid temperature as a function of time, and use it to estimate how long it takes the temperature to reach 120°F.

■ Solution
We will model the liquid as a single lumped thermal mass with a representative temperature T. From conservation of heat energy we have

$$\frac{dE}{dt} = -\frac{T - T_o}{R}$$

where E is the heat energy in the liquid, $T_o = 68°F$ is the temperature of the air surrounding the cup, and R is the total thermal resistance of the cup. We have $E = mc_p(T - T_o) = C(T - T_o)$, where m is the liquid mass, c_p is its specific heat, and $C = mc_p$ is the thermal capacitance. Assuming that m, c_p, and T_o are constant, we obtain

$$C\frac{dT}{dt} = -\frac{T - T_o}{R}$$

If we let $\Delta T = T - T_o$ and note that

$$\frac{d(\Delta T)}{dt} = \frac{dT}{dt}$$

we obtain

$$RC\frac{d(\Delta T)}{dt} + \Delta T = 0 \tag{1}$$

The time constant is $\tau = RC$, and the solution has the form

$$\Delta T(t) = \Delta T(0)e^{-t/\tau}$$

Thus,

$$\ln \Delta T(t) = \ln \Delta T(0) - \frac{t}{\tau} \tag{2}$$

The transformed data $\ln \Delta T(t)$ are plotted in Figure 9.4.2a. Because they fall near a straight line, we can use equation (2) to fit the data. The values obtained are $\ln \Delta T(0) = 4.35$ and $\tau = 2792$ sec. This gives $\Delta T(0) = 77°F$. Thus the model is

$$T(t) = 68 + 77e^{-t/2792} \tag{3}$$

The computed time to reach $120°F$ is

$$t = -2792 \ln \frac{120 - 68}{77} = 1112 \text{ sec}$$

The plot of equation (3), along with the data and the estimated point (1112, 120) marked with a "+" sign, is shown in part (b) of Figure 9.4.2. Because the graph of our model lies near the data points, we can treat its prediction of 1112 sec with some confidence.

USING THE STEP RESPONSE OF THE FIRST-ORDER MODEL

The free response of the model $m\dot{v} + cv = f(t)$ enables us to estimate τ, but does not give enough information to find both m and c separately. However, we may use the step response if available. The step response of (9.4.1) for a step input of magnitude F is

$$v(t) = \left[v(0) - \frac{F}{c}\right]e^{-t/\tau} + \frac{F}{c} \tag{9.4.7}$$

Assume that we know F and $v(0)$ accurately and that we can measure the step response long enough to estimate accurately the steady-state response v_{ss}. Then we can compute c from the steady-state response $v_{ss} = F/c$; that is, $c = F/v_{ss}$. To estimate m we rearrange the step response (9.4.7) as follows, using the fact that $F/c = v_{ss}$.

$$\ln \left[\frac{v(t) - v_{ss}}{v(0) - v_{ss}}\right] = -\frac{t}{\tau}$$

Figure 9.4.2 Temperature data. (a) Plot of transformed data. (b) Plot of the fitted equation.

(a)

(b)
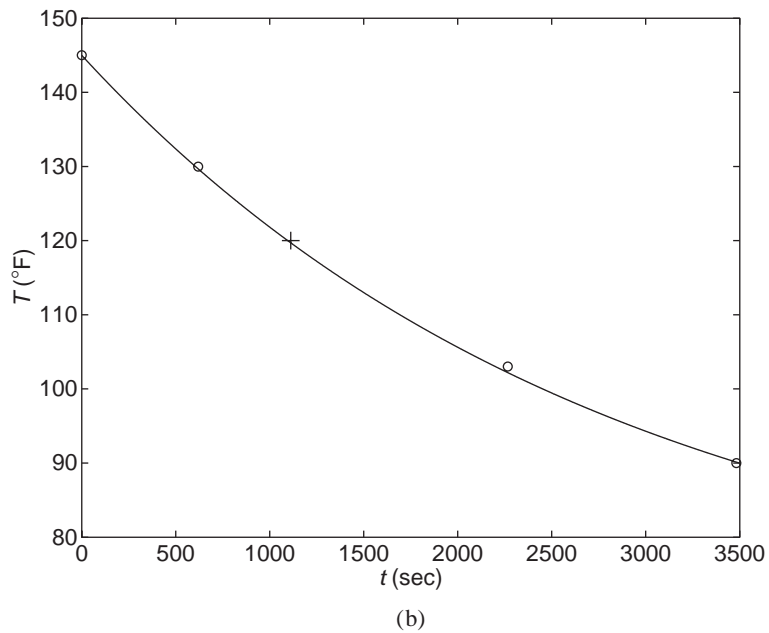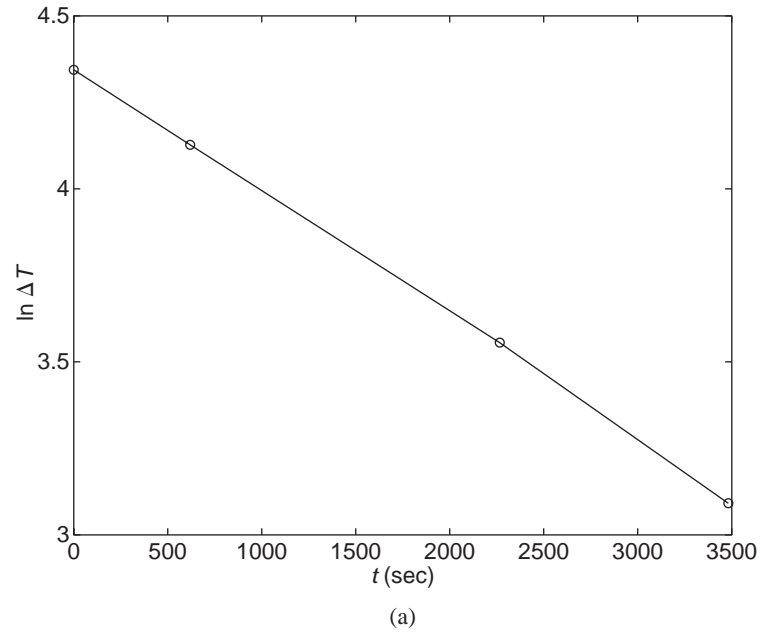
Transform the data $v(t)$ using

$$z(t) = \ln \left[\frac{v(t) - v_{ss}}{v(0) - v_{ss}} \right] \qquad (9.4.8)$$

This gives the zero-intercept, linear model: $z(t) = -t/\tau$, and we can use (9.4.6) to find τ. Assuming we have calculated c from the steady-state response, we can find m from $m = c\tau$.

PARAMETER ESTIMATION FOR THE SECOND-ORDER MODEL

Depending on the available experimental method, we can use either the free or the step response formulas to estimate the parameters of a second-order model.

| Estimating Mass, Stiffness, and Damping from the Step Response | EXAMPLE 9.4.3 |

■ Problem

Figure 9.4.3 shows the response of a system to a step input of magnitude 6×10^3 N. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Estimate the values of m, c, and k.

■ Solution

From the graph we see that the steady-state response is $x_{ss} = 6$ cm. At steady state, $x_{ss} = f_{ss}/k$, and thus $k = 6 \times 10^3 / 6 \times 10^{-2} = 10^5$ N/m.

The peak value from the plot is $x = 8.1$ cm, so the maximum percent overshoot is $M_\% = [(8.1 - 6)/6]100 = 35\%$. From Table 9.3.2, we can compute the damping ratio as follows:

$$R = \ln \frac{100}{35} = 1.0498 \qquad \zeta = \frac{R}{\sqrt{\pi^2 + R^2}} = 0.32$$

The peak occurs at $t_p = 0.32$ s. From Table 9.3.2,

$$t_p = 0.32 = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{3.316}{\omega_n}$$

Thus, $\omega_n^2 = 107$ and

$$m = \frac{k}{\omega_n^2} = \frac{10^5}{107} = 930 \text{ kg}$$
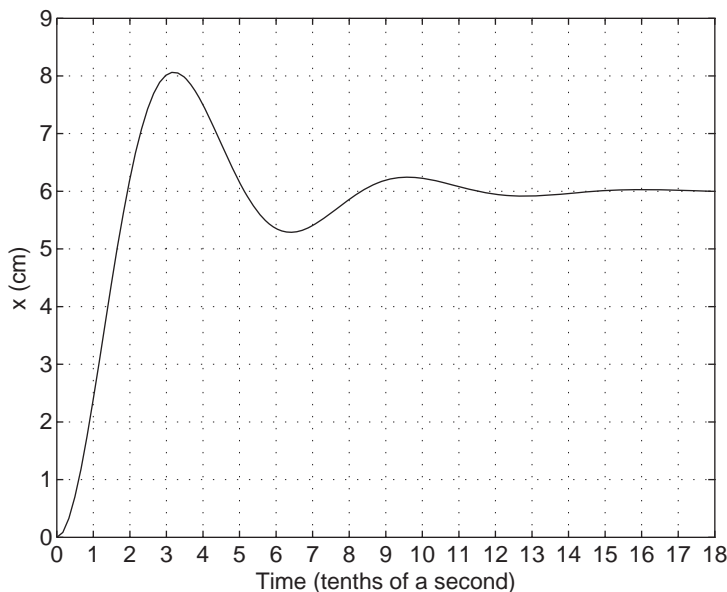
Figure 9.4.3 Measured step response.

From the expression for the damping ratio,

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{c}{2\sqrt{930(10^5)}} = 0.32$$

Thus $c = 6170 \text{ N} \cdot \text{s/m}$, and the model is

$$930\ddot{x} + 6170\dot{x} + 10^5 x = f(t)$$

THE LOGARITHMIC DECREMENT

Usually the damping coefficient c is the parameter most difficult to estimate. Mass m and stiffness k can be measured with static tests, but measuring damping requires a dynamic test. If the system exists and dynamic testing can be done with it, then the *logarithmic decrement* provides a good way to estimate the damping ratio ζ, from which we can compute c ($c = 2\zeta\sqrt{mk}$). To see how this is done, use the form $s = -\zeta\omega_n \pm \omega_d j$ for the characteristic roots, and write the free response for the underdamped case as follows:

$$x(t) = Be^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{9.4.9}$$

The frequency of the oscillation is ω_d, and thus the period P is $P = 2\pi/\omega_d$. The logarithmic decrement δ is defined as the natural logarithm of the ratio of two successive amplitudes; that is,

$$\delta = \ln \frac{x(t)}{x(t + P)} \tag{9.4.10}$$

Using (9.4.9) this becomes

$$\delta = \ln \frac{Be^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}{Be^{-\zeta\omega_n (t+P)} \sin(\omega_d t + \omega_d P + \phi)} \tag{9.4.11}$$

Note that $e^{-\zeta\omega_n(t+P)} = e^{-\zeta\omega_n t}e^{-\zeta\omega_n P}$. In addition, because $\omega_d P = 2\pi$ and $\sin(\theta + 2\pi) = \sin\theta$, $\sin(\omega_d t + \omega_d P + \phi) = \sin(\omega_d t + \phi)$, and (9.4.11) becomes

$$\delta = \ln e^{\zeta\omega_n P} = \zeta\omega_n P$$

Because $P = 2\pi/\omega_d = 2\pi/\omega_n\sqrt{1 - \zeta^2}$, we have

$$\delta = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \tag{9.4.12}$$

We can solve this for ζ to obtain

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{9.4.13}$$

If we have a plot of $x(t)$ from a test, we can measure two values x at two times t and $t + P$. These values can be measured at two successive peaks in x. The x values are then substituted into (9.4.10) to compute δ. Equation (9.4.13) gives the value of ζ, from which we compute $c = 2\zeta\sqrt{mk}$.

The plot of $x(t)$ will contain some measurement error, and for this reason, the preceding method is usually modified to use measurements of two peaks n cycles apart (Figure 9.4.4). Let the peak values be denoted B_1, B_2, etc. and note that

$$\ln \left(\frac{B_1}{B_2} \frac{B_2}{B_3} \frac{B_3}{B_4} \cdots \frac{B_n}{B_{n+1}} \right) = \ln \left(\frac{B_1}{B_{n+1}} \right)$$
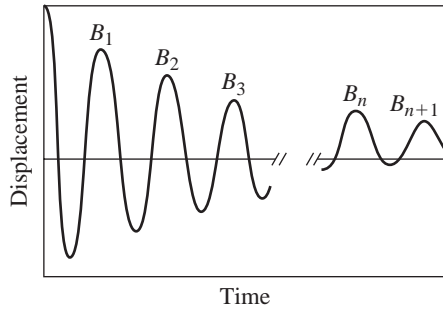
Figure 9.4.4 Terminology for logarithmic decrement.

or

$$\ln \frac{B_1}{B_2} + \ln \frac{B_2}{B_3} + \ln \frac{B_3}{B_4} \cdots \ln \frac{B_n}{B_{n+1}} = \ln \frac{B_1}{B_{n+1}}$$

Thus

$$\delta + \delta + \delta + \cdots + \delta = n\delta = \ln \frac{B_1}{B_{n+1}}$$

or

$$\delta = \frac{1}{n} \ln \frac{B_1}{B_{n+1}} \qquad (9.4.14)$$

We normally take the first peak to be B_1, because this is the highest peak and least subject to measurement error, but this is not required. The above formula applies to any two points n cycles apart.

Estimating Damping and Stiffness | **EXAMPLE 9.4.4**

■ **Problem**

Measurement of the free response of a certain system whose mass is 500 kg shows that after six cycles the amplitude of the displacement is 10% of the first amplitude. Also, the time for these six cycles to occur was measured to be 30 s. Estimate the system's damping c and stiffness k.

■ **Solution**

From the given data, $n = 6$ and $B_7/B_1 = 0.1$. Thus, from (9.4.14),

$$\delta = \frac{1}{6} \ln \left(\frac{B_1}{B_7} \right) = \frac{1}{6} \ln 10 = \frac{2.302}{6} = 0.384$$

From (9.4.13),

$$\zeta = \frac{0.384}{\sqrt{4\pi^2 + (0.384)^2}} = 0.066$$

Because the measured time for six cycles was 30 s, the period P is $P = 30/6 = 5$ s. Thus $\omega_d = 2\pi/P = 2\pi/5$. The damped frequency is related to the undamped frequency as

$$\omega_d = \frac{2\pi}{5} = \omega_n \sqrt{1 - \zeta^2} = \omega_n \sqrt{1 - (0.066)^2}$$

Thus, $\omega_n = 1.26$ and

$$k = m\omega_n^2 = 500(1.26)^2 = 794 \text{ N/m}$$

The damping constant is calculated as follows:

$$c = 2\zeta\sqrt{mk} = 2(0.066)\sqrt{500(794)} = 83.2 \text{ N} \cdot \text{s/m}$$

9.5 INTRODUCTION TO BLOCK DIAGRAMS

We can use the transfer functions of a model to construct a visual representation of the dynamics of the model. Such a representation is a *block diagram*. Block diagrams can be used to describe how system components interact with each other. Unlike a hardware schematic diagram, which shows the physical connections, the block diagram shows the cause and effect relations between the components, and thus helps us to understand the system's dynamics.

Block diagrams can also be used to obtain transfer functions for a given system, for cases where the describing differential equations are not given. In addition, block diagrams can be used to develop *simulation diagrams* for use with computer tools such as Simulink.

BLOCK DIAGRAM SYMBOLS

Block diagrams are constructed from the four basic symbols shown in Figure 9.5.1:

1. The arrow, which is used to represent a variable and the direction of the cause-and-effect relation;
2. The block, which is used to represent the input-output relation of a transfer function;
3. The circle, generically called a *summer,* which represents addition as well as subtraction, depending on the sign associated with the variable's arrow; and
4. The *takeoff point,* which is used to obtain the value of a variable from its arrow, for use in another part of the diagram.

The takeoff point does not modify the value of a variable; a variable has the same value along the entire length of an arrow until it is modified by a circle or a block. You may think of a takeoff point as the tip of a voltmeter probe used to measure a voltage at a point on a wire. If the voltmeter is well-designed, it will not change the value of the voltage it is measuring.

SOME SIMPLE BLOCK DIAGRAMS

The simplest block diagram is shown in Figure 9.5.1b. Inside the block is the system transfer function $T(s)$. The arrow going into the block represents the transform of the input, $F(s)$; the arrow coming out of the block represents the transform of the output, $X(s)$. Thus, the block diagram is a graphical representation of the cause-and-effect relations operating in a particular system. A specific case is shown in Figure 9.5.2a in which the constant transfer function K represents multiplication and the block is called a *multiplier* or a *gain* block. The corresponding equation in the time domain is $x(t) = Kf(t)$. Another simple case is shown in Figure 9.5.2b in which the transfer function $1/s$ represents integration. The corresponding equation in the time domain is $x(t) = \int f(t)\,dt$. Thus, such a block is called an *integrator*. Note that this relation corresponds to the differential equation $\dot{x} = f(t)$.

Figure 9.5.1 The four basic symbols used in block diagrams.

| (a) | (b) | (c) | (d) |
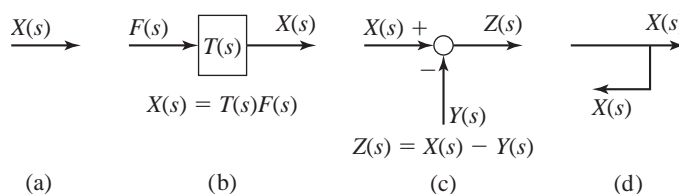
Figure 9.5.2 Two types of blocks. (a) Multiplier. (b) Integrator.
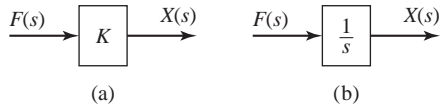
Figure 9.5.3 Diagrams representing the equation $\dot{x} + 7x = f(t)$.
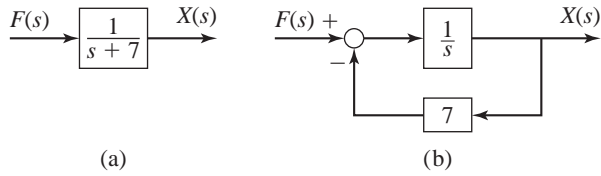
EQUIVALENT BLOCK DIAGRAMS

Figure 9.5.3 shows how more than one diagram can represent the same model, which in this case is $\dot{x} + 7x = f(t)$. The transfer function is $X(s)/F(s) = 1/(s+7)$, and the corresponding diagram is shown in part (a) of the figure. However, we can rearrange the equation as follows:

$$\dot{x} = f(t) - 7x \qquad \text{or} \qquad x = \int [f(t) - 7x]\, dt$$

which gives

$$X(s) = \frac{1}{s}[F(s) - 7X(s)]$$

In this arrangement the equation corresponds to the diagram shown in part (b) of the figure. Note the use of the takeoff point to feed the variable $X(s)$ to the multiplier. The circle symbol is used to represent addition of the variable $F(s)$ and subtraction of $7X(s)$. The diagram shows how $\dot{x}$, the rate of change of x, is affected by x itself. This is shown by the path from $X(s)$ through the multiplier block to the summer, which changes the sign of $7X(s)$. This path is called a *negative feedback path* or a *negative feedback loop*.

SERIES ELEMENTS AND FEEDBACK LOOPS

Figure 9.5.4 shows two common forms found in block diagrams. In part (a) the two blocks are said to be in *series*. It is equivalent to the diagram in part (b) because we may write

$$B(s) = T_1(s)F(s) \qquad X(s) = T_2(s)B(s)$$

These can be combined algebraically by eliminating $B(s)$ to obtain $X(s) = T_1(s) T_2(s)F(s)$. Note that block diagrams obey the rules of algebra. Therefore, any rearrangement permitted by the rules of algebra is a valid diagram.
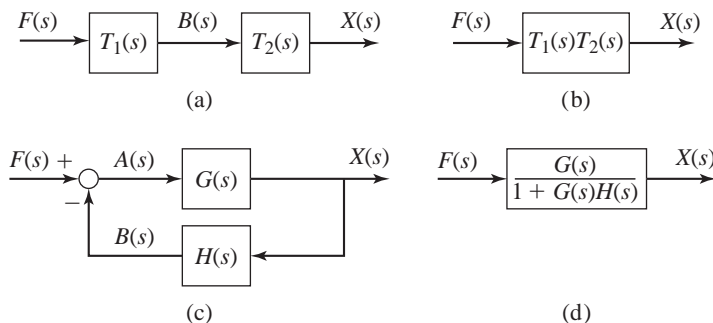
Figure 9.5.4 (a) and (b) Simplification of series blocks. (c) and (d) Simplification of a feedback loop.

Figure 9.5.4c shows a negative feedback loop. From the diagram, we can obtain the following.

$$A(s) = F(s) - B(s) \qquad B(s) = H(s)X(s) \qquad X(s) = G(s)A(s)$$

We can eliminate $A(s)$ and $B(s)$ to obtain

$$X(s) = \frac{G(s)}{1 + G(s)H(s)}F(s) \tag{9.5.1}$$

This is a useful formula for simplifying a feedback loop to a single block.

REARRANGING BLOCK DIAGRAMS

Now consider the second-order model $\ddot{x} + 7\dot{x} + 10x = f(t)$. The transfer function is $X(s)/F(s) = 1/(s^2 + 7s + 10)$, and the simplest diagram for this model is shown in Figure 9.5.5a. However, to show how x and $\dot{x}$ affect the dynamics of the system, we can construct a diagram that contains the appropriate feedback paths for x and $\dot{x}$. To do this, rearrange the equation by solving for the highest derivative.

$$\ddot{x} = f(t) - 7\dot{x} - 10x$$

The transformed equation is

$$X(s) = \frac{1}{s}\left(\frac{1}{s}\{F(s) - 7[sX(s)] - 10X(s)\}\right)$$

With this arrangement we can construct the diagram shown in Figure 9.5.5b. Recall that $sX(s)$ represents $\dot{x}$. The term within the pair of curly braces is the output of the summer and the input to the leftmost integrator. The output of this integrator is shown within the outermost pair of parentheses and is the input to the rightmost integrator.

We may use two summers instead of one, and rearrange the diagram as shown in Figure 9.5.5c. This form shows more clearly the negative feedback loop associated with the derivative $\dot{x}$. Referring to Figure 9.5.3, we see that we may replace this inner loop with its equivalent transfer function $1/(s + 7)$. The result is shown in Figure 9.5.5d, which displays only the feedback loop associated with x.

Figure 9.5.5 Diagrams representing the equation $\ddot{x} + 7\dot{x} + 10x = f(t)$.
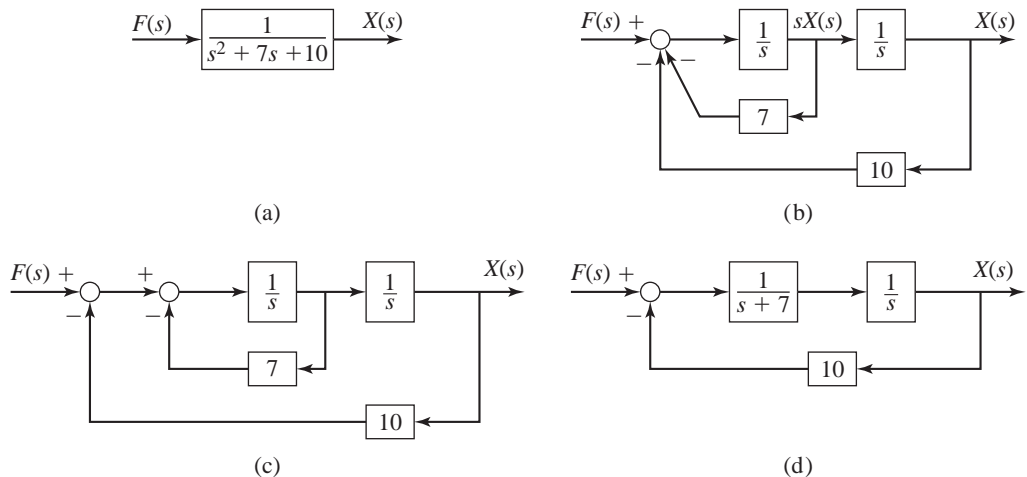
Two important points can be drawn from these examples.

1. More than one correct diagram can be drawn for a given equation; the desired form of the diagram depends on what information we want to display.

2. The form of the resulting diagram depends on how the equation is arranged. A useful procedure for constructing block diagrams is to first solve for the highest derivative of the dependent variable; the terms on the right side of the resulting equation represent the input to an integrator block.

It is important to understand that the block diagram is a "picture" of the algebraic relations obtained by applying the Laplace transform to the differential equations, assuming that the initial conditions are zero. Therefore, a number of different diagrams can be constructed for a given set of equations and they will all be valid as long as the algebraic relations are correctly represented.

Figure 9.5.6 gives more insight into how to modify diagrams. In part (a), if we follow the "flow" of the variable $F(s)$ through the original diagram on the left, it has been multiplied by $T_1(s)T_3(s)$ before reaching the output $X(s)$. In the diagram on the right, the same operations act on $F(s)$, although in a different order. The same can be said about the flow of the input $G(s)$ through both diagrams. Algebraically, moving the summer is equivalent to applying the distributive law of multiplication, as shown in the following equation:

$$X(s) = T_3(s)[T_1(s)F(s) + T_2(s)G(s)] = T_3(s)T_1(s)F(s) + T_3(s)T_2(s)G(s)$$

So the diagram on the right is equivalent to the one on the left.

A similar technique can be applied to part (b) of Figure 9.5.6 by following the flow of the input $F(s)$ to each output. The lefthand and righthand figures are equivalent

Figure 9.5.6 (a) Relocating a summer.
(b) Relocating a take-off point.

(a)

(b)
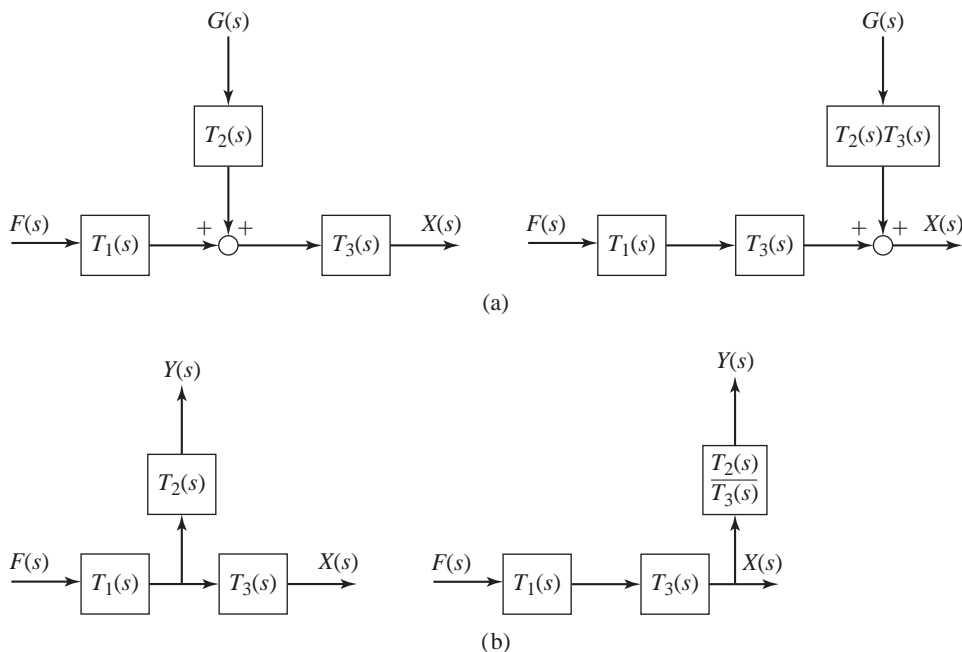
because each results in the same input-output equations. For the figure on the left,

$$X(s) = T_3(s)T_1(s)F(s) \qquad Y(s) = T_2(s)T_1(s)F(s)$$

For the figure on the right,

$$X(s) = T_3(s)T_1(s)F(s) \qquad Y(s) = \frac{T_2(s)}{T_3(s)}T_3(s)T_1(s)F(s) = T_2(s)T_1(s)F(s)$$

Rearranging diagrams sometimes simplifies the process of obtaining the transfer functions. When doing so, following the flow of a variable helps to avoid mistakes.

Block diagrams are especially useful when the model consists of more than one differential equation or has more than one input or output. For example, consider the model

$$\dot{x} = -3y + f(t) \qquad \dot{y} = -5y + 4x + g(t)$$

which has two inputs, $f(t)$ and $g(t)$. Suppose we are interested in the variable y as the output. Then the diagram in Figure 9.5.7a is appropriate. Notice that it shows how y affects itself through the feedback loop with the gain of 3, by first affecting x.

Usually we try to place the output variable on the right side of the diagram, with its arrow pointing to the right. We try to place one input on the left side with its arrow point to the right, with a second input, if any, placed at the top of the diagram. The diagram shown in Figure 9.5.7a follows these conventions, which have been established to make it easier for others to interpret your diagrams. Just as in the English language we read from left to right, so the main "flow" of the cause and effect in a diagram (from input to output) should be from left to right if possible.

If instead, we choose the output to be x, then Figure 9.5.7b is more appropriate.

TRANSFER FUNCTIONS FROM BLOCK DIAGRAMS

Sometimes we are given a block diagram and asked to find either the system's transfer function or its differential equation. There are several ways to approach such a problem; the appropriate method depends partly on the form of the diagram and partly on personal preference. The following examples illustrate the process.

Figure 9.5.7 A diagram with two inputs.

(a)

(b)
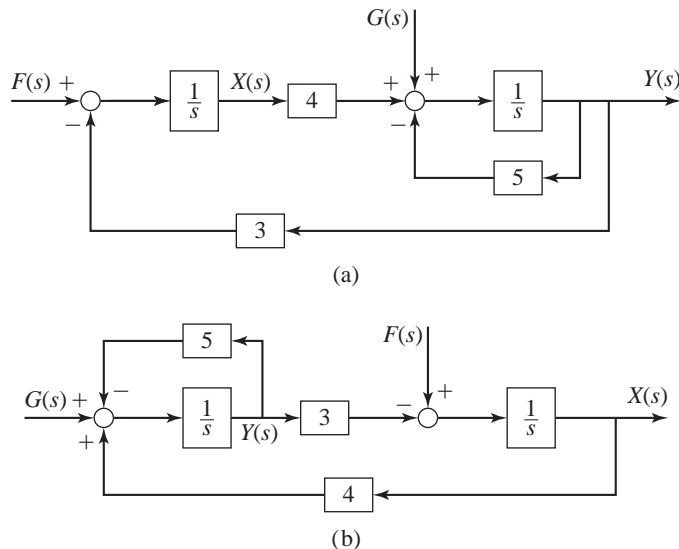
| Series Blocks and Loop Reduction | **EXAMPLE 9.5.1** |

■ Problem

Determine the transfer function $X(s)/F(s)$ for the system whose diagram is shown in Figure 9.5.8a.

■ Solution

When two blocks are connected by an arrow, they can be combined into a single block that contains the product of their transfer functions. The result is shown in part (b) of the figure. This property, which is called the *series* or *cascade* property, is easily demonstrated. In terms of the variables $X(s)$, $Y(s)$, and $Z(s)$ shown in the diagram, we can write

$$X(s) = \frac{1}{s+12}Y(s) \qquad Y(s) = \frac{1}{s+6}Z(s)$$

Eliminating $Y(s)$ we obtain

$$X(s) = \frac{1}{s+6}\frac{1}{s+12}Z(s)$$

This gives the diagram in part (b) of the figure. So we see that combining two blocks in series is equivalent to eliminating the intermediate variable $Y(s)$ algebraically.

To find the transfer function $X(s)/F(s)$, we can write the following equations based on the diagram in part (b) of the figure:

$$X(s) = \frac{1}{(s+6)(s+12)}Z(s) \qquad Z(s) = F(s) - 8X(s)$$

Eliminating $Z(s)$ from these equations gives the transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 18s + 80}$$
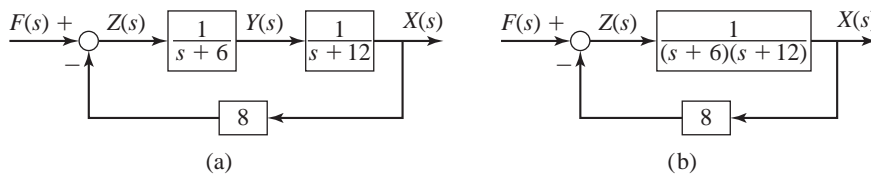
Figure 9.5.8 An example of series combination and loop reduction.

| Using Integrator Outputs | **EXAMPLE 9.5.2** |

■ Problem

Determine the model for the output x for the system whose diagram is shown in Figure 9.5.9.
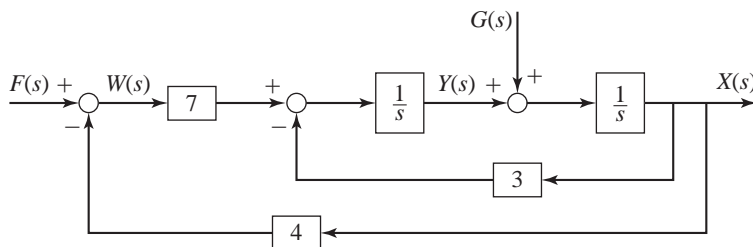
Figure 9.5.9 Diagram for Example 9.5.2.

■ Solution

The input to an integrator block $1/s$ is the time derivative of the output. Thus, by examining the inputs to the two integrators shown in the diagram we can immediately write the time-domain equations as follows.

$$\dot{x} = g(t) + y \qquad \dot{y} = 7w - 3x \qquad w = f(t) - 4x$$

We can eliminate the variable w from the last two equations to obtain $\dot{y} = 7f(t) - 31x$. Thus, the model in differential equation form is

$$\dot{x} = g(t) + y \qquad \dot{y} = 7f(t) - 31x$$

To obtain the model in transfer function form we first transform the equations.

$$sX(s) = G(s) + Y(s) \qquad sY(s) = 7F(s) - 31X(s)$$

Then we eliminate $Y(s)$ algebraically to obtain

$$X(s) = \frac{7}{s^2 + 31}F(s) + \frac{s}{s^2 + 31}G(s)$$

There are two transfer functions, one for each input-output pair. They are

$$\frac{X(s)}{F(s)} = \frac{7}{s^2 + 31} \qquad \frac{X(s)}{G(s)} = \frac{s}{s^2 + 31}$$

Sometimes, we need to obtain the expressions not for just the output variables, but also for some internal variables. The following example illustrates the required method.

EXAMPLE 9.5.3

Deriving Expressions for Internal Variables

■ Problem

Derive the expressions for $C(s)$, $E(s)$, and $M(s)$ in terms of $R(s)$ and $D(s)$ for the diagram in Figure 9.5.10.

■ Solution

Start from the right-hand side of the diagram and work back to the left until *all* blocks and comparators are accounted for. This gives

$$C(s) = \frac{7}{s+3}[M(s) - D(s)] \tag{1}$$

$$M(s) = \frac{K}{4s+1}E(s) \tag{2}$$

$$E(s) = R(s) - C(s) \tag{3}$$

Figure 9.5.10 Block diagram for Example 9.5.3.
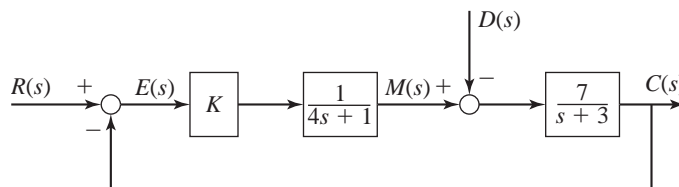
Multiply both sides of equation (1) by $s + 3$ to clear fractions, and substitute $M(s)$ and $E(s)$ from equations (2) and (3).

$$(s + 3)C(s) = 7M(s) - 7D(s)$$

$$= 7\frac{K}{4s + 1}E(s) - 7D(s) = \frac{7K}{4s + 1}[R(s) - C(s)] - 7D(s)$$

Multiply both sides by $4s + 1$ to clear fractions, and solve for $C(s)$ to obtain:

$$C(s) = \frac{7K}{4s^2 + 13s + 3 + 7K}R(s) - \frac{7(4s + 1)}{4s^2 + 13s + 3 + 7K}D(s) \tag{4}$$

The characteristic polynomial is found from the denominator of either transfer function. It is $4s^2 + 13s + 3 + 7K$.

The equation for $E(s)$ is

$$E(s) = R(s) - C(s)$$

$$= R(s) - \frac{7K}{4s^2 + 13s + 3 + 7K}R(s) + \frac{7(4s + 1)}{4s^2 + 13s + 3 + 7K}D(s)$$

$$= \frac{4s^2 + 13s + 3}{4s^2 + 13s + 3 + 7K}R(s) + \frac{7(4s + 1)}{4s^2 + 13s + 3 + 7K}D(s)$$

Because $4s^2 + 13s + 3$ can be factored as $(4s + 1)(s + 3)$, the equation for $M(s)$ can be expressed as

$$M(s) = \frac{K}{4s + 1}E(s)$$

$$= \frac{K}{4s + 1}\left[\frac{(4s + 1)(s + 3)}{4s^2 + 13s + 3 + 7K}R(s) + \frac{7(4s + 1)}{4s^2 + 13s + 3 + 7K}D(s)\right]$$

$$= \frac{K(s + 3)}{4s^2 + 13s + 3 + 7K}R(s) + \frac{7K}{4s^2 + 13s + 3 + 7K}D(s)$$

Note the cancellation of the term $4s + 1$. You should always look for such cancellations. Otherwise, the denominator of the transfer functions can appear to be of higher order than the characteristic polynomial.

9.6 MODELING SYSTEMS WITH BLOCK DIAGRAMS

One application of transfer functions is to provide a graphical representation of the system's dynamics in the form of a block diagram that illustrates the cause-and-effect relations operating in a particular system. The algebraic representation of the system's equations in terms of transfer functions enables easier manipulation for analysis and design purposes, and the graphical representation allows the engineer to see the interaction between the system's components.

OP-AMP CIRCUITS

A block diagram helps to represent the function of op amp circuits. The multiplier circuit treated in Chapter 6 consists of a feedback resistor and an input resistor. The multiplier can be modified to act as an *adder* (Figure 9.6.1a). Its block diagram is shown in Figure 9.6.1b. The output relation for the adder is

$$v_o = \left(\frac{R_3}{R_1}v_1 + \frac{R_3}{R_2}v_2\right)$$

Figure 9.6.1 An op-amp adder. (a) Circuit. (b) Block diagram.

(a)

(b)
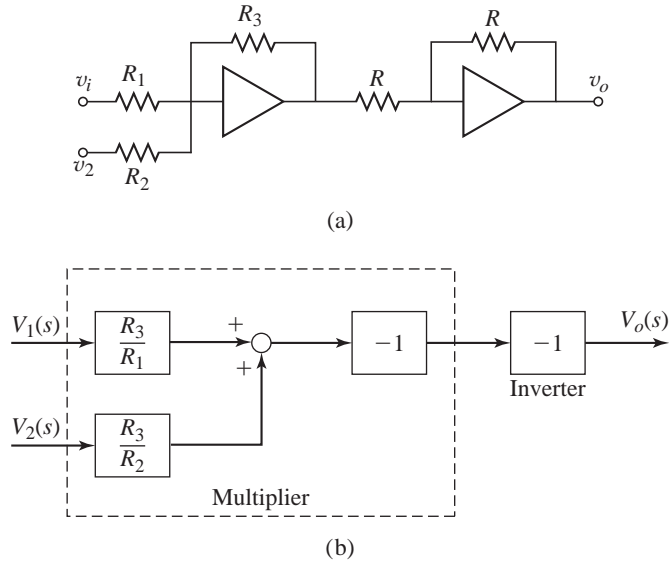
Figure 9.6.2 An op-amp subtractor. (a) Circuit. (b) Block diagram.

(a)

(b)
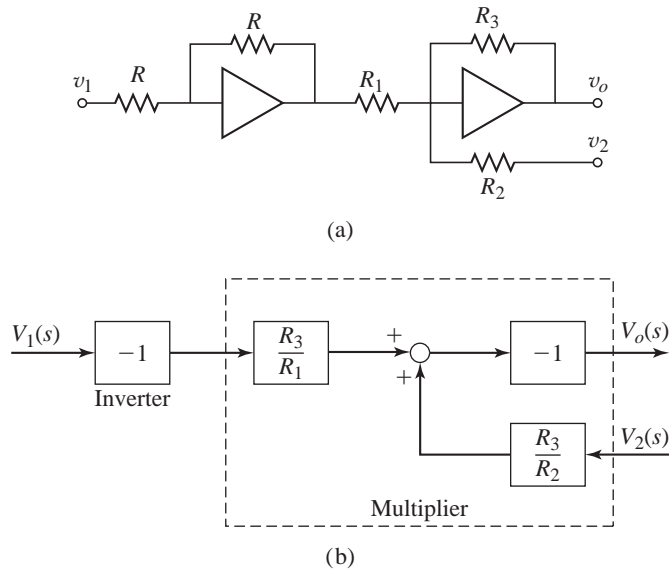
The multiplier can be modified to act as a *subtractor* if the inverter is used at the input v_1 (Figure 9.6.2a). Its block diagram is shown in Figure 9.6.2b. The output relation for the subtractor is

$$v_o = \left(\frac{R_3}{R_1} v_1 - \frac{R_3}{R_2} v_2 \right)$$

We can make a *comparator* by selecting $R_1 = R_2 = R_3$, in which case

$$v_o = v_1 - v_2$$

In Chapter 10 we will see how this is used in a control system.

Figure 9.6.3 A feedback amplifier.

Figure 9.6.4 Block diagram of a feedback amplifier.

Figure 9.6.5 An *RLC* circuit with two voltage sources.
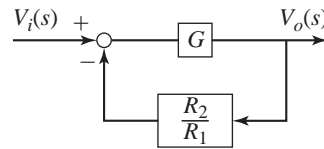
Figure 9.6.6 Block diagram of an *RLC* circuit with two voltage sources.
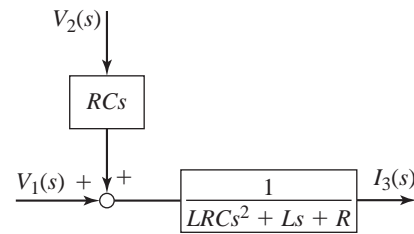
FEEDBACK AMPLIFIER

The equation for the feedback amplifier discussed in Example 6.2.6 and shown again in Figure 9.6.3 is

$$v_o = G\left(v_i - R_2 v_o / R_1\right)$$

The block diagram of this device is given in Figure 9.6.4. It shows the presence and action of the negative feedback loop, by which the output voltage v_o is used to modify the behavior of the device.

CIRCUIT MODELS

The following model of the circuit shown in Figure 9.6.5 was derived in Example 6.2.13.

$$LRC\frac{d^2 i_3}{dt^2} + L\frac{di_3}{dt} + Ri_3 = v_1 + RC\frac{dv_2}{dt}$$

The transfer function for the two inputs v_1 and v_2 are

$$\frac{I_3(s)}{V_1(s)} = \frac{1}{LRCs^2 + Ls + R} \qquad \frac{I_3(s)}{V_2(s)} = \frac{RCs}{LRCs^2 + Ls + R}$$

Figure 9.6.6 shows the corresponding block diagram.

Two Coupled *RC* Loops | **EXAMPLE 9.6.1**

■ Problem

Draw the block diagram for the circuit shown in Figure 9.6.7. This circuit was analyzed in Example 6.3.1. The input is v_s and the output is v_o.

Figure 9.6.7 Coupled *RC* loops.
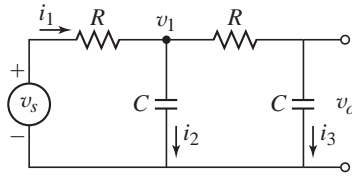
■ Solution

From equation (6) in Example 6.3.1

$$\frac{dv_1}{dt} = \frac{1}{RC}(v_s - 2v_1 + v_o)$$

This gives

$$(RCs + 2)V_1(s) = V_s(s) + V_o(s)$$

or

$$V_1(s) = \frac{1}{RCs + 2}[V_s(s) + V_o(s)] \tag{1}$$

From equation (3) in Example 6.3.1

$$\frac{dv_o}{dt} = \frac{1}{RC}(v_1 - v_o)$$

This gives

$$(RCs + 1)V_1(s) = V_1(s)$$

or

$$V_0(s) = \frac{1}{RCs + 1}V_1(s) \tag{2}$$

Equations (1) and (2) form the basis of the block diagram shown in Figure 9.6.8. The diagram can be reduced to find the transfer function, which is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1}$$

Figure 9.6.8 Block diagrams of coupled *RC* loops.
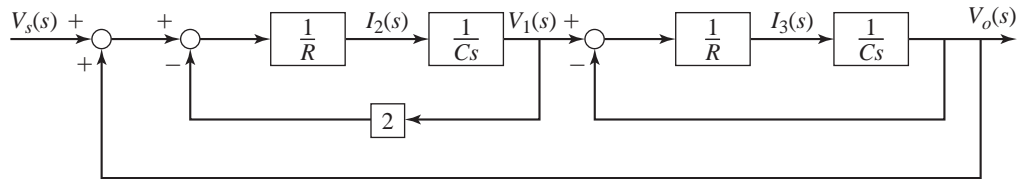
(a)
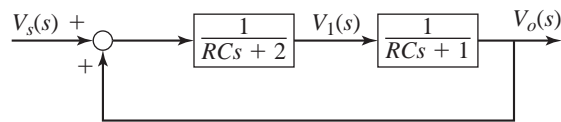
(b)
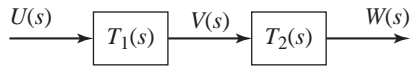
Figure 9.6.9 Block diagram of two series elements.

LOADING EFFECTS AND BLOCK DIAGRAMS

Suppose two elements whose individual transfer functions are $T_1(s)$ and $T_2(s)$ are physically connected end-to-end so that the output of the lefthand element becomes the input to the righthand element. We can represent this connection by the block diagram shown in Figure 9.6.9, *only if* the output w of the righthand element does *not* affect the inputs u and v or the behavior of the lefthand element. If it does, the righthand element is said to "load" the lefthand element. The following example illustrates this point.

Two Uncoupled RC Loops

EXAMPLE 9.6.2

■ Problem
The circuit shown in Figure 9.6.10a consists of two series RC circuits wired so that the output voltage of the first circuit is the input voltage to an isolation amplifier. The output voltage of the amplifier is the input voltage to the second RC circuit. The amplifier has a voltage gain G; that is, $v_2(t) = Gv_1(t)$. Derive the transfer function $V_o(s)/V_s(s)$ for this circuit, and for the case $G = 1$ compare it with the transfer function of the circuit shown in Figure 9.6.7.

■ Solution
The amplifier isolates the first RC loop from the effects of the second loop; that is, the amplifier prevents the voltage v_1 from being affected by the second RC loop. This in effect creates two separate loops with an intermediate voltage source $v_2 = GV_1$, as shown in Figure 9.6.10b. Thus, using the results from Example 6.2.8 for the lefthand RC loop, we obtain

$$\frac{V_1(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

For the righthand RC loop,

$$\frac{V_o(s)}{V_2(s)} = \frac{1}{RCs + 1}$$

For the amplifier with gain G,

$$V_2(s) = GV_1(s)$$

Figure 9.6.10 Two RC loops with an isolation amplifier.

(a)   (b)
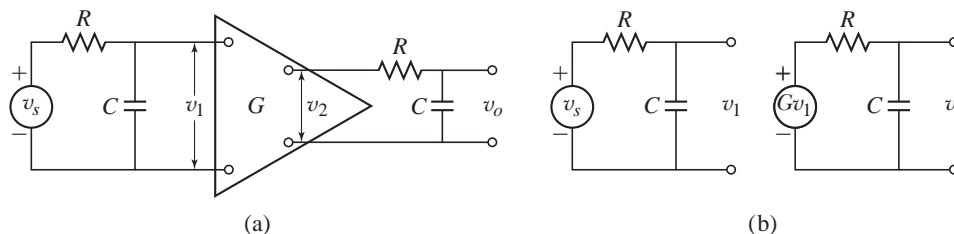
Figure 9.6.11 Block diagrams of two *RC*
loops with an isolation amplifier.

(a)   (b)
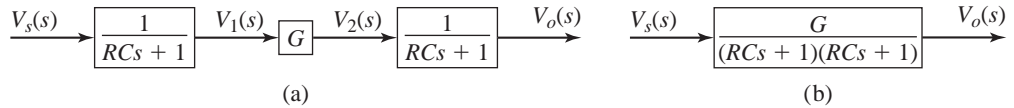
To obtain the transfer function $V_o(s)/V_s(s)$, eliminate the variables $V_1(s)$ and $V_2(s)$ from these
equations as follows:

$$\frac{V_o(s)}{V_s(s)} = \frac{V_o(s)}{V_2(s)}\frac{V_2(s)}{V_1(s)}\frac{V_1(s)}{V_s(s)} = \frac{1}{RCs+1}G\frac{1}{RCs+1} = \frac{G}{R^2C^2s^2 + 2RCs + 1} \tag{1}$$

This procedure is described graphically by the block diagram shown in Figure 9.6.11a. The three
blocks can be combined into one block as shown in part (b) of the figure.

The transfer function of the circuit shown in Figure 9.6.7 was derived in Example 9.6.2.
It is

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1} \tag{2}$$

Note that it is *not* the same as the transfer function given by equation (1) with $G = 1$.

A common mistake is to obtain the transfer function of physical elements connected end-
to-end by multiplying their transfer functions. This is equivalent to treating them as independent
devices, which they may not be if each device "loads" the adjacent device. Here the "device" is
an *RC* loop, and we have seen that in Figure 9.6.7 the righthand loop "loads" the lefthand loop,
thus changing the current and voltage in that loop. An isolation amplifier prevents a loop from
loading an adjacent loop, and when such amplifiers are used, we may multiply the loop transfer
functions to obtain the overall transfer function.

This mistake is sometimes made when drawing block diagrams. The circuit of Figure 9.6.10
can be represented by the block diagram of Figure 9.6.11, where the transfer function of each
block can be multiplied to obtain the overall transfer function $V_o(s)/V_i(s)$. However, the circuit
of Figure 9.6.7 *cannot* be represented by a simple series of blocks because the output voltage
v_o affects the voltage v_1. To show this effect requires a feedback loop, as shown previously in
Figure 9.6.8.

In general, even though elements are physically connected end-to-end, we cannot represent
them by a series of blocks if the output of one element affects its input or the behavior of any
preceding elements.

ARMATURE-CONTROLLED MOTOR

The equations of an armature-controlled dc motor were derived in Section 6.4. They
are (6.4.5) and (6.4.6) and are repeated here.

$$v_a - R_a i_a - L_a\frac{di_a}{dt} - K_b\omega = 0 \tag{9.6.1}$$

$$I\frac{d\omega}{dt} = K_T i_a - c\omega - T_L \tag{9.6.2}$$

Figure 9.6.12 is a block diagram of an armature-controlled motor with motor speed
as the output. The equations used to obtain the diagram can be obtained by transforming
the motor equations using zero initial conditions. Solve (9.6.1) for $I_a(s)$.

$$I_a(s) = \frac{1}{L_a s + R_a}[V_a(s) - K_b\Omega(s)] \tag{9.6.3}$$
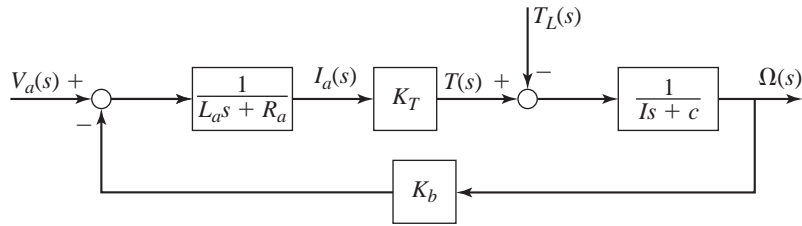
This equation is the basis for the left half of the diagram. Now solve (9.6.2) for $\Omega(s)$:

$$\Omega(s) = \frac{1}{Is + c}[K_T I_a(s) - T_L(s)] \tag{9.6.4}$$

This equation is the basis for the right half of the diagram.

The diagram shows how the back emf acts as a negative feedback loop to slow down the motor's speed. It also shows that the dynamics of the motor are affected by the dynamics of the armature circuit, whose time constant is L_a/R_a, and by the dynamics of the mechanical subsystem, whose time constant is I/c.

FIELD-CONTROLLED MOTOR

The following equations of a field-controlled dc motor were derived in Section 6.4, Example 6.4.2.

$$v_f = R_f i_f + L_f \frac{di_f}{dt} = 0 \tag{9.6.5}$$

$$I \frac{d\omega}{dt} = K_T i_f - c\omega - T_L \tag{9.6.6}$$

The block diagram of a field-controlled motor is shown in Figure 9.6.13. To see how it was obtained, take the transform of (9.6.5) with zero initial conditions, and solve for $I_f(s)$ to obtain

$$I_f(s) = \frac{1}{L_f s + R_f} V_f(s) \tag{9.6.7}$$

This equation was used to obtain the left side of the block diagram. Take the transform of (9.6.6) with zero initial conditions and solve for $\Omega(s)$ to obtain

$$\Omega(s) = \frac{1}{Is + c}\left[K_T I_f(s) - T_L(s)\right]$$

This equation was used to obtain the right side of the block diagram. Note that this motor has no feedback loop because it does not have back emf.

From (9.6.7) we see that the transfer function for the field current is

$$\frac{I_f(s)}{V_f(s)} = \frac{1}{L_f s + R_f} \tag{9.6.8}$$
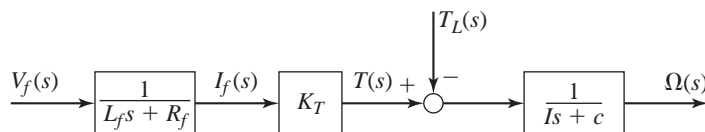
The characteristic root is $s = -R_f/L_f$, and the time constant is L_f/R_f. Thus, if $v_f(t)$ is a step function, the field current will take approximately $4L_f/R_f$ to reach the constant value of $i_f = v_f/R_f$. Note that, unlike the armature-controlled motor, the current in the field-controlled motor is not affected by the load torque T_L, because there is no feedback loop due to back emf.

From the block diagram we can easily obtain the transfer functions for the speed. Setting $T_L(s) = 0$ temporarily and using the series law for diagram reduction, we find that

$$\frac{\Omega(s)}{V_f(s)} = \frac{K_T/R_f c}{(\frac{L_f}{R_f}s + 1)(\frac{I}{c}s + 1)} \tag{9.6.9}$$

The characteristic roots are $s = -R_f/L_f$ and $s = -c/I$, which are real. So the speed ω will not oscillate if the applied voltage v_f is a step function. Its response time is governed by the larger of the two time constants, I/c and L_f/R_f, which are due to the mechanical and electrical subsystems respectively. In most cases, the largest time constant is I/c. The real roots are due to the fact that the current equation (9.6.5) does not contain the speed ω, and thus is not coupled to the speed equation (9.6.6).

Setting $V(s) = 0$ in the block diagram, we obtain

$$\frac{\Omega(s)}{T_L(s)} = -\frac{1}{Is + c} \tag{9.6.10}$$

The time constant is I/c.

If the time constant of the electrical subsystem is small compared to that of the mechanical subsystem, the speed response can be approximately described by the first-order model

$$\frac{\Omega(s)}{V_f(s)} = \frac{K_T/R_f c}{\frac{I}{c}s + 1} = \frac{K_T/R_f}{Is + c}, \qquad \frac{L_f}{R_f} << \frac{I}{c} \tag{9.6.11}$$

where the motor torque is now given approximately by

$$T(s) = \frac{K_T}{R_f}V_f(s) \tag{9.6.12}$$

BLOCK DIAGRAMS OF LIQUID-LEVEL SYSTEMS

The importance of the nonloading assumption is easily seen by examining some liquid-level systems. The single capacitance system shown in Figure 9.6.14a is analogous to a single RC loop. If q and q_o are volume flow rates, then the model is

$$A\frac{dh}{dt} = q - \frac{g}{R}h$$

Figure 9.6.14 Block diagrams for a single-capacitance fluid system. a) Liquid-level system showing the volume flow rates q and q_o. b) Block diagram with height h as output. c) Block diagram with height q_o as output.
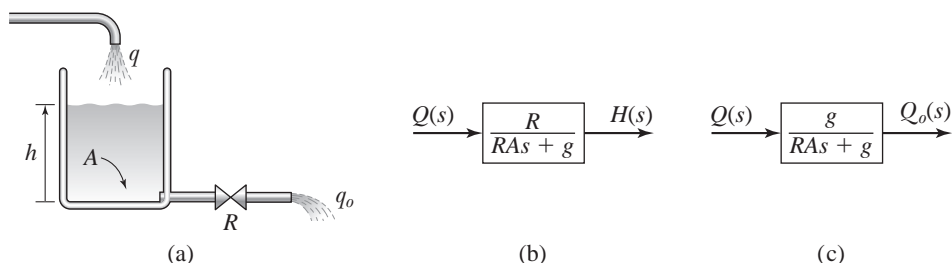
and the transfer function for the height h as the output is

$$\frac{H(s)}{Q(s)} = \frac{R}{RAs + g} \tag{9.6.13}$$

The transfer function for the volume flow rate $q_o = gh/R$ as the output is

$$\frac{Q_o(s)}{Q(s)} = \frac{g}{RAs + g} \tag{9.6.14}$$

The corresponding block diagrams are shown in Figures 9.6.14b and 9.6.14c.

Now consider the two-capacitance system shown in Figure 9.6.15a. This is analogous to the two uncoupled RC loops shown in Figure 9.6.10a because the height h_1 and the volume flow rate q_1 are not affected by the height h_2. Thus we may represent the system by connecting two block diagrams of the types shown in 9.6.14b and 9.6.14c. The result is shown in Figure 9.6.15b.

Next consider the two-capacitance system shown in Figure 9.6.16a. This is analogous to the two coupled RC loops shown in Figure 9.6.7 because the height h_1 and the volume flow rate q_1 are affected by the height h_2. The equations are

$$A_1 \frac{h_1}{dt} = q - q_1$$

$$q_1 = \frac{g}{R_1}(h_1 - h_2)$$

$$A_2 \frac{h_2}{dt} = q_1 - \frac{g}{R_2}h_2$$

Transformed, these become

$$H_1(s) = \frac{1}{A_1 s}[Q(s) - Q_1(s)]$$

$$Q_1(s) = \frac{g}{R_1}[H_1(s) - H_2(s)]$$

$$H_2(s) = \frac{R_2}{R_2 A_2 s + g}Q_1(s)$$
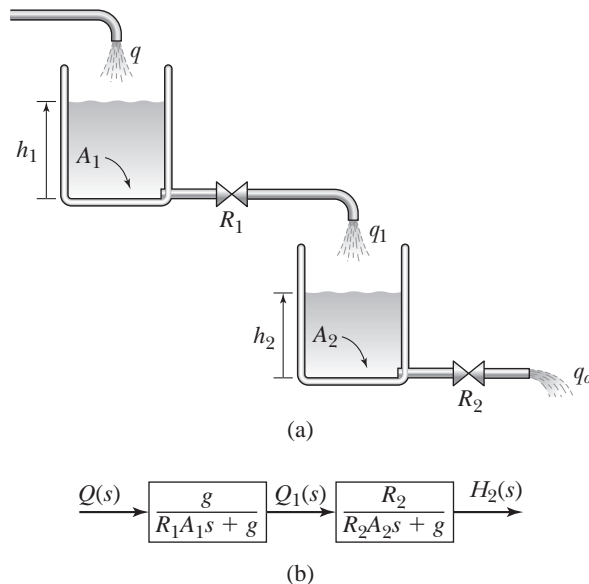
Figure 9.6.15 Block diagram for an unconnected, double-capacitance fluid system. a) Liquid-level system showing the volume flow rates q, q_1, and q_o. b) Block diagram with the height h_2 as output.

Figure 9.6.16 Block diagram for a connected, double-capacitance fluid system. a) Liquid-level system showing the volume flow rates q, q_1, and q_o. b) Block diagram with the height h_2 as output.

(a)

(b)
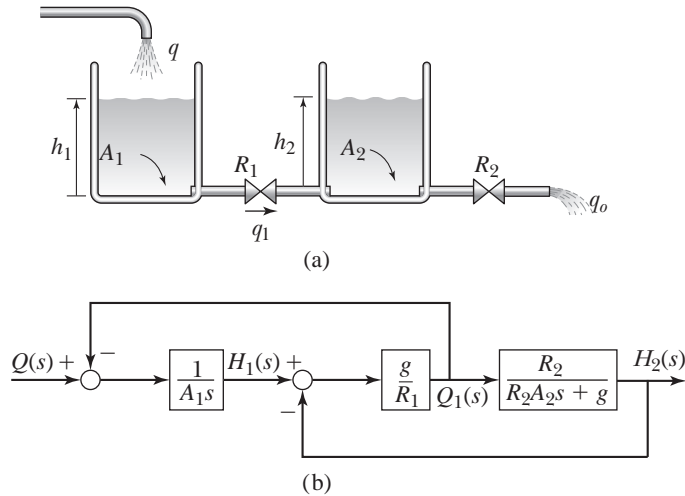
These latter three equations form the basis of the diagram shown in Figure 9.6.16b. Such a diagram will be useful for designing a system to control the heights h_1 or h_2. This will be covered in Chapter 10.

9.7　MATLAB APPLICATIONS

MATLAB has several features for determining response characteristics, such as peak time and maximum overshoot, as well as functions for doing block-diagram algebra.

COMPUTING RESPONSE CHARACTERISTICS WITH MATLAB

When the `step(sys)` function puts a plot on the screen, you may use the plot to calculate the settling time, the 10% to 90% rise time, the maximum overshoot, and the peak time by right-clicking anywhere within the plot area. This brings up a menu. Choose "Characteristics" to obtain a submenu that contains the response characteristics. When you select a specific characteristic, say "Peak Response," MATLAB puts a large dot on the peak and displays dashed lines indicating the value of the overshoot and the peak time. Move the cursor over this dot to see a display of the values. You can use the other solvers in the same way, although the menu choices may be different. For example, peak response and settling time are available when you use the `impulse(sys)` function, but not the rise time.

　　You can read values off the curve by placing the cursor on the curve at the desired point. You can also move the cursor along the curve and read the values as they change.

BLOCK DIAGRAM ALGEBRA USING MATLAB

MATLAB can be used to perform block diagram algebra if all the gains and transfer function coefficients have numerical values. You can combine blocks in series or in feedback loops using the `series` and `feedback` functions to obtain the transfer function and the state-space model.

　　If the LTI models `sys1` and `sys2` represent blocks in series, their combined transfer function can be obtained by typing `sys3 = series(sys1,sys2)`. A simple gain need not be converted to a LTI model, and does not require the `series`

function. For example, if the first system is a simple gain K, use the multiplication symbol * and type sys3 = K*sys2.

If the LTI model sys2 is in a negative feedback loop around the LTI model sys1, then type sys3 = feedback(sys1,sys2) to obtain the LTI model of the closed-loop system. If the feedback loop is positive, use the syntax sys3 = feedback(sys1,sys2,+1).

You can extract the numerator and denominator of the closed-loop transfer function by typing [num,den] = tfdata(sys3,'v'). You can then find the characteristic roots by typing roots(den).

9.8 SIMULINK APPLICATIONS

As we have seen throughout our study, linear or linearized models are useful for predicting the behavior of dynamic systems because powerful analytical techniques are available for such models, especially when the inputs are relatively simple functions such as the impulse, step, ramp, and sine. Often in the design of an engineering system, however, we must eventually deal with nonlinearities in the system and with more complicated inputs such as trapezoidal functions, and this must often be done with simulation. In this section, we introduce four additional Simulink elements that enable us to model a wide range of nonlinearities and input functions.

VEHICLE SUSPENSION RESPONSE

As our example, we will use the single-mass suspension model shown in Figure 9.8.1, where the spring and damper elements have the nonlinear models shown in Figures 9.8.2 and 9.8.3. These models represent a hardening spring and a degressive damper. In addition, the damper model is asymmetric. It represents a damper whose damping during rebound is higher than during jounce (to minimize the force transmitted to the passenger compartment when the vehicle strikes a bump). The bump is represented by the trapezoidal function $y(t)$ shown in Figure 9.8.4. This function corresponds approximately to a vehicle traveling at 30 mi/h over a 0.2-m-high road surface elevation 48 m long.

The system model from Newton's law is

$$m\ddot{x} = f_s + f_d \qquad (9.8.1)$$

where $m = 400$ kg, f_s is the nonlinear spring function shown in Figure 9.8.2, and f_d is the nonlinear damper function shown in Figure 9.8.3. The corresponding simulation diagram is shown in Figure 9.8.5.
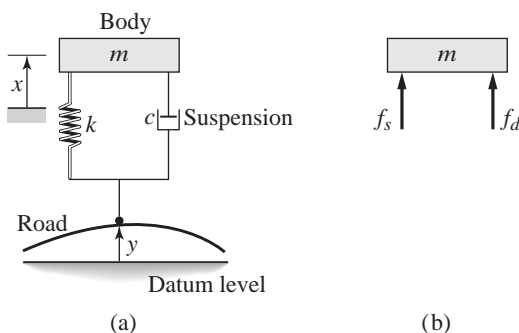
Figure 9.8.1 Single-mass suspension model.

Figure 9.8.2 Hardening spring model.

Spring force (N)

4500

500

−0.5 −0.3

−500   0.3   0.5

Deflection
$y - x$ (m)

−4500
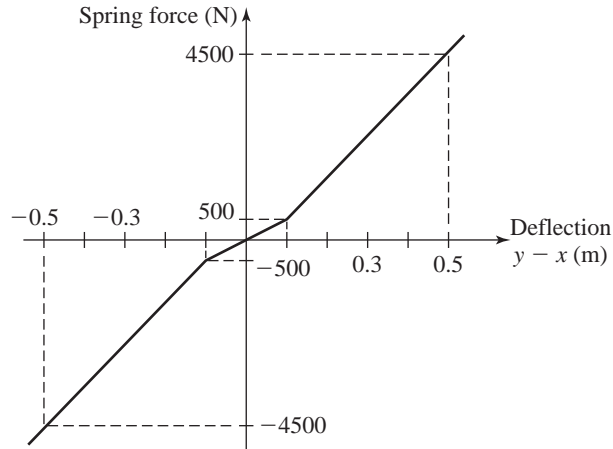
Figure 9.8.3 Degressive damper model.

Damper force f (N)

0

$f = 200v^{0.6}$

$f = -800|v|^{0.6}$

0

$v = dy/dt - dx/dt$ (m/s)
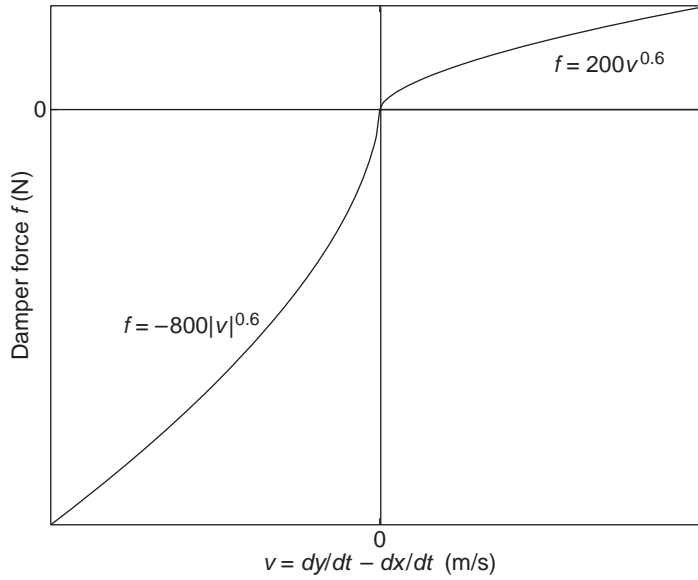
Figure 9.8.4 Bump profile.

$y(t)$ (m)

0.2

0.1   3.0   3.1

t(s)
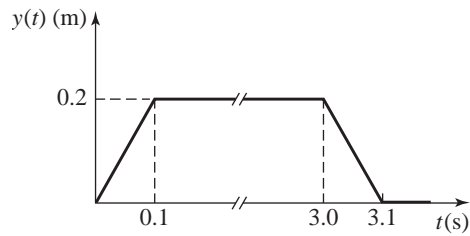
Figure 9.8.5 Simulation diagram for the suspension model.

$\dfrac{d}{dt}$ +   −

f_d

y +   +   +
      − f_s $\dfrac{1}{m}$ $\dfrac{1}{s}$ $\dot{x}$ $\dfrac{1}{s}$ x
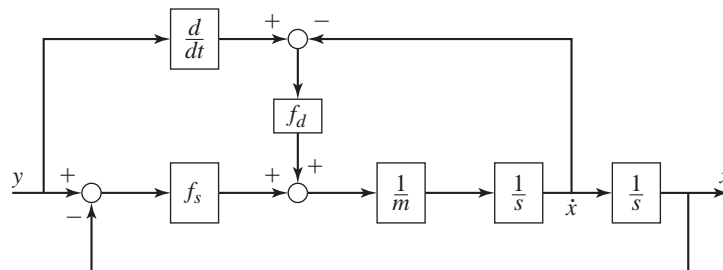
Figure 9.8.6 Simulink diagram for
the suspension model.
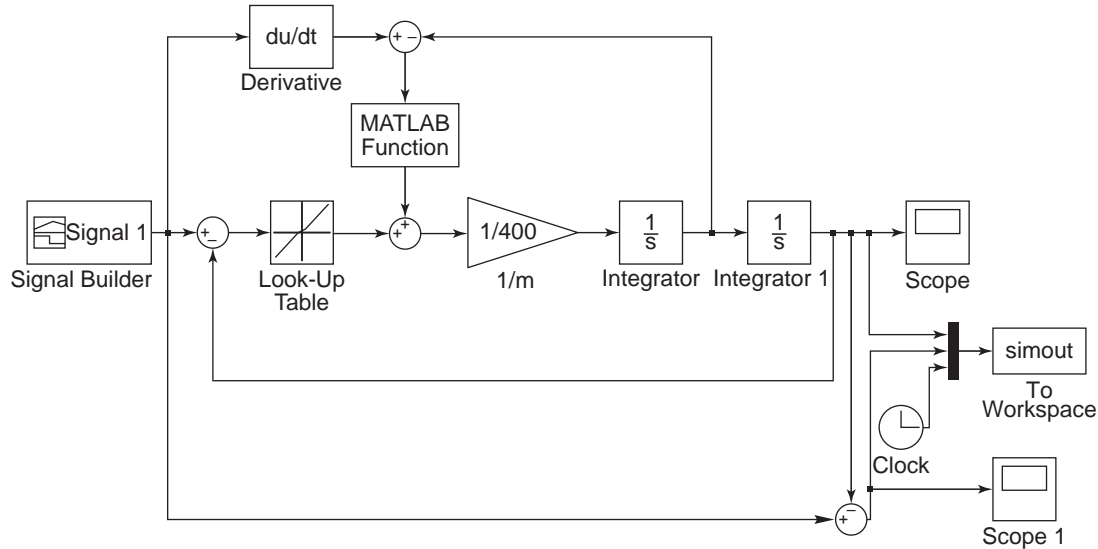
This diagram shows that we need to compute $\dot{y}$. Because Simulink uses numerical and not analytical methods, it computes derivatives only approximately, using the Derivative block, which is in the Continuous library. We must keep this in mind when using rapidly changing or discontinuous inputs. The Derivative block has no settings, so merely place it in the Simulink diagram as shown in Figure 9.8.6.

Next place the Signal Builder block, which is in the Sources library, then double-click on it. A plot window appears that enables you to place points to define the input function. Follow the directions in the window to create the function shown in Figure 9.8.4. The spring function f_s is created with the Lookup Table block, which is in the Lookup Tables library. After placing it, double-click on it and enter [-0.5, -0.1, 0, 0.1, 0.5] for the Vector of input values and [-4500, -500, 0, 500, 4500] for the Vector of output values. Use the default settings for the remaining parameters.

Place the two integrators as shown, and make sure the initial values are set to 0. Then place the Gain block and set its gain to 1/400. The To Workspace block and the Clock will enable us to plot $x(t)$ and $y(t) - x(t)$ versus t in the MATLAB Command window.

CREATING FUNCTIONS

In Section 7.10 we used the Fcn block to implement the signed square-root function. We cannot use this block for the damper function shown in Figure 9.8.3 because we must write a user-defined function to describe it. This function is as follows.

```
function f = damper(v)
if v <= 0
   f = -800*(abs(v)).^(0.6);
else
   f = 200*v.^(0.6);
end
```

Figure 9.8.7 Response plot
for the suspension model.
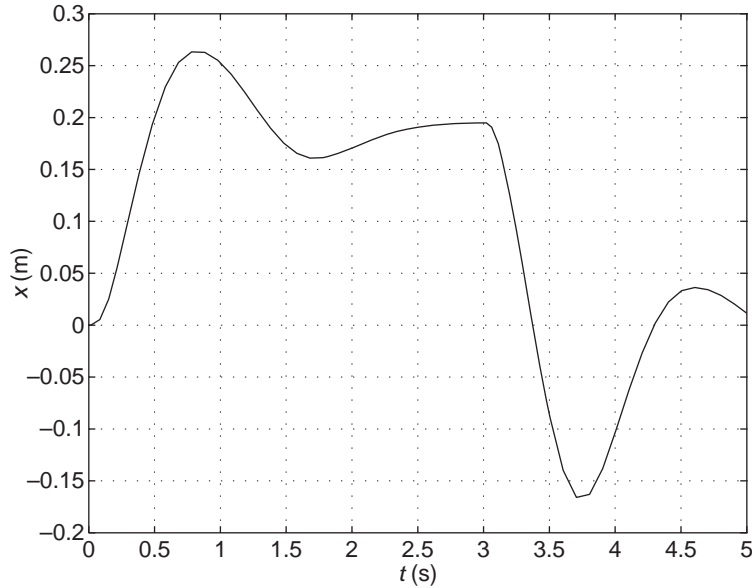
Create and save this function file. After placing the MATLAB Function block, double-click on it and enter its name `damper`. Make sure `Output dimensions` is set to `-1` and the `Output signal type` is set to `auto`.

The Fcn, MATLAB Function, Math Function, and S-Function blocks can be used to implement functions, but each has its advantages and limitations. The Fcn block can contain an expression, as we saw in Section 7.10, but its output must be a scalar, and it cannot call a function file. The MATLAB Function block is slower than the Fcn block, but its output can be an array and it can call a function file. The Math Function block can produce an array output but it is limited to a single MATLAB function and cannot use an expression or call a file. The S-Function block provides more advanced features, such as the ability to use C language code.

The Simulink model when completed should look like Figure 9.8.6. After running it, you can plot the response $x(t)$ in the Command window as follows:

```
≫x = simout(:,1);
≫t = simout(:,3);
≫plot(t,x),grid,xlabel('t (s)'),ylabel('x (m)')
```

The result is shown in Figure 9.8.7. The maximum overshoot is seen to be $(0.26 - 0.2) = 0.06$ m, but the maximum *under*shoot is seen to be much greater, -0.168 m.

9.9 CHAPTER REVIEW

This chapter emphasizes understanding system behavior in the time domain. The forcing functions commonly used to model real inputs or to test a system's response in the time domain are the impulse, the step, and the ramp functions. The impulse models a suddenly applied and suddenly removed input. The step function models a suddenly applied input that remains constant. The ramp models an input that is changing at a constant rate.

Sections 9.1 and 9.2 treated the response of first- and second-order systems. The time constant τ, the damping ratio ζ, and the undamped natural frequency ω_n are

important for assessing system response. In Section 9.3 we introduced the concepts of maximum overshoot M_p, peak time t_p, delay time t_d, rise time t_r, and settling time t_s. These are useful for describing and for specifying the step response.

Response prediction cannot be made unless we have numerical values for the model's coefficients, and Section 9.4 treated parameter estimation in the time domain.

Section 9.5 introduced block diagrams, which are useful for understanding how subsystems interact with one another. Section 9.6 presented a variety of applications of block diagram models.

Section 9.7 showed how to use MATLAB to determine transient response characteristics and how to do block diagram algebra with MATLAB. Finally, Section 9.8 introduced several Simulink blocks that are useful for simulating nonlinear systems or systems with complicated inputs.

Now that you have completed Chapter 9, you should be able to do the following:

1. Obtain and interpret the free, step, ramp, and impulse response of linear models.
2. Compute and apply the time constant τ, the damping ratio ζ, and the undamped natural frequency ω_n to assess system response.
3. Compute and apply maximum overshoot M_p, peak time t_p, delay time t_d, rise time t_r, and settling time t_s to describe and assess system response.
4. Use time-domain response data to estimate coefficients in dynamic models.
5. Interpret and draw block diagrams, given a system's equations or transfer functions.
6. Use MATLAB to determine transient response characteristics and to do block diagram algebra.
7. Use Simulink to simulate nonlinear systems and systems with complicated inputs.

PROBLEMS

Section 9.1 Response of First-Order Systems

9.1 A rocket sled has the following equation of motion: $6\dot{v} = 2700 - 24v$. How long must the rocket fire before the sled travels 2000 m? The sled starts from rest.

9.2 Suppose the rocket motor in Problem 9.1 takes 0.04 s to reach a constant thrust of 2700 N. Is a step function a good representation of this input? Support your answer with a calculation.

9.3 For each of the following models, obtain the free response and the time constant, if any.

    a. $16\dot{x} + 14x = 0$, $x(0) = 6$
    b. $12\dot{x} + 5x = 15$, $x(0) = 3$
    c. $13\dot{x} + 6x = 0$, $x(0) = -2$
    d. $7\dot{x} - 5x = 0$, $x(0) = 9$

9.4 For the model $2\dot{x} + x = 10f(t)$,

    a. If $x(0) = 0$ and $f(t)$ is a unit step, what is the steady-state response x_{ss}? How long does it take before 98% of the difference between $x(0)$ and x_{ss} is eliminated?
    b. Repeat part (a) with $x(0) = 5$.
    c. Repeat part (a) with $x(0) = 0$ and $f(t) = 20u_s(t)$.

9.5   Obtain the steady-state response of each of the following models, and estimate how long it will take the response to reach steady-state.
   a. $6\dot{x} + 5x = 20u_s(t)$, $x(0) = 0$
   b. $6\dot{x} + 5x = 20u_s(t)$, $x(0) = 1$
   c. $13\dot{x} - 6x = 18u_s(t)$, $x(0) = -2$

9.6   Obtain the total response of the following models.
   a. $6\dot{x} + 5x = 20u_s(t)$, $x(0) = 0$
   b. $6\dot{x} + 5x = 20u_s(t)$, $x(0) = 1$
   c. $13\dot{x} - 6x = 18u_s(t)$, $x(0) = -2$

9.7   A certain rotational system has the equation of motion

$$100\frac{d\omega}{dt} + 5\omega = T(t)$$

where $T(t)$ is the torque applied by an electric motor, as shown in Figure 9.1.6. The model of the motor's field current i_f in amperes is

$$0.002\frac{di_f}{dt} + 4i_f = v(t)$$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 15\,\text{N} \cdot \text{m/A}$. Suppose the applied voltage is $12u_s(t)$ V. Determine the steady-state speed of the inertia and estimate the time required to reach that speed.

9.8   The RC circuit shown in Figure 9.1.2c has the parameter values $R = 3 \times 10^6\,\Omega$ and $C = 1\,\mu\text{F}$. If the inital capacitor voltage is 6 V and the applied voltage is $v_s(t) = 12u_s(t)$, obtain the expression for the capacitor voltage response $v(t)$.

9.9   The liquid-level system shown in Figure 9.1.2d has the parameter values $A = 50\,\text{ft}^2$ and $R = 60\,\text{ft}^{-1}\text{sec}^{-1}$. If the inflow rate is $q_v(t) = 10u_s(t)$ ft³/sec, and the initial height is 2 ft, how long will it take for the height to reach 15 ft?

9.10   The immersed object shown in Figure 9.1.2e is steel and has a mass of 100 kg and a specific heat of $c_p = 500\,\text{J/kg} \cdot \text{°C}$. Assume the thermal resistance is $R = 0.09\text{°C} \cdot \text{s/J}$. The inital temperature of the object is 20° when it is dropped into a large bath of temperature 80°C. Obtain the expression for the temperature $T(t)$ of the object after it is dropped into the bath.

9.11   Compare the responses of $2\dot{v} + v = \dot{g}(t) + g(t)$ and $2\dot{v} + v = g(t)$ if $g(t) = 10u_s(t)$ and $v(0) = 5$.

9.12   Compare the responses of $5\dot{v} + v = \dot{g} + g$ and $5\dot{v} + v = g$ for $t \geq 0$ if $v(0) = 5$ and $g = 10$ for $-\infty \leq t \leq \infty$.

9.13   Consider the following model:

$$6\dot{v} + 3v = \dot{g}(t) + g(t)$$

where $v(0) = 0$.
   a.   Obtain the response $v(t)$ if $g(t) = u_s(t)$.
   b.   Obtain the response $v(t)$ to the approximate step input $g(t) = 1 - e^{-5t}$ and compare with the results of part (a).

9.14   Obtain the response of the model $2\dot{v} + v = f(t)$, where $f(t)$ is an impulse of strength 5 and $v(0) = 3$.

9.15   Obtain the response of the model $2\dot{v} + v = f(t)$, where $f(t) = 5t$ and $v(0) = 0$. Identify the transient and steady-state responses.

9.16 Obtain the response of the model $9\dot{v} + 3v = f(t)$, where $f(t) = 7t$ and $v(0) = 0$. Is steady-state response parallel to $f(t)$?

Section 9.2 Response of Second-Order Systems

9.17 Obtain the oscillation frequency and amplitude of the response of the model $3\ddot{x} + 12x = 0$ for (a) $x(0) = 5$ and $\dot{x}(0) = 0$ and (b) $x(0) = 0$ and $\dot{x}(0) = 5$.

9.18 Obtain the response of the following models with the initial conditions: $x(0) = 0$ and $\dot{x}(0) = 1$.

   a. $\ddot{x} + 4\dot{x} + 8x = 0$

   b. $\ddot{x} + 8\dot{x} + 12x = 0$

   c. $\ddot{x} + 4\dot{x} + 4x = 0$

9.19 Obtain the response of the following models with zero initial conditions:

   a. $\ddot{x} + 4\dot{x} + 8x = 2u_s(t)$

   b. $\ddot{x} + 8\dot{x} + 12x = 2u_s(t)$

   c. $\ddot{x} + 4\dot{x} + 4x = 2u_s(t)$

9.20 Obtain the response of the following models with the initial conditions: $x(0) = 1$ and $\dot{x}(0) = -1$.

   a. $3\ddot{x} + 21\dot{x} + 30x = 0$

   b. $5\ddot{x} + 20\dot{x} + 20x = 0$

   c. $2\ddot{x} + 8\dot{x} + 58x = 0$

9.21 Obtain the unit-step response of the following models with zero initial conditions:

   a. $3\ddot{x} + 21\dot{x} + 30x = f(t)$

   b. $5\ddot{x} + 20\dot{x} + 20x = f(t)$

   c. $2\ddot{x} + 8\dot{x} + 58x = f(t)$

9.22 Find the response for the following models. The initial conditions are zero.

   a. $3\ddot{x} + 21\dot{x} + 30x = 4t$

   b. $5\ddot{x} + 20\dot{x} + 20x = 7t$

   c. $2\ddot{x} + 8\dot{x} + 58x = 5t$

9.23 Suppose the input $f(t)$ to the following model is a ramp function: $f(t) = at$. Assuming that the model is stable, for what values of a, m, c, and k will the steady-state response be parallel to the input? For this case, what is the steady-state difference between the input and the response?

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

9.24 If applicable, compute ζ, τ, ω_n, and ω_d for the following roots, and find the corresponding characteristic polynomial.

   1. $s = -2 \pm 6j$

   2. $s = 1 \pm 5j$

   3. $s = -10, -10$

   4. $s = -10$

9.25 If applicable, compute ζ, τ, ω_n, and ω_d for the dominant root in each of the following sets of characteristic roots.

   1. $s = -2, -3 \pm j$

   2. $s = -3, -2 \pm 2j$

9.26 A certain fourth-order model has the roots

$$s = -2 \pm 4j, -10 \pm 7j$$

Identify the dominant roots and use them to estimate the system's time constant, damping ratio, and oscillation frequency.

9.27 Given the model

$$\ddot{x} - (\mu + 2)\dot{x} + (2\mu + 5)x = 0$$

a. Find the values of the parameter μ for which the system is
   1. Stable
   2. Neutrally stable
   3. Unstable
b. For the stable case, for what values of μ is the system
   1. Underdamped?
   2. Overdamped?

9.28 The characteristic equation of a certain system is $3s^2 + cs + 27 = 0$. Obtain the free response for the following values of damping: $c = 0, 9, 18,$ and 22. Use the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

9.29 The characteristic equation of a certain system is $4s^2 + 6ds + 25d^2 = 0$, where d is a constant. (a) For what values of d is the system stable? (b) Is there a value of d for which the free response will consist of decaying oscillations?

9.30 The characteristic equation of a certain system is $s^2 + 6bs + 5b - 10 = 0$, where b is a constant. (a) For what values of b is the system stable? (b) Is there a value of b for which the free response will consist of decaying oscillations?

9.31 A certain system has two coupled subsystems. One subsystem is a rotational system with the equation of motion:

$$50\frac{d\omega}{dt} + 10\omega = T(t)$$

where $T(t)$ is the torque applied by an electric motor, Figure 9.1.6. The second subsystem is a field-controlled motor. The model of the motor's field current i_f in amperes is

$$0.001\frac{di_f}{dt} + 5i_f = v(t)$$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 25$ N $\cdot$ m/A. Obtain the damping ratio ζ, time constants, and undamped natural frequency ω_n of the combined system.

9.32 A certain armature-controlled dc motor has the characteristic equation

$$L_a I s^2 + (R_a I + cL_a)s + cR_a + K_b K_T = 0$$

Using the following parameter values:

$$K_b = K_T = 0.1 \text{ N} \cdot \text{m/A} \qquad I = 6 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

$$R_a = 0.6 \text{ } \Omega \qquad L_a = 4 \times 10^{-3} \text{ H}$$

obtain the expressions for the damping ratio ζ and undamped natural frequency ω_n in terms of the damping c. Assuming that $\zeta < 1$, obtain the expression for the time constant.

9.33   Obtain the response for the model $\ddot{x} + 5\dot{x} + 4x = \dot{y} + 3y$ if $y(t)$ is a unit-step function and the initial conditions are zero.

9.34   Obtain the response for the model $\ddot{x} + 4\dot{x} + 85x = \dot{y} + 3y$ if $y(t)$ is a unit-step function and the initial conditions are zero.

9.35   Obtain the response for the model $\ddot{x} + 5\dot{x} + 4x = 3y$ if $y(t)$ is an impulse of strength 5 and the initial conditions are zero.

Section 9.3 Description and Specification of Step Response

9.36   Compute the maximum percent overshoot, the maximum overshoot, the peak time, the 100% rise time, the delay time, and the 2% settling time for the following model. The initial conditions are zero. Time is measured in seconds.

$$\ddot{x} + 4\dot{x} + 8x = 2u_s(t)$$

9.37   A certain system is described by the model

$$\ddot{x} + c\dot{x} + 4x = u_s(t)$$

Set the value of the damping constant c so that both of the following specifications are satisfied. Give priority to the overshoot specification. If both cannot be satisfied, state the reason. Time is measured in seconds.

1. Maximum percent overshoot should be as small as possible and no greater than 20%.
2. 100% rise time should be as small as possible and no greater than 3 s.

9.38   A certain system is described by the model

$$9\ddot{x} + c\dot{x} + 4x = u_s(t)$$

Set the value of the damping constant c so that both of the following specifications are satisfied. Give priority to the overshoot specification. If both cannot be satisfied, state the reason. Time is measured in seconds.

1. Maximum percent overshoot should be as small as possible and no greater than 20%.
2. 100% rise time should be as small as possible and no greater than 3 s.

9.39   Derive the fact that the peak time is the same for all characteristic roots having the same imaginary part.

9.40   For the two systems shown in Figure 9.3.7, the displacement $y(t)$ is a given input function. Obtain the response for each system if $y(t) = u_s(t)$ and $m = 3$, $c = 18$, and $k = 10$, with zero initial conditions.

9.41   For the two systems shown in Figure 9.3.7, the displacement $y(t)$ is a given input function. Suppose that $y(t) = u_s(t)$ and $m = 3$, $c = 18$, and $k = 10$, with zero initial conditions. Use MATLAB to evaluate the 2% settling time, the 10% to 90% rise time, the maximum overshoot, and the peak time.

Section 9.4 Parameter Estimation in the Time Domain

9.42   Suppose that the resistance in the circuit of Figure 9.4.1a is $3 \times 10^6\ \Omega$. A voltage is applied to the circuit and then is suddenly removed at time $t = 0$. The measured voltage across the capacitor is given in the following table. Use the data to estimate the value of the capacitance C.

Time t (s)	Voltage v_C (V)
0	12.0
2	11.2
4	10.5
6	9.8
8	9.2
10	8.6
12	8.0
14	7.5
16	7.0
18	6.6
20	6.2

9.43 The temperature of liquid cooling in a cup at room temperature (68°F) was measured at various times. The data are given next.

Time t (sec)	Temperature T (°F)
0	178
500	150
1000	124
1500	110
2000	97
2500	90
3000	82

Develop a model of the liquid temperature as a function of time, and use it to estimate how long it will take the temperature to reach 135°F.

9.44 Figure P9.44 shows the response of a system to a step input of magnitude 1000 N. The equation of motion is $m\ddot{x} + c\dot{x} + kx = f(t)$. Estimate the values of m, c, and k.

Figure P9.44
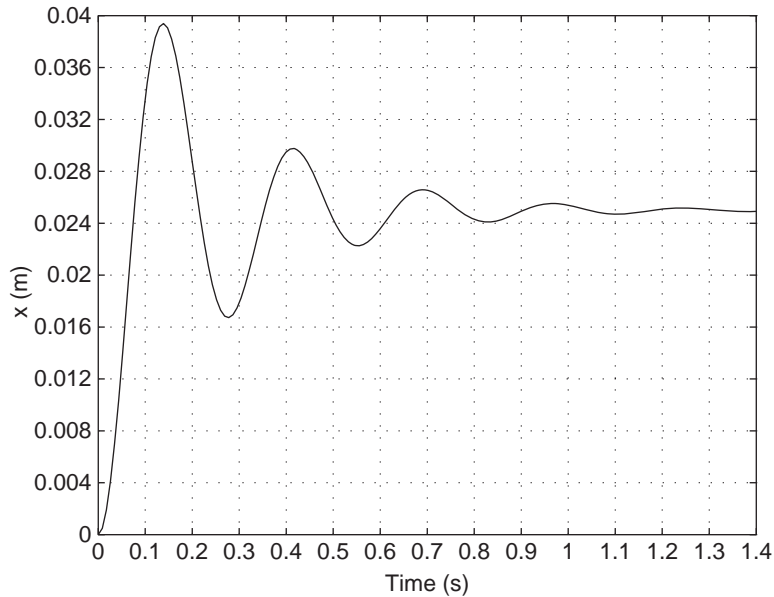
9.45 A mass-spring-damper system has a mass of 100 kg. Its free response amplitude decays such that the amplitude of the 30th cycle is 20% of the amplitude of the 1st cycle. It takes 60 s to complete 30 cycles. Estimate the damping constant c and the spring constant k.

Section 9.5 Introduction to Block Diagrams

9.46 Obtain the transfer function $X(s)/F(s)$ from the block diagram shown in Figure P9.46.

Figure P9.46
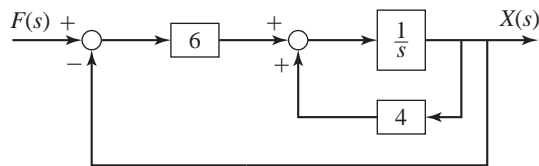
Figure P9.47
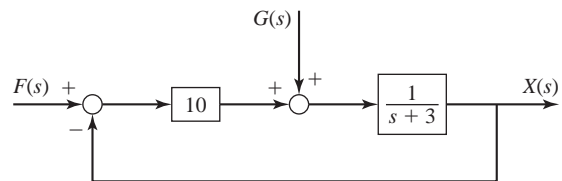
9.47 Obtain the transfer function $X(s)/F(s)$ from the block diagram shown in Figure P9.47.

9.48 Obtain the transfer function $X(s)/F(s)$ from the block diagram shown in Figure P9.48.
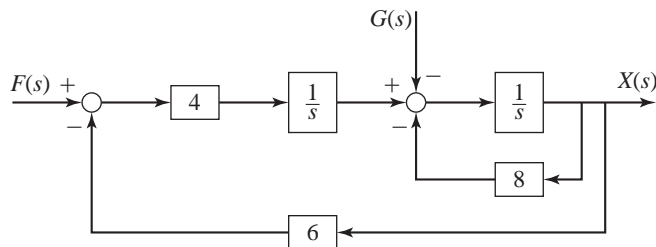
Figure P9.48

9.49 Draw a block diagram for the following equation. The output is $X(s)$; the inputs are $F(s)$ and $G(s)$.

$$5\ddot{x} + 3\dot{x} + 7x = 10f(t) - 4g(t)$$

9.50 Draw a block diagram for the following model. The output is $X(s)$; the inputs are $F(s)$ and $G(s)$. Indicate the location of $Y(s)$ on the diagram.

$$\dot{x} = y - 5x + g(t) \qquad \dot{y} = 10f(t) - 30x$$

9.51 Referring to Figure P9.51, derive the expressions for the variables $C(s)$, $E(s)$, and $M(s)$ in terms of $R(s)$ and $D(s)$.
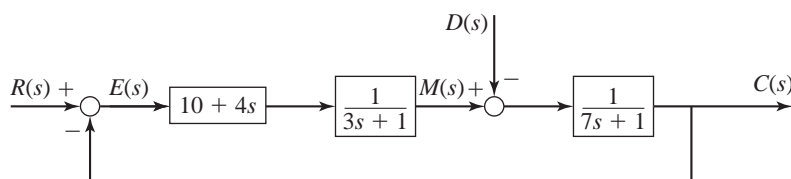
Figure P9.51

9.52   Referring to Figure P9.52, derive the expressions for the variables $C(s)$, $E(s)$, and $M(s)$ in terms of $R(s)$ and $D(s)$.

Figure P9.52
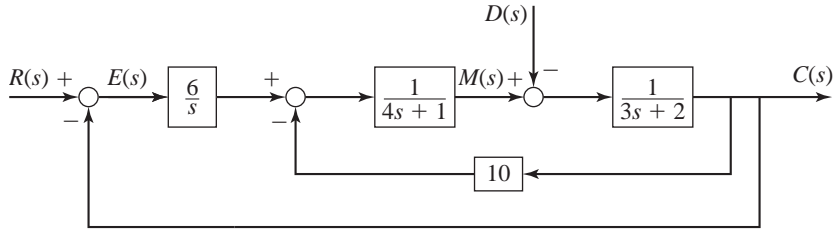
Section 9.6 Modeling Systems with Block Diagrams

9.53   Draw a block diagram of the circuit shown in Figure P9.53. The inputs are v_1 and v_2. The output is i_2.

Figure P9.53
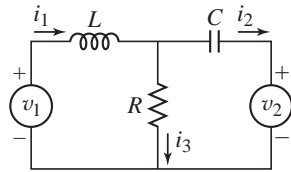
9.54   Draw a block diagram of the circuit shown in Figure P9.54. The inputs are v_1 and v_2. The output is v_3.

Figure P9.54
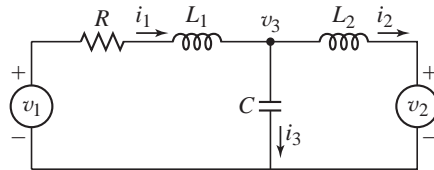
9.55   Modify the motor block diagram shown in Figure 9.6.13 to account for a gear pair between the motor shaft and the load. The ratio of motor speed to load speed ω_L is N. The motor inertia is I_m and the motor damping is c_m. The load inertia is I_L and the load damping is c_L. The load torque acts directly on the load inertia.

Section 9.8 Simulink Applications

9.56   Redo the Simulink suspension model developed in Section 9.8, using the spring relation and input function shown in Figure P9.56, and the following damper relation.

$$f_d(v) = \begin{cases} -500|v|^{1.2} & v \leq 0 \\ 50v^{1.2} & v > 0 \end{cases}$$

Use the simulation to plot the response. Evaluate the overshoot and undershoot.

Figure P9.56

(a)

(b)
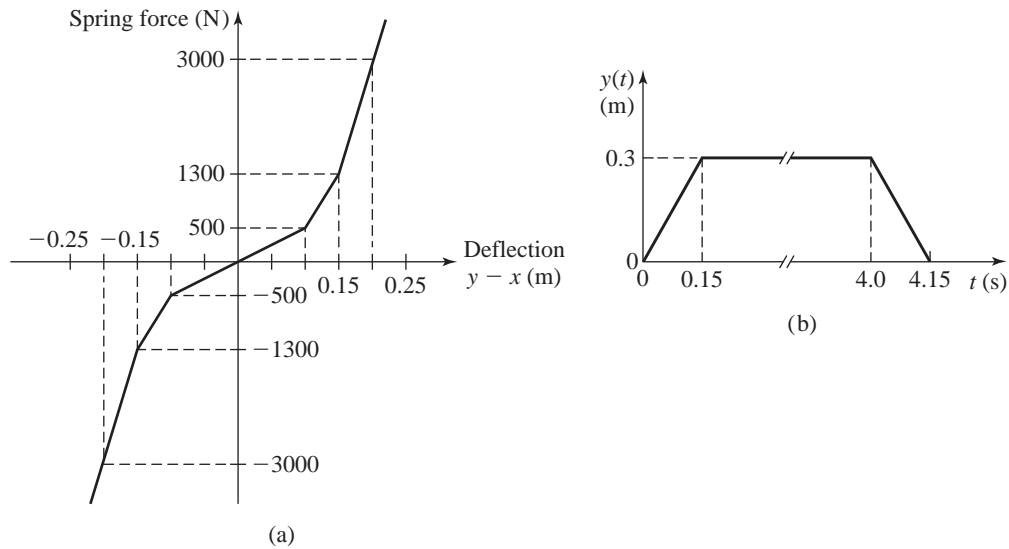
9.57 Consider the system shown in Figure P9.57. The equations of motion are

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = f(t)$$

Suppose that $m_1 = m_2 = 1$, $c_1 = 3$, $c_2 = 1$, $k_1 = 1$, and $k_2 = 4$.

a. Develop a Simulink simulation of this system. In doing this, consider whether to use a state-variable representation or a transfer function representation of the model.

b. Use the Simulink program to plot the response $x_1(t)$ for the following input. The initial conditions are zero.

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \ge 2 \end{cases}$$

Figure P9.57

Introduction to Feedback Control Systems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Model common control system components.

2. Select an appropriate control algorithm of the PID type or one of its variations, for a given application and for given steady-state and transient performance specifications.

3. Analyze the performance of a control algorithm using transfer functions, block diagrams, and computer methods, in light of given performance specifications.

4. Use MATLAB and Simulink to analyze and simulate control systems.

A major application of the methods of system dynamics is the design of control systems. In this chapter we introduce the basic concepts of feedback control and show how to model control systems so that we can analyze their performance. To this end we will use many of the modeling and analysis techniques developed in earlier chapters.

Section 10.1 introduces the concept of feedback control, also called closed-loop control. Control system terminology is covered in Section 10.2, and Section 10.3 shows how the various components in control systems are modeled. The most common control algorithm is the so-called PID algorithm, which is introduced in Section 10.4. A general control system analysis procedure is presented in Section 10.5.

The performance of PID control and variations of it are analyzed in Sections 10.6 and 10.7 for a variety of system models. Additional examples showing specific applications are covered in Section 10.8. Computer analysis and simulation of control systems are the topics of Sections 10.9 and 10.10, which use MATLAB and Simulink, respectively. Section 10.11 summarizes the major topics of the chapter. ■

10.1 CLOSED-LOOP CONTROL

A common example of a control system is a timer used to turn room lights on and off. Such a control system is called an *open-loop* system because it does not use a sensor to alter its behavior. If a light sensor is used to adjust the timing, the system would be *closed-loop*. The term derives from the block diagram of such a system, in which the sensor measurement is represented by a feedback loop.

While open-loop systems have their applications and can be used to supplement closed-loop systems, the emphasis in this text is on closed-loop control, because of the numerous and great advantages afforded by the use of a sensor to measure the output of the object or process to be controlled.

OPEN-LOOP CONTROL

Consider the liquid-level system shown in Figure 10.1.1a. Its model is

$$RA\dot{h} + gh = Rq_c(t) + Rq_d(t)$$

where $q_c(t)$ is a volume flow rate that is under our control, and the disturbance $q_d(t)$ is a volume flow rate that is not under our control and is somewhat unpredictable. The tank's bottom area is A and the outlet resistance is R.

The block diagram is shown in Figure 10.1.1b. Note that

$$\frac{H(s)}{Q_c(s)} = \frac{R}{RAs + g} \tag{10.1.1}$$

Suppose we want to control the rate q_c so that the liquid height $h(t)$ is some specified or requested function of time, $h_r(t)$. A control valve used for this purpose is illustrated in Figure 10.1.2. The valve opening and therefore the flow rate are controlled by the current-to-pressure transducer that regulates the air pressure applied to the valve.

Assuming that 1) we know the value of g accurately, 2) $q_d = 0$, and 3) we have estimates A_e and R_e of the area A and resistance R, we can calculate the required flow rate q_c from the model as follows:

$$q_c(t) = A_e\dot{h}_r + \frac{g}{R_e}h_r \tag{10.1.2}$$
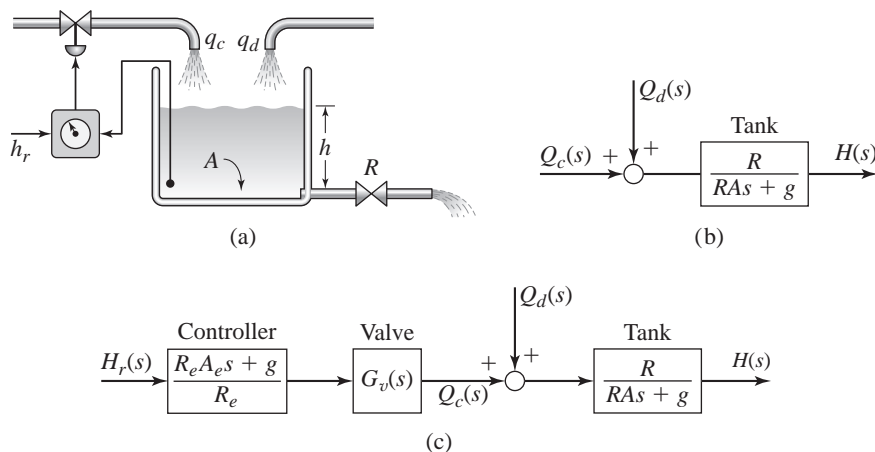
Figure 10.1.1 Open-loop control of a liquid-level system.

Figure 10.1.2 A pneumatic control valve.
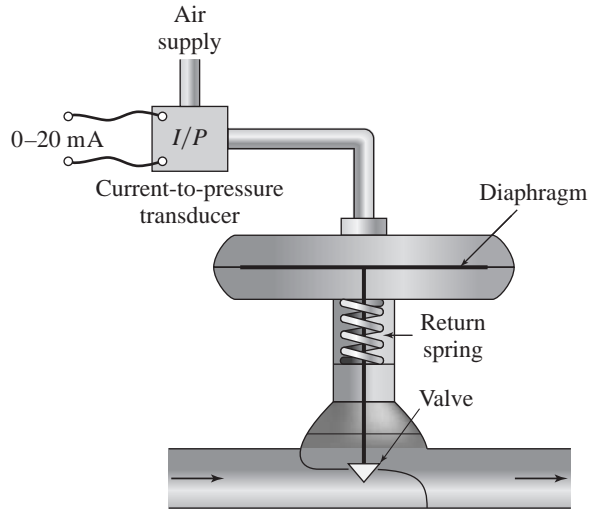
The transfer function of this algorithm is

$$\frac{Q_c(s)}{H_r(s)} = \frac{R_e A_e s + g}{R_e} \qquad (10.1.3)$$

Note that (10.1.3) is the inverse of (10.1.1) with R_e and A_e replacing R and A. This scheme is illustrated in Figure 10.1.1c. If we assume that the valve instantaneously gives the flow rate $q_c(t)$ calculated and commanded by the controller, then $G_v(s) = 1$.

If we substitute (10.1.3) into (10.1.1), we obtain

$$H(s) = Q_c(s)\frac{R}{RAs + g} = \frac{R_e A_e s + g}{R_e}\frac{R}{RAs + g}H_r(s) \qquad (10.1.4)$$

Now if our estimates R_e and A_e are exact, then $A_e = A$, $R_e = R$, and (10.1.4) becomes

$$H(s) = \frac{RAs + g}{R}\frac{R}{RAs + g}H_r(s) = 1H_r(s)$$

This implies that the actual height $h(t)$ will equal the desired height $h_r(t)$. In practice, there are several reasons why this approach can fail:

1. We rarely would have exact values for A_e and R_e.
2. The controller, which is the physical device making the calculation required by (10.1.2) cannot compute the derivative $\dot{h}_r$ exactly. This is true for any type of controller, whether it be an analog controller, composed of op amps, for example, or a digital computer.
3. The model (10.1.1) only approximately describes the real system, which may be nonlinear, for example.
4. This scheme will not work whenever the disturbance flow q_d is present.

This example illustrates the difficulties with using open-loop control.

CLOSED-LOOP CONTROL

Suppose we use a sensor to measure the liquid height h and compare the measured height with the desired height h_r, as shown in Figure 10.1.3. The *error signal, $e = h_r - h$*, is
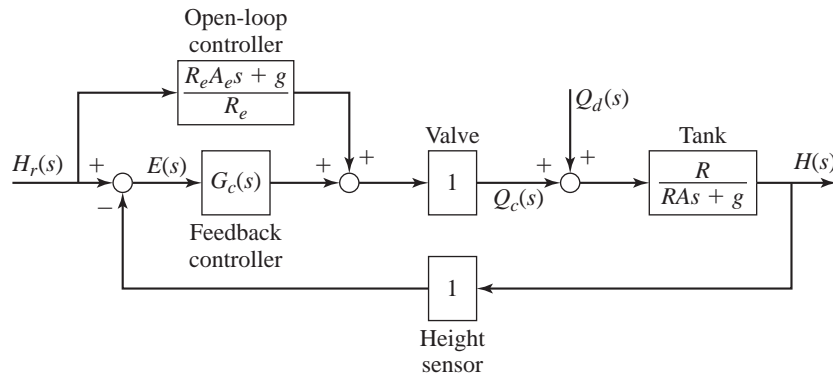
Figure 10.1.3 Feedforward compensation for a liquid-level controller.

used to adjust the signal going to the valve that produces the flow rate q_c. This part of the controller is the *feedback,* or *closed-loop* controller. The total signal to the valve is the sum of this signal and the signal produced by the open-loop part of the controller. We may thus think of the feedback controller as providing a correction to the open-loop calculation. This correction, which is based on the measured height, enables the controller to adjust for inaccuracies in the model or in the estimates A_e and R_e, and for changes caused by the disturbance flow q_d. The proper design of the algorithm used in the feedback controller is the major topic of this chapter and Chapters 11 and 12.

The use of open-loop control in this manner is called *feedforward compensation of the command input*. It has gained renewed interest since the advent of computer control systems. It is based on the principle that if there are no disturbances, you can effectively control a system if you have an accurate enough model of it. Some modern systems are so complex that a simple differential equation model cannot be obtained that completely describes their behavior. Computer simulation models, however, can often be developed to describe such plants. These models can then be used as command compensation in control systems having digital computers.

For example, computers can store lookup tables that can be used to control systems that are difficult to model entirely with differential equations and analytical functions. Figure 10.1.4 shows a fuel control system designed for an internal combustion engine.
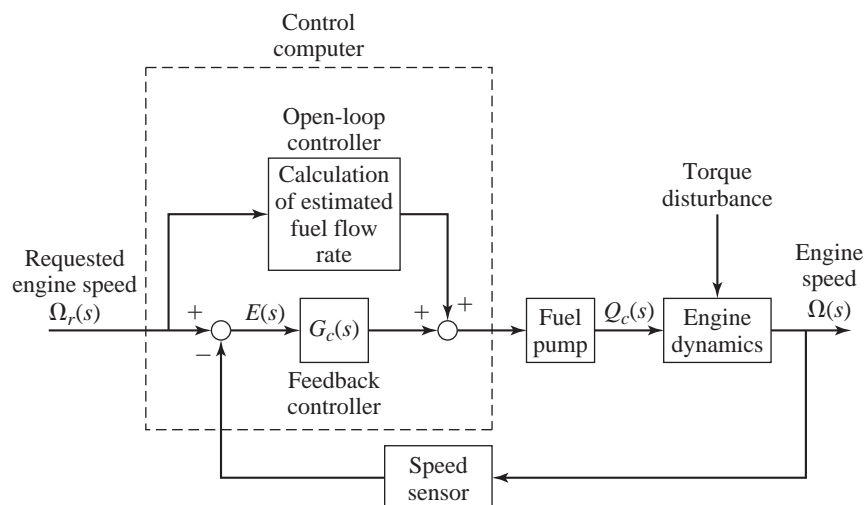
Figure 10.1.4 Use of feedforward compensation in a fuel control system.

The fuel flow rate required to achieve a desired speed depends in a complicated way on the desired speed and other variables. This dependence was measured in engine tests and was summarized in a table stored in the control computer. This table is used to estimate the required fuel flow rate. A feedback control algorithm is used to adjust the flow rate estimate based on the speed error (because the lookup table model will not be exact) and to counteract the effects of disturbances such as a load torque on the engine.

For now, we will concentrate on how to analyze the performance of a feedback control algorithm. Section 10.8 discusses feedforward compensation in more detail.

10.2 CONTROL SYSTEM TERMINOLOGY

Figure 10.2.1 is a block diagram that represents the structure of many, but not all, feedback control systems. To simplify the example, we have not included feedforward compensation.

The feedback controller can be thought of as a logic element that compares the command signal with the measurement of the output and decides what should be done. The control logic elements produce the control signal, which is sent to the actuator or *final control element*. This is the device that develops enough torque, pressure, heat, etc., to influence the elements under control.

The object to be controlled is the *plant*. The *manipulated variable* is generated by the final control elements for this purpose. In the liquid-level control system treated in Section 10.1, the manipulated variable is the flow rate q_c, the plant is the tank of liquid, and the actuator is the control valve. To make a human analogy, the brain is the controller; the arm, hand, and whatever is being held in the hand is the plant; and the arm muscle is the actuator. The eye plays the role of the sensing element.

The sensor transfer function is $H(s)$. Note that if the sensor gives error-free measurements instantaneously, then $H(s) = 1$. This implies that the sensor responds quickly compared to the rest of the system (i.e., its time constant is small compared to the other time constants of the system).

The disturbance input also acts on the plant. This is an input over which the designer has no control, and perhaps for which little information is available about the magnitude, functional form, or time of occurrence. The disturbance can be a random input, such as wind gusts on a radar antenna, or deterministic, such as Coulomb friction effects. In the latter case, we can include the friction force in the system model by using a nominal value for the coefficient of friction. The disturbance input would then be the deviation of the friction force from this estimated value and would represent the uncertainty in our estimate. The *command* transfer function is $C(s)/R(s)$. The *disturbance* transfer function is $C(s)/D(s)$.

Figure 10.2.1 Control system terminology.
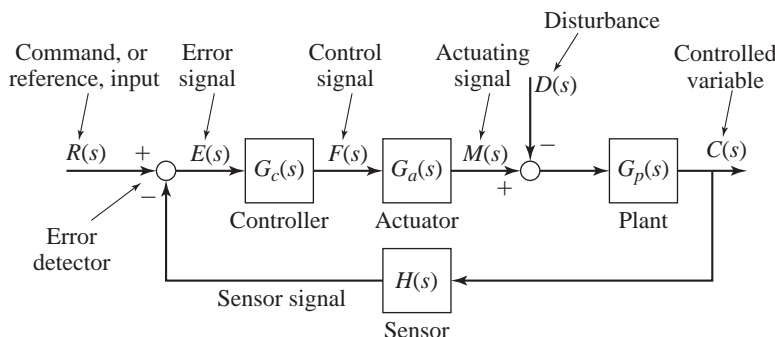
10.3 MODELING CONTROL SYSTEMS

Modeling the control system with all its components is the first step in the design process. The key to successful design often lies in the engineer's ability to develop a model that describes the important behavior of the system without including so much detail that the required mathematical analysis is cumbersome. In this section, we show some examples of how such modeling is accomplished.

MODELING ACTUATOR AND SENSOR RESPONSE

Every actuator and every sensor requires time to respond to its input. However, if actuator and sensor response times are small compared to the response times of the controller and the plant, then we usually model the actuator and the sensor with constant gains. The purpose of this approximation is to reduce the order of the system and thus simplify the required mathematics.

Consider the control system shown in Figure 10.3.1a. The controller transfer function is a constant, K_p called the *proportional gain*. The control signal $f(t)$ is proportional to the error signal $e(t)$, as $f(t) = K_p e(t)$. The actuator and the sensor have both been modeled as first-order systems having the time constants τ_a and τ_s. The command transfer function for this system is

$$\frac{C(s)}{R(s)} = \frac{K_P K_a K (\tau_s s + 1)}{(\tau_a s + 1)(\tau s + 1)(\tau_s s + 1) + K_s K_P K_a K}$$

We thus see that the system model is third order.

The sensor gain K_s will be unity if the sensor gives an accurate measurement. If the sensor time constant τ_s is very small compared to that of the plant, τ, then we can set $\tau_s = 0$ and approximate the sensor transfer function as a simple gain of 1. We will usually use this sensor model.

Similarly, if the actuator time constant τ_a is very small compared to that of the plant, then we may set $\tau_a = 0$ and approximate the actuator transfer function as a simple gain K_a, which may be absorbed into the proportional gain, as shown in Figure 10.3.1(b). The command transfer function of this simplified model is

$$\frac{C(s)}{R(s)} = \frac{K_P K_a K}{\tau s + 1 + K_P K_a K}$$

which is first order and therefore much easier to analyze.

Figure 10.3.1 (a) A feedback control system. (b) Approximate model obtained by neglecting the time constants of the sensor and actuator.
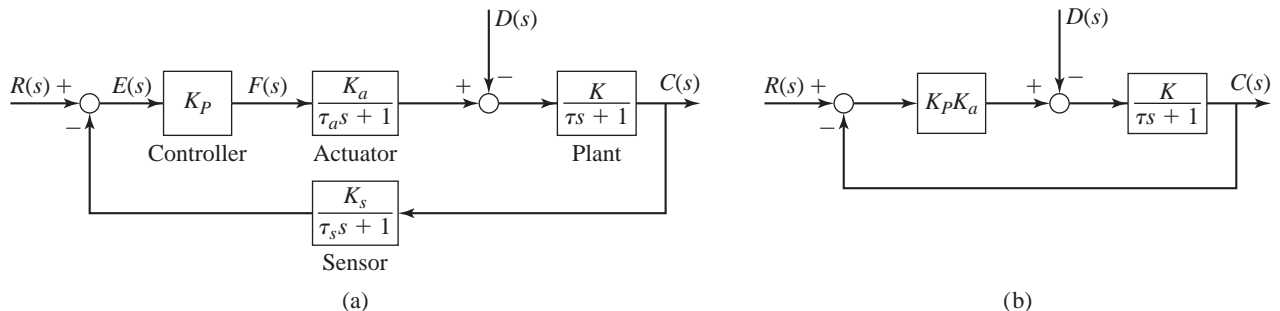
In most cases, the sensor time constant is the smallest of the three, so we will usually neglect it in our examples. The actuator time constant may not always be negligible, however. Quite often in our examples we assume that the actuator gain K_a is 1 and do not show its block to simplify the examples. In these cases, the hidden block has units that convert the controller output (e.g., volts) to the actuator output (e.g., force). If the actuator gain is not 1 in practice, it can be absorbed into the controller gain.

EXAMPLE 10.3.1

Figure 10.3.2 Effects of neglecting the actuator time constant.

Effect of Gain on Modeling Approximations

■ Problem
For the system shown in Figure 10.3.2a the plant time constant is 2 and the nominal value of the actuator time constant is $\tau_a = 0.02$. Investigate the effects of neglecting this time constant as the gain K_P is increased.
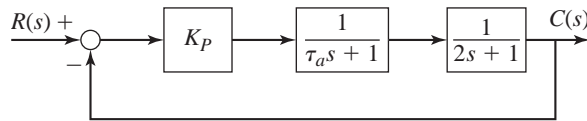
■ Solution
If we set $\tau_a = 0$, there is one characteristic root, which is $s = -(1 + K_P)/2$. As K_P is increased, the root remains real and moves to the left of $s = -0.5$. The predicted step response will not oscillate.

If we set $\tau_a = 0.02$, there are two characteristic roots and the characteristic equation is $0.04s^2 + 2.02s + 1 + K_P = 0$.

$$s = -50.5 \pm \sqrt{2450.25 - 100K_p}$$

As K_P is increased, the dominant root is real and moves to the left of $s = -0.5$, just as it does with the approximate model. But for $K_P > 24.5$, $\zeta < 1$, the roots are complex, the response will oscillate, and so the behavior of the approximate model with $\tau_a = 0$ does not resemble the behavior of the more accurate model with $\tau_a = 0.02$.

Note that as we increase K_P, the dominant time constant decreases and becomes closer to τ_a, and the response oscillates. Thus the approximate model becomes less accurate.

In general, as we increase the gain in a system to obtain a small dominant time constant, any neglected time constants become more significant and should be included in the model. Doing this, however, increases the order of the model and the complexity of the required mathematics. So, in practice, the control engineer begins the design of a system by neglecting the smaller time constants, just as we neglected the actuator time constant τ_a in Example 10.3.1. Once a preliminary design has been made, the more accurate model is used to refine the design and check the effect of the approximations, including the effect of high gain, often by simulation if the model is highly detailed.

MODELING DISTURBANCES

The disturbance inputs experienced by real systems are usually not simple or deterministic functions of time like steps, ramps, and sine waves. In fact, we often do not know the exact functional form of the disturbance. It is often a random function, such

as the disturbance torque due to wind gusts. The analysis of random inputs is beyond the scope of this text. However, the frequency response plot of the disturbance transfer function is often very useful for evaluating the effects of a disturbance. Although such a plot describes only the steady-state response for sinusoidal and cosinusoidal inputs, periodic disturbance inputs can be represented as a sum of sines and cosines, as we saw in Chapter 8. In addition, measurements of an actual disturbance can be analyzed to determine its frequency content. If these frequencies lie outside the bandwidth of the disturbance transfer function, then we may conclude that the control system adequately handles the disturbance.

Some system inputs may or may not be treated as a disturbance, depending on whether or not they are known. The following example illustrates this.

Controlling a Thermal Process | **EXAMPLE 10.3.2**

■ Problem
The inside temperature T of the oven shown in Figure 10.3.3a is to be controlled. The oven wall has a thermal resistance R. The only significant thermal capacitance is that of the oven contents, C. The air temperature outside the oven is T_o. The heater supplies a heat flow rate q.

A power amplifier supplies current to the resistance-type heater. The amplifier inputs are the voltage from the command potentiometer, which represents the desired value T_r of the

Figure 10.3.3 (a) An oven temperature control system. (b) Model of the control system if T_o is a known constant temperature. (c) Outside temperature T_o modeled as a disturbance.
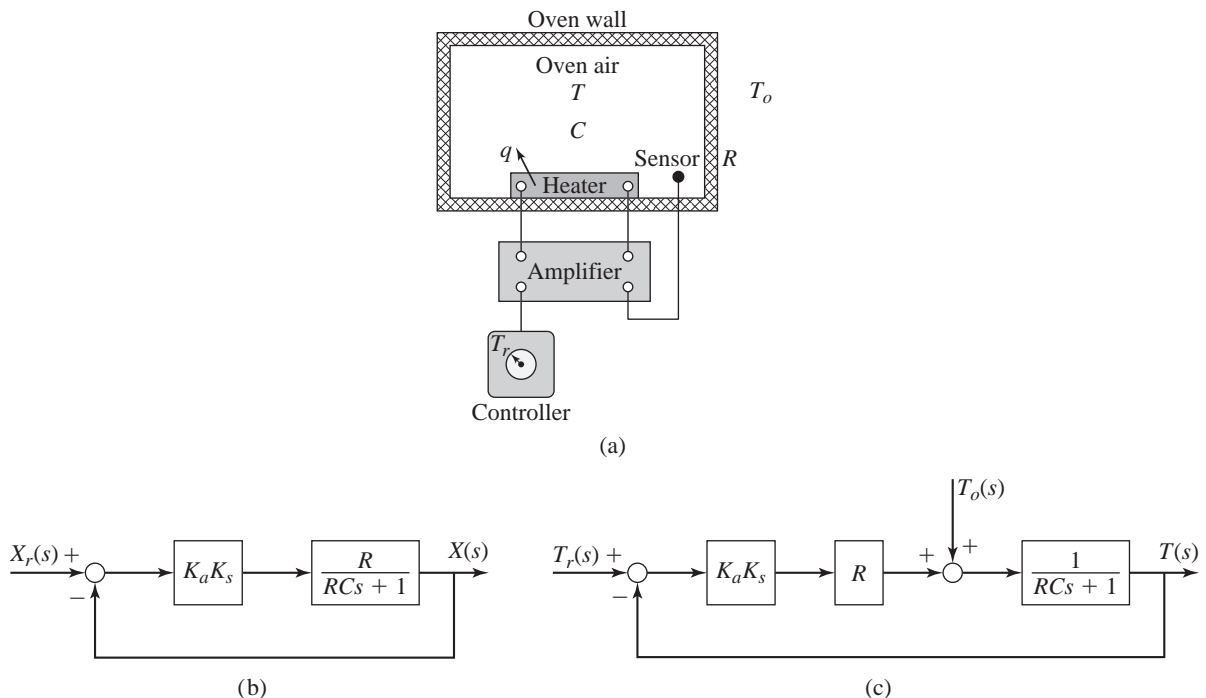
temperature T, and the voltage from the temperature sensor. These voltages are $v_r = K_r T_r$ and $v_s = K_s T$, respectively.

Develop a model of the system.

■ **Solution**

Using the methods of Section 7.8, the model of the plant is found to be

$$C\frac{dT}{dt} = q - \frac{1}{R}(T - T_o)$$

We should choose the scale factor of the potentiometer so that $K_r = K_s$. This means that the voltage difference $v_r - v_s$ will be proportional to the temperature difference $T_r - T$, as $v_r - v_s = K_s(T_r - T)$. The heat flow rate q from the heater is equal to $K_a(v_r - v_s)$, where K_a is the amplifier gain. Thus the system model is

$$C\frac{dT}{dt} = q - \frac{1}{R}(T - T_o) = K_a K_s(T_r - T) - \frac{1}{R}(T - T_o) \qquad (1)$$

If T_o is a known constant value, then we can define two new variables x and x_r such that

$$x = T - T_o \qquad x_r = T_r - T_o$$

Then $T = x + T_o$ and $T_r = x_r + T_o$. Substitution of these into the system model gives

$$C\frac{dx}{dt} = K_a K_s(x_r - x) - \frac{1}{R}x$$

The corresponding block diagram is shown in part (b) of the figure. The effect of the new variables x and x_r is to remove T_o from the equation model and the block diagram, thus simplifying both.

If, however, the outside temperature T_o fluctuates in an unknown manner, then it is a disturbance. In this case, the variable transformation $x = T - T_o$ is useless because we would not know the value of T_o. The system model is equation (1) and its block diagram is shown in part (c) of the figure. We will analyze the performance of this system later in this chapter.

MODELING DC MOTOR CONTROL

DC motors are frequently used in motion control applications. In such applications, we need to control either the speed or the position of a load. Figure 10.3.4 is a schematic diagram of a system for controlling the rotational speed of a load. Applications of such a system include electric-vehicle speed control, tape drives, conveyor belts, robot arms, and mobile robots. The load speed is sensed by a tachometer. The tachometer outputs a voltage that is proportional to its angular velocity; thus $v_{tach} = K_{tach}\omega_L$, where ω_L is the load velocity. The figure shows the tachometer connected to the load shaft; if this is not possible, then a gear pair or similar connection can be used.

Somehow the system must be given the desired load speed. Here we assume that an operator turns a rotary potentiometer to the desired speed marked on a calibrated dial face. Thus the potentiometer produces a voltage $K_d K_{pot}\omega_r$ that is proportional to the desired load speed ω_r, where K_d is the dial factor and K_{pot} is the potentiometer gain.

An op-amp system consisting of a comparator and a multiplier computes the difference between the potentiometer voltage and the tachometer voltage and multiplies this difference by a gain K_1, whose value we must set. The voltage output from the multiplier is connected to the input of a power amplifier. This amplifier is used to boost the current in the signal to provide enough power to drive the motor. Thus we assume that the amplifier can provide whatever current is needed by the motor. The amplifier
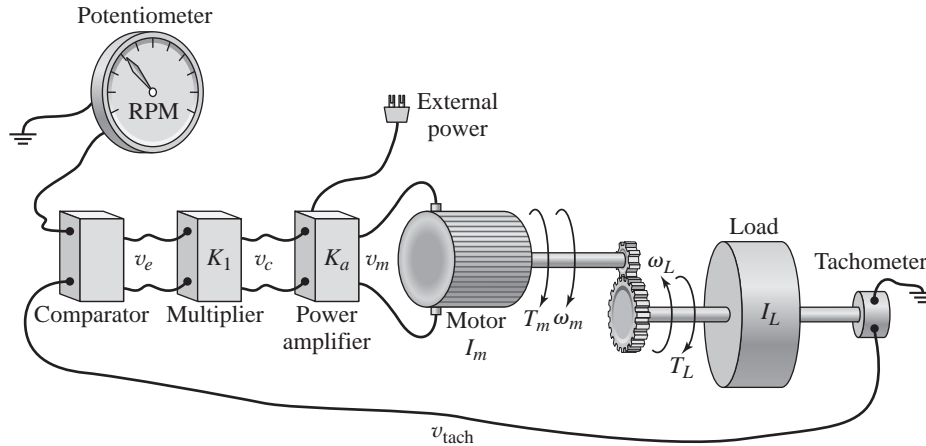
Figure 10.3.4 A speed control system.

also increases the voltage by the gain K_a. The diagram does not show the external power connections required to drive the op amp and the potentiometer.

The voltage difference v_e is given by

$$v_e = K_d K_{\text{pot}} \omega_r - K_{\text{tach}} \omega_L$$

where ω_r is the desired value of the load speed. This is sometimes called the *requested* or *commanded* speed. It is important to understand that the voltage difference v_e must be proportional to the speed error $\omega_r - \omega_L$. Therefore, we must set the dial factor K_d so that $K_d K_{\text{pot}} = K_{\text{tach}}$. If we do this, then $v_e = K_{\text{tach}}(\omega_r - \omega_L)$ as required. (Note that it is easier to set K_d by calibrating the dial face than it is to find a potentiometer or tachometer with the necessary gain.)

Reflecting the load and tachometer inertias and the load damping back to the motor shaft using the concepts of Chapter 2, we obtain the following equivalent inertia and damping values:

$$I_e = I_m + \frac{I_L + I_t}{N^2} \qquad c_e = \frac{c}{N^2}$$

where I_m is the motor inertia, I_L is the load inertia, I_t is the tachometer inertia, and c is the torsional viscous damping constant acting on the load shaft. Here, N is the gear ratio ($N = \omega_m/\omega_L$, where ω_m is the motor speed). Thus, the dynamics of the mechanical subsystem are described by

$$I_e \frac{d\omega_m}{dt} = T_m - \frac{T_L}{N} - c_e \omega_m$$

Thus,

$$\Omega_m(s) = \frac{1}{I_e s + c_e} \left[T_m(s) - \frac{T_L(s)}{N} \right]$$

Use of a Field-Controlled Motor | **EXAMPLE 10.3.3**

■ **Problem**

Assuming that the motor is field-controlled, draw the block diagram of the speed-control system shown in Figure 10.3.4 and obtain its command and disturbance transfer functions.

Figure 10.3.5 Block diagram of a speed control system using a field-controlled motor.
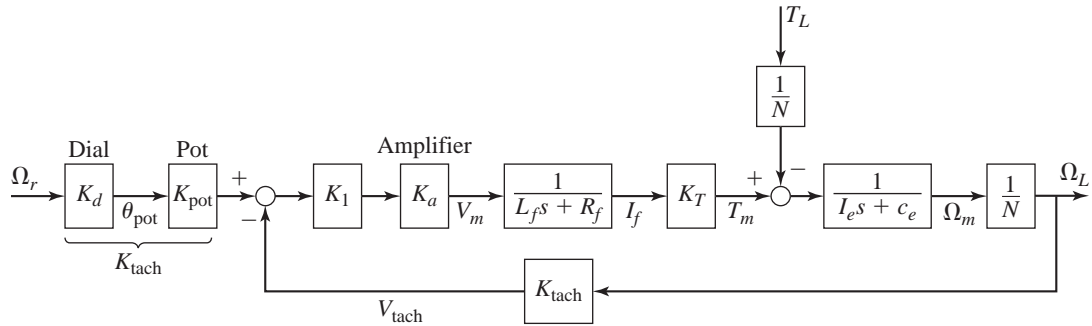
■ Solution

A field-controlled motor is controlled by varying the field current i_f while keeping the armature current constant. Figure 10.3.5 shows the block diagram of a proportional control system for controlling the speed of such a motor and was obtained by modifying the motor diagram in Figure 9.6.13 in Chapter 9.

The transfer functions can be obtained either by reducing the block diagram or by transforming the equations and eliminating all variables except the inputs and output. Recalling that $K_d K_{pot} = K_{tach}$, we can write the following equation from the diagram

$$V_m(s) = K_1 K_a K_{tach}[\Omega_r(s) - \Omega_L(s)]$$

From the diagram we can see that the motor torque is given by

$$T_m(s) = \frac{K_T}{L_f s + R_f} V_m(s)$$

Finally, we have $\Omega_L(s) = \Omega_m(s)/N$. We obtain the transfer functions by eliminating $V_m(s)$, $T_m(s)$, and $\Omega_m(s)$ from these equations. Let

$$K_P = K_{tach} K_1 K_a \tag{1}$$

Using the parameter K_P, the diagram can be simplified as shown in Figure 10.3.6. The transfer functions are

$$\frac{\Omega_L(s)}{\Omega_r(s)} = \frac{K_P K_T}{N L_f I_e s^2 + N(R_f I_e + c_e L_f)s + N c_e R_f + K_P K_T} \tag{2}$$

$$\frac{\Omega_L(s)}{T_L(s)} = -\frac{(L_f s + R_f)/N}{N L_f I_e s^2 + N(R_f I_e + c_e L_f)s + N c_e R_f + K_P K_T} \tag{3}$$

EXAMPLE 10.3.4

Approximate First-Order Model

■ Problem

Derive the transfer functions of the system shown in Figure 10.3.6 for the case where the field time constant L_f/R_f is small.

■ Solution

If $T_L = 0$, Figure 10.3.6 shows that the transfer function $\Omega_m(s)/V_m(s)$ is

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{K_T}{(L_f s + R_f)(I_e s + c_e)} = \frac{K_T/R_f c_e}{(\tau_f s + 1)(\tau_m s + 1)} \tag{1}$$
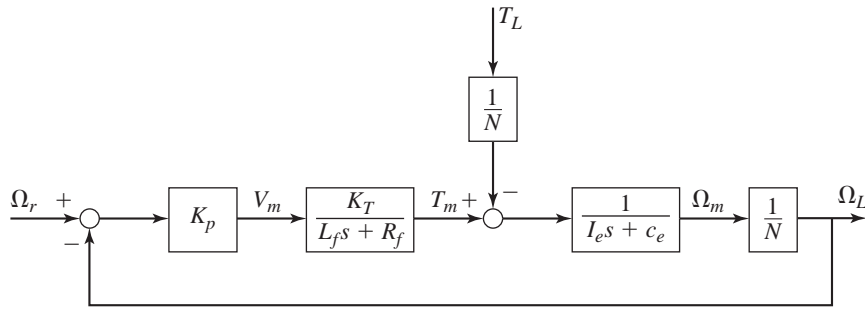
Figure 10.3.6 Simplified block diagram of a speed control system using a field-controlled motor.

where $\tau_f = L_f/R_f$, which is called the field time constant, and $\tau_m = I_e/c_e$, which is the mechanical time constant. Quite often $\tau_f \ll \tau_m$, in which case equation (1) reduces to a first-order model:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{K_T/R_f c_e}{\tau_m s + 1} = \frac{K_T/R_f}{I_e s + c_e}$$

The approximation $\tau_f \ll \tau_m$ is equivalent to setting $L_f = 0$. If this approximation is valid, equations (1) and (2) of Example 10.3.3 reduce to the following first-order models:

$$\frac{\Omega_L(s)}{\Omega_r(s)} = \frac{K_P K_T}{N R_f I_e s + N c_e R_f + K_P K_T}$$

$$\frac{\Omega_L(s)}{T_L(s)} = -\frac{R_f/N}{N R_f I_e s + N c_e R_f + K_P K_T}$$

These correspond to the block diagram shown in Figure 10.3.7a. Moving the factor $1/N$ gives the diagram shown in part (b). Note that we may thus absorb the factor $N K_T/R_f$ into the

Figure 10.3.7 Model simplification for a speed control system with negligible field time constant.

(a)

(b)

(c)
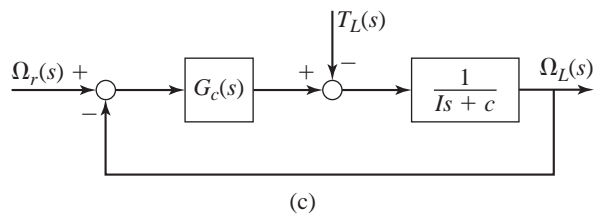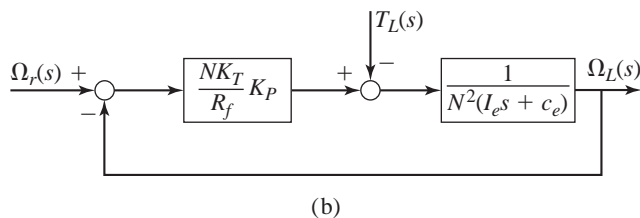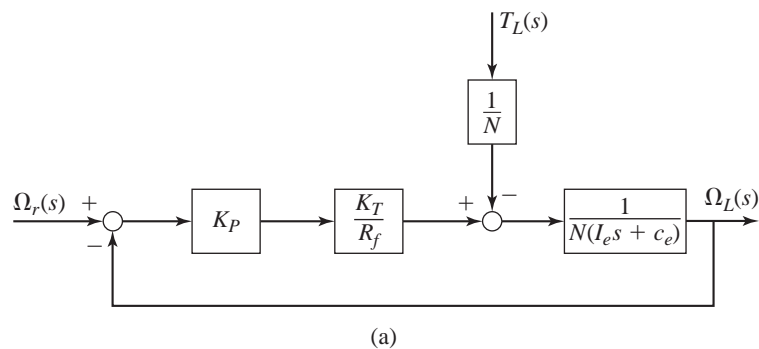
controller gain K_P. Part (c) shows the case where the control algorithm is some arbitrary transfer function $G_c(s)$, where $I = N^2 I_e$ and $c = N^2 c_e$. We will use this simplified diagram quite often in discussing the use of various control algorithms in Sections 10.6 and 10.7.

EXAMPLE 10.3.5

Use of an Armature-Controlled Motor

■ Problem
Assuming that the motor is armature-controlled, draw the block diagram of the speed-control system shown in Figure 10.3.4 and obtain its transfer functions.

■ Solution
The system inputs are the desired or required value ω_r of the load speed ω_L and the load torque T_L. The block diagram is shown in Figure 10.3.8 and was obtained by modifying the motor diagram in Figure 9.6.12 in Chapter 9.

The transfer functions can be obtained either by reducing the block diagram or by transforming the equations and eliminating all variables except the inputs and the output. The latter method is perhaps the easier one when the diagram has multiple inputs and loops containing comparators within them. Recalling that $K_d K_{pot} = K_{tach}$, we can write the following equation from the diagram

$$V_m(s) = K_{tach} K_1 K_a [\Omega_r(s) - \Omega_L(s)]$$

From the diagram we see that the motor torque is given by

$$T_m(s) = \frac{K_T}{L_a s + R_a} [V_m(s) - K_b \Omega_m(s)]$$

Finally, we have $\Omega_L(s) = \Omega_m(s)/N$.

Figure 10.3.8 Block diagram of a speed control system using an armature-controlled motor.
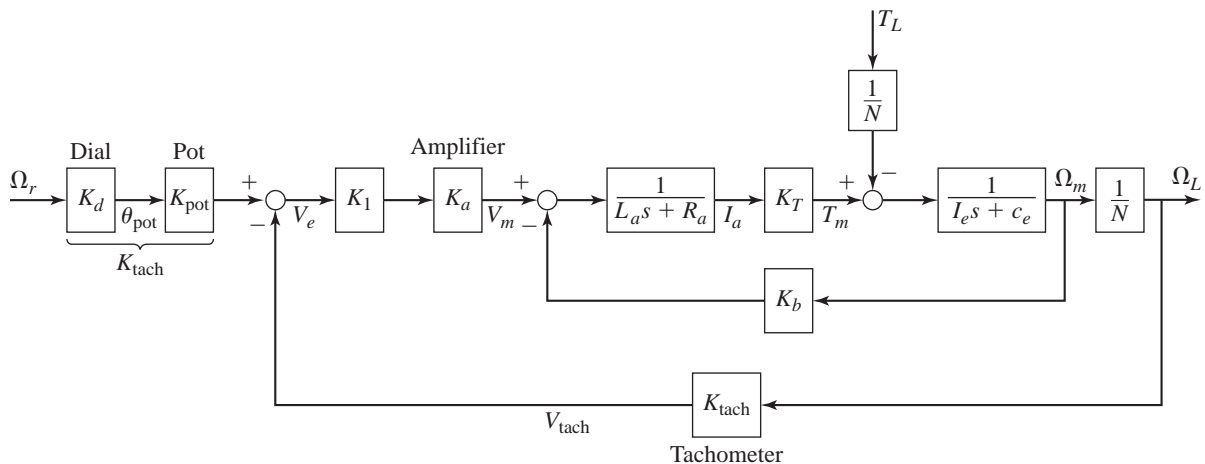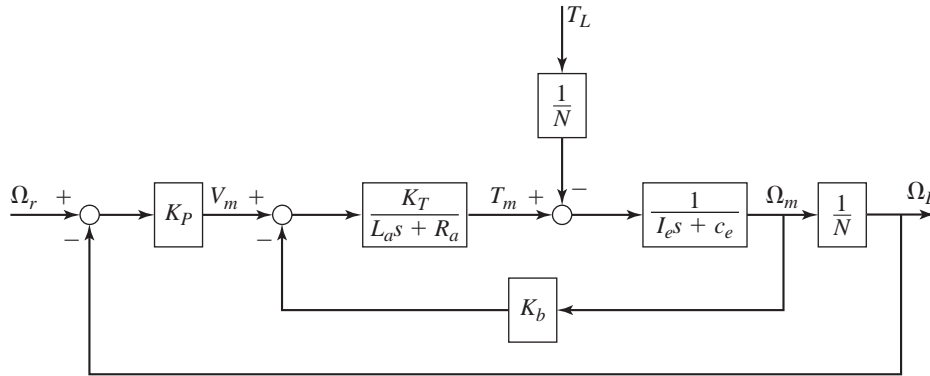
Using algebra to eliminate $V_m(s)$, $T_m(s)$, and $\Omega_m(s)$ from these equations, we obtain the following transfer functions. To simplify the expressions, we have defined a new constant $K_P = K_1 K_a K_{\text{tach}}$.

$$\frac{\Omega_L(s)}{\Omega_r(s)} = \frac{K_P K_T}{D(s)} \tag{1}$$

$$\frac{\Omega_L(s)}{T_L(s)} = -\frac{(L_a s + R_a)/N}{D(s)} \tag{2}$$

where the denominator is

$$D(s) = N L_a I_e s^2 + N(R_a I_e + c_e L_a)s + N R_a c_e + N K_T K_b + K_P K_T \tag{3}$$

Using the parameter K_P, the block diagram can be simplified as shown in Figure 10.3.9. These transfer functions can be analyzed to determine whether or not the system is able to control the speed successfully and to determine an appropriate value for the gain K_1. Such analysis would consider the system's stability, steady-state error, and transient response. We will illustrate this analysis in Sections 10.6, 10.7, and 10.8.

Effect of Negligible Armature Time Constant | **EXAMPLE 10.3.6**

■ Problem
Derive the transfer functions for the case where the armature time constant L_a/R_a is small.

■ Solution
If $T_L = 0$, we obtain Figure 10.3.10a. We can reduce the inner loop containing K_b to obtain part (b) of the figure. Setting $L_a = 0$ we obtain the diagram in part (c). Part (d) is obtained by replacing K_P with the arbitrary controller transfer function $G_c(s)$ and defining

$$I = \frac{N R_a I_e}{K_T}$$

$$c = N \frac{R_a c_e + K_T K_b}{K_T}$$

The block diagram in part (d) is obviously a simpler representation than the diagram in part (a). This simplified diagram will help us to demonstrate the use of various control algorithms in Sections 10.6 and 10.7. Note that it does not include the disturbance. To analyze the disturbance response, we need to use the diagram in Figure 10.3.9 or the disturbance transfer function given by equations (2) and (3) in Example 10.3.5.

(a)

(b)

(c)

(d)

EXAMPLE 10.3.7 | Controlling the Speed of a Conveyor

■ **Problem**

A conveyor drive system to produce translation of the load is shown in Figure 10.3.11. To translate the load a specified distance, the drive wheels must rotate through a required angle, and this can be accomplished by controlling the speed, often with a trapezoidal speed profile. The equivalent inertia of the load and all the drive components felt at the motor shaft is I_e. The effect of Coulomb friction in the system produces an opposing torque T_{Fe} at the motor shaft, and the damping in the system is negligible. Develop the block diagram of a proportional control system using an armature-controlled motor for this application. Assume that the drive wheel speed ω_L is measured by a tachometer and that the motor speed ω_m is related to the drive wheel speed by $\omega_m = N\omega_L$, where N is the speed ratio due to the reducer and the chain drive.

■ **Solution**

The dynamics of the mechanical subsystem are described by

$$I_e \frac{d\omega_m}{dt} = T - T_{Fe} \tag{1}$$

where the motor torque is $T = K_T i_a$. The system is like that shown in Figure 10.3.4. The block diagram can be obtained by modifying Figure 10.3.8 using equation (1) and collecting the various gains into one gain: $K_P = K_{\text{tach}} K_1 K_a$. The resulting diagram is shown in Figure 10.3.12.

Figure 10.3.11 A conveyor system.
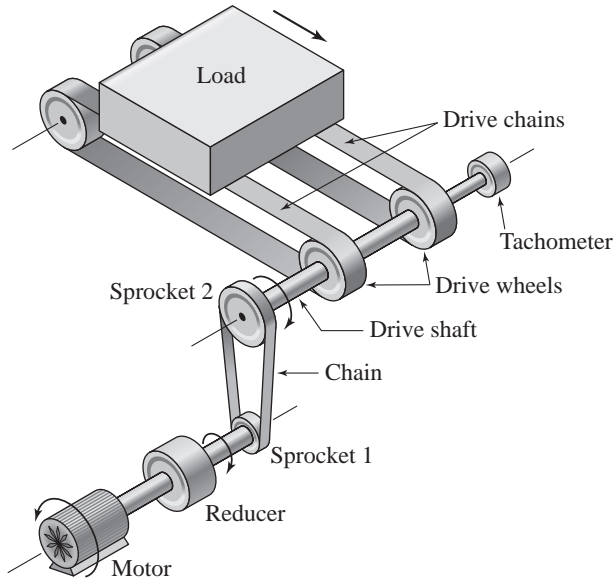
Figure 10.3.12 Block diagram of a speed control system for a conveyor.
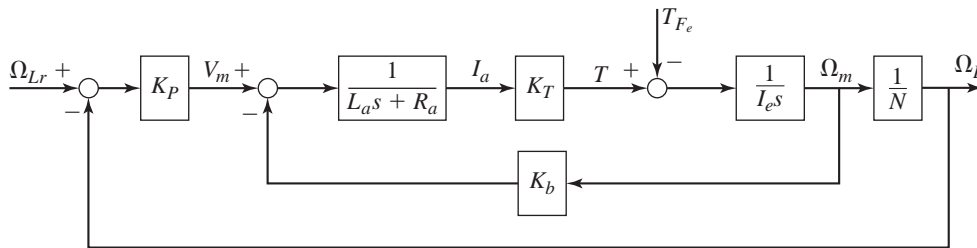
Controlling the Position of a Robot Arm Link | **EXAMPLE 10.3.8**

■ **Problem**

The drive system for one link of a robot arm is illustrated in Figure 10.3.13. The equivalent inertia of the link and all the drive components felt at the motor shaft is I_e. Gravity produces an opposing torque that is proportional to $\sin \theta$ but which we model as a constant torque T_d felt at the motor shaft (this is a good approximation if the change in θ is small). Neglect friction and damping in the system. Develop the block diagram of a proportional control system using an armature-controlled motor for this application. Assume that motor rotation angle θ_m is measured by a sensor and is related to the arm rotation angle θ by $\theta_m = N\theta$, where N is the gear ratio.

■ **Solution**

The dynamics of the mechanical subsystem are described by

$$I_e \frac{d^2\theta_m}{dt^2} = T - \frac{1}{N}T_d \tag{1}$$

where the motor torque is $T = K_T i_a$. The system is like that shown in Figure 10.3.4. The block diagram can be obtained by modifying Figure 10.3.8 using equation (1) and collecting the various gains into one gain: $K_P = K_{\text{tach}} K_1 K_a$. The resulting diagram is shown in Figure 10.3.14. Note

Figure 10.3.13 A robot arm link.
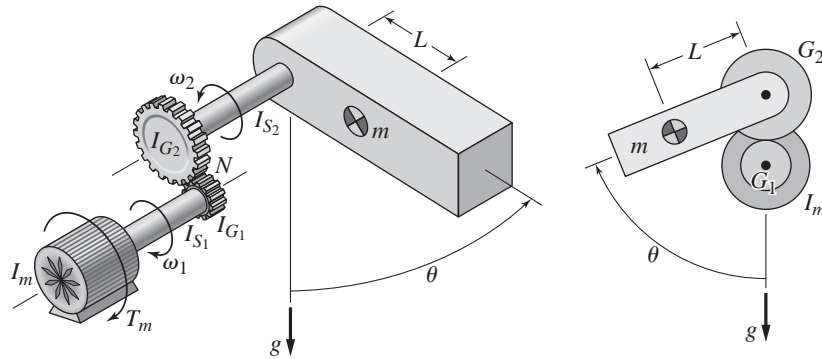
Figure 10.3.14 Block diagram of a position control system for a robot arm link.
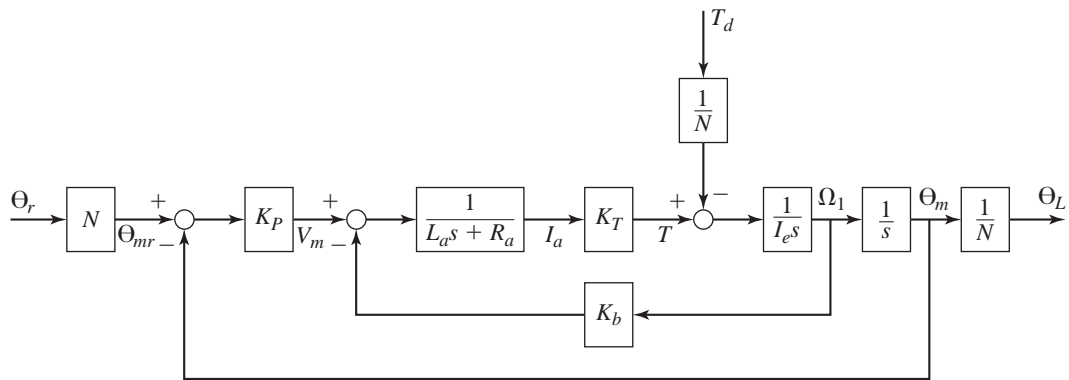
that the system actually controls the *motor* rotation angle, so the arm angle command θ_r must be converted to the motor angle command θ_{mr}.

CURRENT AND TORQUE LIMITATIONS

Obviously there is a limit to how large we can make the gain K_P. The limit depends on the power available in the actuator system, which consists of the motor and the power amplifier.

In practice, because the motor torque depends on the armature current, as $T_m = K_T i_a$, the maximum torque available is limited by the armature current. This current is limited by two factors: (1) the maximum current the armature can withstand without demagnetizing and (2) the maximum current available from the power supply (these values should be given by the motor and power supply manufacturers). Think of your household electrical outlets. If an outlet is protected by a 15-A circuit breaker (to prevent a fire hazard), then it can supply no more than 15 A. Example 10.8.6 in Section 10.8 examines the current requirements of this control system in more detail.

To guard against demagnetization and power supply damage, some amplifiers have built-in current overload protection. The engineer should be aware of this protection, because it will change the dynamics of the system. When too much current is called for, the amplifier will supply a constant current, and therefore the motor torque will be constant, until the controller calls for less than the maximum allowable current. The

effect on the system's dynamics is best studied with numerical methods, which we will demonstrate with Simulink in Section 10.10.

DIGITAL CONTROL SYSTEMS

In many modern control systems, a digital computer, rather than an op amp, performs the calculations required to implement the control algorithm.

An *analog signal* is one that is continuous in both time and amplitude. Given a large enough plot of a signal $y(t)$, we can theoretically obtain a value of y from the plot for any value of t over its plotted range. Thus, both y and t have an infinite number of values. Digital computers, however, can handle mathematical relations and operations only when expressed as a finite set of numbers rather than as infinite-valued functions. Thus, any continuous measurement signal must be converted into a set of pulses by *sampling*—the process by which a continuous-time variable is measured at distinct, separated instants a time T apart. The interval T is called the *sampling period*. The sampling frequency is $1/T$.

The sampled sequence of measurements is rounded off to one of a finite number of levels for storage in the computer memory. This process is called *quantization*. A *digital signal* is one that is quantized in both time and amplitude. For the computer to do calculations, the quantized amplitude must be assigned a binary value. This process is called *coding*.

So the process of converting an analog signal into computer-usable form consists of sampling, quantization, and coding. A device for doing this is called an *analog-to-digital converter* (abbreviated A/D converter or ADC).

It is necessary to convert binary outputs from the digital device into a form usable by the hardware being controlled. A *digital-to-analog converter* (abbreviated D/A converter or DAC) performs this function. A common mathematical algorithm used in a D/A converter is the *zero-order hold (ZOH)*. The binary form of the output from the digital device is first converted to a sequence of short-duration voltage pulses. However, it is usually not possible to drive a load, such as a motor, with short pulses. To deliver sufficient energy, the pulse amplitude might have to be so large that it is infeasible to be generated. Also, large-voltage pulses might saturate or even damage the system being driven. The solution to this problem is to smooth the output pulses to produce a signal in analog form. The simplest way of converting a pulse sequence into a continuous signal is to hold the value of the pulse until the next one arrives. A device called a *zero-order hold* accomplishes this task.

DIGITAL CONTROL STRUCTURES

There are two types of applications of digital computers to control problems. The first is *supervisory control,* where analog controllers are directly involved with the plant to be controlled, while the digital computer provides command signals to the controllers. If this were the case for the system shown in Figure 10.3.4, the op-amp comparator and multiplier would still be present, but the command input ω_r would be provided by the computer through a D/A converter, instead of by the potentiometer. In the second application, *direct digital control (DDC),* the digital device acts at the lowest levels of the system, replacing the op-amp comparator and multiplier, in direct control of the amplifier-motor system.

Figure 10.3.15 shows two possible structures of a single-loop DDC controller. In both structures, the computer with its internal clock drives the D/A and A/D converters. It compares the command signals with the feedback signals and generates the control

Figure 10.3.15 Possible structures for
digital control systems.

(a)

(b)
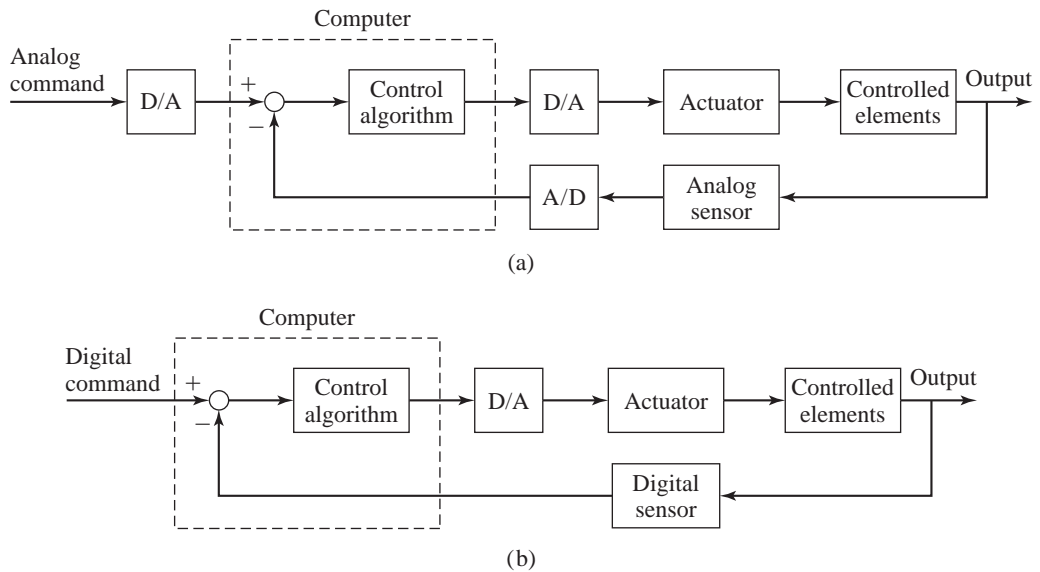
signals to be sent to the D/A and eventually to the actuator. These control signals are
computed from the control algorithm stored in the memory. In part (a) of the figure, the
feedback signal is an analog signal (from an analog sensor such as a tachometer), which
must be fed to an ADC. To convert the controller shown in Figure 10.3.4 to such a system,
we must replace the op-amp comparator and multiplier with a computer and DAC, and
use an ADC on the tachometer signal. An alternative is shown in Figure 10.3.15b,
which illustrates the use of a digital sensor (such as an optical encoder), whose signal
can be communicated directly to the computer without using an ADC. If we retain the
potentiometer, we must use an ADC on its voltage signal. An alternative is to replace
the potentiometer with a command entered into the computer, perhaps via a keyboard.

PULSE-WIDTH MODULATION

The voltage output of the zero-order hold is constant over the sampling interval T
but varies in amplitude over successive sampling intervals. The task of providing a
variable-amplitude voltage can be somewhat difficult because of the dynamics caused
by the motor inductance. Instead of varying the amplitude, an alternative to the zero-
order hold is to output a constant voltage but vary the amount of time the voltage is
applied over the sampling interval. This is the basis of *pulse-width modulation (PWM)*.
Because it requires only a voltage source that can be switched between two constant
values, say $v_{min} = 0$ and $v_{max} = V$, pulse-width modulation is particularly well suited
for use with digital computers. If we consider the voltage v_c computed from the control
algorithm to be the average required voltage over the interval T, then the pulse time t_i
can be calculated from

$$t_i = \frac{v_c}{V} T$$

For the modulated voltage to be fully equivalent to v_c, t_i should be calculated so that
the pulse of duration t_i delivers the same power as v_c over the interval T. However, this

is difficult to calculate because of the dynamics of the circuit. The formula for t_i can be extended to include the case where $v_{min} < 0$ (negative voltage is required to slow down or reverse the motor's speed). Note that it is possible for the computed value of t_i to be greater than T. This unacceptable condition may occur if the control gains are too large.

10.4 THE PID CONTROL ALGORITHM

The control element is designed to act on the error signal to produce the control signal. The logic that is used for this purpose is also called the *control law,* the *control action,* or the *control algorithm.* A nonzero error signal results from either a change in command or a disturbance. The general function of the controller is to keep the controlled variable near its desired value when these occur. More specifically, the control objectives might be stated as follows:

1. Minimize the steady-state error.
2. Minimize the settling time.
3. Achieve other transient specifications, such as minimizing the overshoot.

In practice, the design specifications for a controller are more detailed. For example, the bandwidth might also be specified along with a safety margin for stability. We never know the numerical values of the system's parameters with true certainty, and some controller designs can be more sensitive to such parameter uncertainties than other designs. So a parameter sensitivity specification might also be included.

TWO-POSITION CONTROL

Two-position control action is the most familiar type perhaps because of its use in home thermostats. The control output takes on one of two values. A special case is an *on-off controller,* in which the controller output is either on or off. Such is the case with a thermostatically controlled heating system, in which the heater or furnace is turned on and off. If the outside conditions (temperature, wind, solar insolation, etc.) were predictable, we could design a building heating system that would operate continuously to supply heat at a predetermined rate just large enough to replace the heat lost to the outside environment. Of course, the real world does not behave so predictably, so we must use a feedback controller to adjust the heat output according to the actual room temperature.

A common example illustrates how the capacitance of the plant affects the suitability of on-off control. Regulation of the temperature of shower water by using on-off control of the hot water valve will obviously be unsuitable, but it is acceptable for a bath, because the thermal capacitance of the bath water is greater than that of the shower water.

Two-position control is acceptable for many applications in which the requirements are not too severe. In the home heating application, two-position control is acceptable because the typical 2°F temperature gap is hardly detectable by the occupants. In situations requiring finer control, however, use of two-position control results in frequent switching of the actuator, which shortens its life span and might waste energy.

PROPORTIONAL CONTROL

We have already seen proportional control, in which the control signal is proportional to the error. Referring to Figure 10.2.1, the transfer function of proportional control

$$\frac{V_o(s)}{V_i(s)} = -\frac{R_f}{R_i}, \qquad K_P = \frac{R_f}{R_i}$$
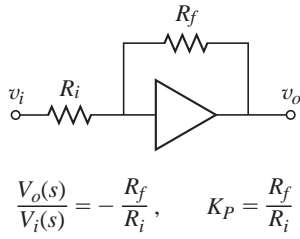
action is $G_c(s) = K_P$, and the controller output is given by

$$F(s) = G_c(s)E(s) = K_P E(s) \tag{10.4.1}$$

Sometimes the term "proportional band" is used in industrial controllers, rather than "proportional gain." The percent change in error needed to move the actuator full scale is the proportional band. It is related to the gain as follows:

$$\text{proportional band } (\%) = \frac{100}{K_P} \tag{10.4.2}$$

Thus, a gain $K_P = 1$ corresponds to a band of 100%. A gain $K_P = 2$ corresponds to a band of 50%.

Figure 10.4.1 illustrates a proportional controller using an op amp, where the proportional gain is $K_P = R_f/R_i$. Figure 10.4.2a is a diagram of a system using proportional control, and part (b) of the figure is a circuit diagram showing how op amps can be used to create the controller. An op amp adder with an inverter in the feedback loop implements the comparator. Each op amp inverts its input signal. So the voltage v_r is inverted twice before entering the power amplifier, and there is no net change of sign. Similarly, the sensor voltage is inverted three times, for one net sign change, as required by the minus sign of the comparator.

Figure 10.4.2 (a) Block
diagram representation of a
proportional control system.
(b) Op-amp implementation of
a proportional controller.

(a)

(b)
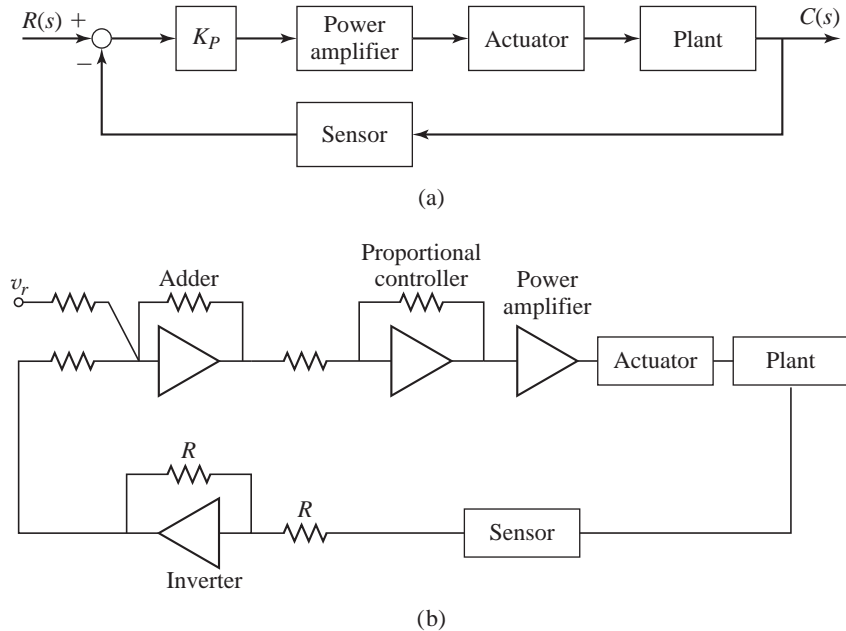
EXAMPLE 10.4.1

Hydraulic Implementation of Proportional Control

■ Problem

Figure 10.4.3 shows a hydraulic implementation of proportional action to control the angle of an aircraft rudder, elevator, or aileron (see [Cannon, 1967]). The input motion y is produced by the motion of the pilot's control stick acting through cables. Analyze its motion assuming that the rudder inertia is small, and show that the system gives proportional action.
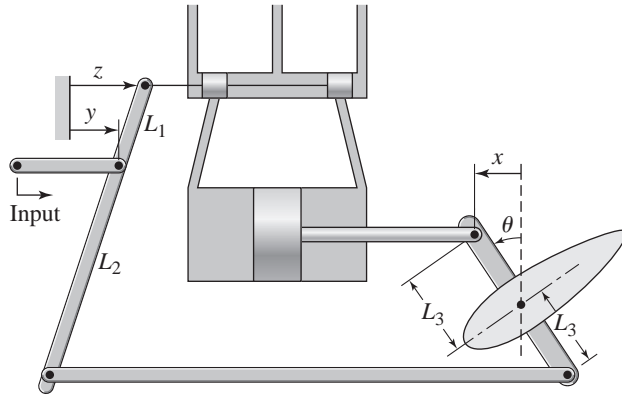
Figure 10.4.3 Hydraulic implementation of a proportional controller for an aerodynamic surface.

■ **Solution**

When the input motion y occurs, the beam pivots about its lower end. For small motions $x = L_3\theta$ and the beam geometry is such that

$$z = \frac{L_1 + L_2}{L_2}y - \frac{L_1}{L_2}x \tag{1}$$

The proof of this equation for small angular motions is as follows. First, suppose that x is fixed to be 0. Then from similar triangles,

$$\frac{z}{L_1 + L_2} = \frac{y}{L_2} \quad \text{or} \quad z = (L_1 + L_2)\frac{y}{L_2}$$

Now suppose that y is fixed to be 0. From similar triangles,

$$\frac{z}{L_1} = -\frac{x}{L_2} \quad \text{or} \quad z = -L_1\frac{x}{L_2}$$

Since the general motion is the superposition of the two individual motions, adding these two relations gives equation (1).

For the servomotor, as shown in Example 7.4.7, if the rudder inertia is small,

$$\frac{X(s)}{Z(s)} = \frac{C_1}{As}$$

These equations give the following transfer function.

$$\frac{\Theta(s)}{Y(s)} = \frac{L_1 + L_2}{L_1 L_3}\frac{1}{\tau s + 1}$$

where

$$\tau = \frac{L_2 A}{L_1 C_1}$$

If the servomotor gain C_1/A is large, then $\tau \approx 0$, and the transfer function becomes

$$\frac{\Theta(s)}{Y(s)} = \frac{L_1 + L_2}{L_1 L_3} \equiv K_P$$

Thus the system implements proportional control with proportional gain K_P if the motion y represents the error signal and θ represents the controller output.

INTEGRAL CONTROL ACTION

We will see that sometimes proportional control results in the system reaching an equilibrium in which the control signal no longer changes, thus allowing a constant

error to exist. If the controller is modified to produce an increasing signal as long as the error is nonzero, the error might be eliminated. This is the principle of *integral control action,* also called *reset action*. In this mode, the control signal is proportional to the integral of the error. In the terminology of Figure 10.2.1, this corresponds to a controller transfer function of $G_c(s) = K_I/s$, or

$$F(s) = G_c(s)E(s) = \frac{K_I}{s}E(s)$$

where K_I is called the *integral gain*. In the time domain, the relation is

$$f(t) = K_I \int_0^t e(t)\,dt \tag{10.4.3}$$

if $f(0) = 0$. In this form, it can be seen that if the error signal $e(t)$ remains positive, the integration will theoretically produce an infinite value of $f(t)$. This implies that special care must be taken to reinitialize the controller.

The integral term does not react instantaneously to the error signal but continues to correct, which tends to cause oscillations if the designer does not take this effect into account.

When integral and proportional actions are combined, we obtain the PI control algorithm:

$$f(t) = K_P e(t) + K_I \int_0^t e(t)\,dt \tag{10.4.4}$$

or

$$F(s) = \left(K_P + \frac{K_I}{s} \right) E(s) \tag{10.4.5}$$

An op-amp implementation of PI action is shown in Figure 10.4.4. With three component values to be chosen, normally the capacitance value is selected first, and the two resistances computed from the desired gain values, using the equations given in the figure.

Figure 10.4.4 Op-amp implementation of PI action.
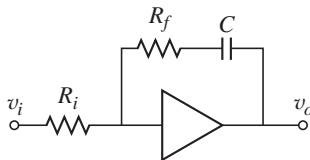
$$\frac{V_o(s)}{V_i(s)} = -\frac{R_f}{R_i} - \frac{1}{R_i C s}$$

$$K_P = \frac{R_f}{R_i} \qquad K_I = \frac{1}{R_i C}$$

EXAMPLE 10.4.2

Hydraulic Implementation of PI Control

■ Problem
Figure 10.4.5 shows a hydraulic system whose input is the motion y. Show that this system implements PI action, if the inertia of the load is small.

Figure 10.4.5 Hydraulic implementation of PI action.
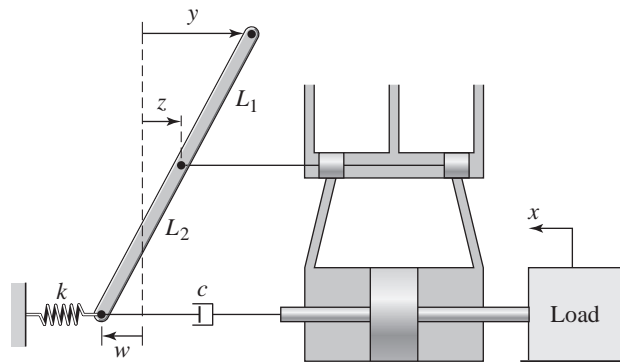
■ Solution

For small motions,

$$z = \frac{L_2}{L_1 + L_2} y - \frac{L_1}{L_1 + L_2} w$$

A force balance at the bottom of the beam gives

$$c\dot{w} + kw = c\dot{x}$$

For the servomotor, if the load inertia is small

$$\frac{X(s)}{Z(s)} = \frac{C_1}{As}$$

If

$$\frac{C_1}{A} \gg \left| \frac{(L_1 + L_2)(c\omega j + k)}{cL_1} \right|$$

for ω lying within the operating range of frequencies, then these equations may be combined to yield

$$\frac{X(s)}{Y(s)} = \frac{L_2}{L_1} \left(1 + \frac{k}{cs} \right) = K_P + \frac{K_I}{s}$$

where

$$K_P = \frac{L_2}{L_1} \qquad K_I = \frac{L_2}{L_1} \frac{k}{c}$$

DERIVATIVE CONTROL ACTION

Integral action produces a control signal even after the error has vanished, and this tends to cause oscillations. This suggests that the controller should be made aware that the error is approaching zero. One way to accomplish this is to design the controller to react to the rate of change of the error. This is the basis of *derivative control* action, in which the controller transfer function is $G_c(s) = K_D s$ and

$$F(s) = G_c(s)E(s) = K_D s E(s) \tag{10.4.6}$$

where K_D is the *derivative gain*. This algorithm is also called *rate action*. It is used to damp out oscillations.

The output of derivative control action depends on the error rate, and thus should never be used alone because it will produce zero output even if the error is large but constant. When used with proportional action, the following PD control algorithm results.

$$f(t) = K_P e(t) + K_D \frac{de}{dt} \tag{10.4.7}$$

or

$$F(s) = (K_P + K_D s)E(s) \tag{10.4.8}$$

An op-amp implementation of PD action is shown in Figure 10.4.6, where αT_D should be chosen to be small.

PID ACTION

When integral and derivative action are used with proportional action, we obtain the following proportional-plus-integral-plus-derivative control algorithm, abbreviated PID.

$$F(s) = \left(K_P + \frac{K_I}{s} + K_D s \right) E(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s \right) E(s) \tag{10.4.9}$$

Figure 10.4.6 Op-amp implementation of PD action.

Figure 10.4.7 Op-amp implementation of PID action.
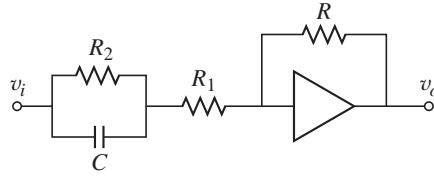
$$\frac{V_o(s)}{V_i(s)} = -\frac{R(R_2CS + 1)}{R_1 + R_2 + R_1R_2Cs} = -\frac{K_P(1 + T_Ds)}{1 + \alpha T_Ds}$$

$$K_P = \frac{R}{R_1 + R_2} \qquad T_D = R_2C \qquad \alpha = \frac{R_1}{R_1 + R_2}$$
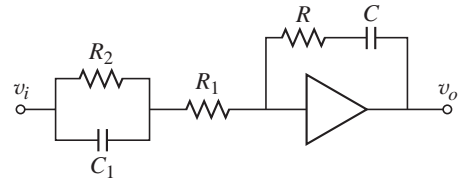
$$\frac{V_o(s)}{V_i(s)} = -\left(K_P + \frac{K_I}{s} + K_Ds\right)\frac{1}{\beta R_1C_1s + 1}$$

$$K_P = \beta\frac{RC + R_2C_1}{R_2C} \qquad K_I = \frac{\beta}{R_2C}$$

$$K_D = \beta RC_1 \qquad \beta = \frac{R_2}{R_1 + R_2}$$

This is called a *three-mode controller*. Some industrial controllers are programmed not in terms of K_I and K_D, but in terms of the *reset time T_I* and the *rate time T_D*, defined as follows:

$$T_I = \frac{K_P}{K_I} \tag{10.4.10}$$

$$T_D = \frac{K_D}{K_P} \tag{10.4.11}$$

The reset time is the time required for the integral action signal to equal that of the proportional term if a constant error exists (a hypothetical situation). The reciprocal of reset time is expressed as repeats per minute and is the frequency with which the integral action repeats the proportional correction signal.

Proportional action is frequently used, but integral or derivative action might or might not be used, depending on the application. An op-amp implementation of PID action is shown in Figure 10.4.7, where $\beta R_1 C_1$ should be chosen to be small.

The typical effects of P, I, and D action on a system's step response are illustrated in Figure 10.4.8, which shows the response to a unit-step command (with no disturbance)

Figure 10.4.8 Typical step response of P, PI, and PID control systems.
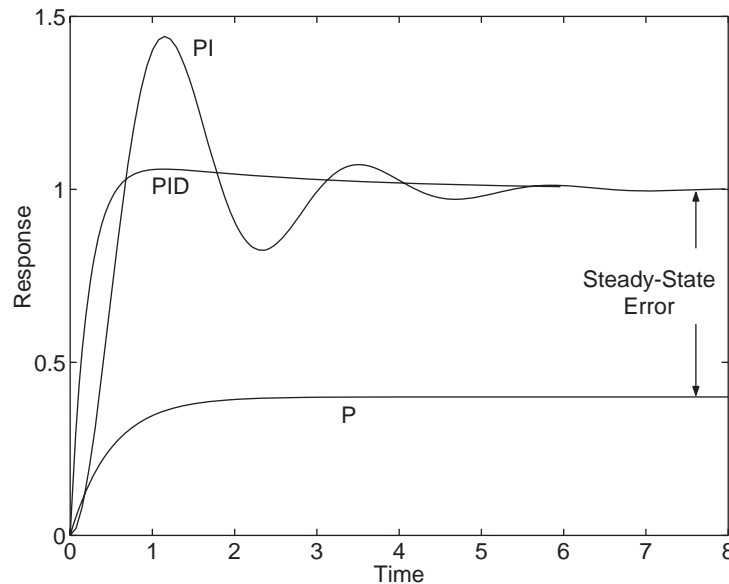
Table 10.4.1 Comparison of control actions.

Control action	Advantages	Disadvantages
On-Off	Simple to implement.	Cannot achieve small error without excessive oscillation.
Proportional	Simple to implement. Improves response time.	Allows steady-state error.
Integral	Reduces steady-state error.	Increases system order, creating potential for overshoot, oscillations, and instability.
Derivative	Reduces oscillations. Improves response time.	Amplifies noise in measurement signals.
Feedforward compensation	Reduces command input error.	Depends on accurate system model.
Disturbance compensation	Reduces disturbance error.	Depends on accurate system model. Depends on usable measurement of the disturbance.

of three systems having the same plant but using P, PI, and PID control, respectively. The response with P control is nonoscillatory and reasonably fast, but it allows a nonzero steady-state error to exist. When I action is added, the response under PI control becomes oscillatory with a large overshoot, but the steady-state error is now zero. When D action is added, the response under PID control has fewer oscillations and a smaller overshoot, and the steady-state error remains zero. In general, but not always, as control action is changed from P to PI to PID, first, the steady-state error is eliminated, and then the oscillations and settling time are reduced. Note that D action can never change a constant steady-state error, because it produces no output for a constant error.

Table 10.4.1 gives a comparison of the various control actions.

Proportional, integral, and derivative actions and their various combinations are not the only control laws possible, but they are the most common. It has been estimated that most controllers have PI or PID as their basic control action, and these actions will remain for some time the standard against which any new designs must compete.

DIGITAL CONTROL ALGORITHMS

The analog PID control algorithm

$$f(t) = K_P e(t) + K_I \int_0^t e(t)\, dt + K_D \frac{de}{dt} \tag{10.4.12}$$

cannot be implemented directly in a digital controller because digital hardware, which is limited to arithmetic operations, cannot perform differentiation and integration. Rather, the algorithm must first be converted to a difference equation. There are several finite difference equivalents used for integral and derivative action. A commonly used form is

$$f(t_k) = K_P e(t_k) + K_I T \sum_{i=0}^{k} e(t_i) + \frac{K_D}{T}[e(t_k) - e(t_{k-1})] \tag{10.4.13}$$

where $f(t_k)$ and $e(t_k)$ are the control and error signals at time $t_k = kT$, and T is the sampling period. The integral has been replaced with a sum of the areas of rectangles of width T. The derivative has been replaced with a simple difference expression.

The algorithm (10.4.13) is the *position* version of the PID control law. The *incremental* or *velocity* version of the algorithm determines the *change* in the control

signal: $f(t_k) - f(t_{k-1})$. To obtain it, decrement k by 1 in (10.4.13) and subtract the resulting equation from (10.4.13). This gives

$$f(t_k) - f(t_{k-1}) = K_P[e(t_k) - e(t_{k-1})] + K_I T e(t_k)$$

$$+ \frac{K_D}{T}[e(t_k) - 2e(t_{k-1}) + e(t_{k-2})] \qquad (10.4.14)$$

Suppose that the control signal $f(k)$ is a valve position. The position version of the algorithm is so named because it specifies the valve position directly as a function of the error signal. The incremental algorithm, on the other hand, specifies the change in valve position. The incremental version has the advantage that the valve will maintain its last position in the event of failure or shutdown of the control computer. Also, the valve will not "saturate" at start-up if the controller is not matched to the current valve position. In addition to having these safety features, the incremental algorithm is also well suited for use with incremental output devices, such as stepper motors.

These approximations for derivative and integral actions are the simplest, but are useful for introducing the concept of a digital control algorithm. Many other, more accurate approximations have been implemented in controllers. These are discussed in more advanced references dealing with digital control.

Throughout this chapter and the next we will concentrate on the analog version of the PID algorithm, but this does not prevent these results from being applied in a digital controller. The important point to keep in mind when doing so is the sampling rate of the controller. For a very fast sampling rate, the sampling time T is very small, and the finite difference approximations for the derivative and the integral approach their analog counterparts. That is,

$$\lim_{T \to 0} T \sum_{i=0}^{k} e(t_i) \to \int_0^t e(t)\, dt$$

and

$$\lim_{T \to 0} \frac{e(t_k) - e(t_{k-1})}{T} \to \frac{de}{dt}$$

Thus, if we assume that the sampling time is small enough, the gains computed for the analog PID algorithm can also be used for digital control.

10.5 CONTROL SYSTEM ANALYSIS

In this section we outline the approach to selecting an appropriate control algorithm and computing its gain values. Central to this process is an understanding of the system's transient and steady-state behavior, especially with regard to the effect of the command and disturbance inputs on the error.

SOURCES OF SYSTEM ERRORS

We have seen that the error $e(t)$ in a control system is the difference between the command input $r(t)$ and the controlled variable $c(t)$; that is, $e(t) = r(t) - c(t)$. The error is obviously a function of time, and when designing a control system we might specify and examine the peak error, the settling time for the error, and the steady-state error, for example. For a system having a command input and a disturbance input, the error is the sum of the errors caused by the command and the disturbance. These errors are called the *command error* and the *disturbance error*. Figure 10.5.1 illustrates the two types of error for the case where both the command and the disturbance are step

Table 10.5.1 The final value theorem.

1. **General Case:** If $T(s)$ is the transfer function and $R(s)$ is the input, the output is $C(s) = T(s)R(s)$. The steady-state value c_{ss} is given by

$$c_{ss} = \lim_{s \to 0} sC(s) = \lim_{s \to 0} sT(s)R(s)$$

if all the roots of the denominator of $sT(s)R(s)$ have negative real parts.

2. **Step Inputs:** If $R(s)$ represents a step of magnitude M, then $R(s) = M/s$ and

$$c_{ss} = \lim_{s \to 0} s\frac{M}{s}T(s) = M \lim_{s \to 0} T(s) = MT(0)$$

only if all the roots of the denominator of $T(s)$ have negative real parts; that is, if $T(s)$ represents a stable system.

3. **Ramp Inputs:** The transformed error given by $E(s) = R(s) - C(s)$ has the steady-state value given by

$$e_{ss} = \lim_{s \to 0} s[R(s) - C(s)] = \lim_{s \to 0} sR(s)[1 - T(s)]$$

if all the roots of the denominator of $sR(s)[1 - T(s)]$ have negative real parts. If $R(s)$ represents a ramp of slope m, then $R(s) = m/s^2$ and

$$e_{ss} = \lim_{s \to 0} s\frac{m}{s^2}[1 - T(s)] = \lim_{s \to 0} \frac{m}{s}[1 - T(s)]$$

Figure 10.5.2 Ramp responses of three control systems.

A similar procedure can be used for the disturbance input. This result is summarized in Table 10.5.1.

Control systems are never expected to respond to a pure ramp command or ramp disturbance, because no real input increases indefinitely. A ramp command function, however, is a good test of a control system's ability to follow a constantly changing

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$$e_{ss} = \lim_{s \to 0} s\frac{m}{s^2}[1 - T(s)] = \lim_{s \to 0} \frac{m}{s}[1 - T(s)] \tag{10.5.4}$$

A similar procedure can be used for the disturbance input. This result is summarized in Table 10.5.1.

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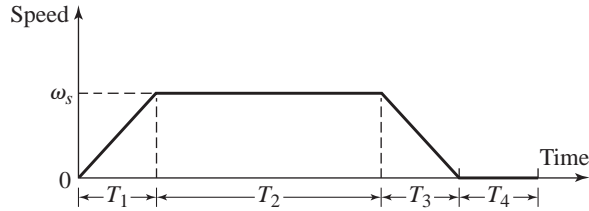
Figure 10.5.3 A trapezoidal commanded-speed profile.

command. Although the ramp function continues to increase without limit, in practice we are interested in the system's response only up to the time the transient response disappears. The reason for this is illustrated by Figure 10.5.3, which shows a command input called the *trapezoidal profile*. This is a typical command to speed control systems; it represents the desired speed in many positioning system applications such as tape drives, conveyor drives, and robotic systems. For example, when controlling the rotational speed of a motor, in order to bring the load up to some desired speed ω_s, called the *slew speed*, as quickly as possible, the drive motor should accelerate at its maximum acceleration α_{max}, which is limited by the maximum available motor torque. The motor then decelerates to bring the load to rest. To follow such a command input, the controller must be able to deal with both step and ramp commands.

DESIGN INFORMATION

When designing a control system, the control systems engineer is usually given the following information.

1. Usually the plant is given, and cannot be changed. The engineer might be given a model of the plant and the actuator, or might be expected to develop a model.
2. Usually the physical type of controller (electronic, pneumatic, hydraulic, or digital) and the actuator type (electric, pneumatic, or hydraulic) is specified, or the choice will be obvious from the application.
3. The command and disturbance inputs might be specified, or the engineer might be expected to develop suitable test inputs based on the application. Step functions are the principal test inputs, because they are the most common and perhaps represent the severest test of system performance. Ramp, trapezoidal, and sinusoidal test inputs are also employed. The type to use should be made clear in the design specifications.

CONTROL SYSTEM DESIGN PROCEDURE

In general, the design steps are as follows:

1. Based on the system model and the performance specifications, choose a control action and obtain the output, error, and actuator equations.
2. Analyze the system for stability. If it cannot be made stable, stop and try a different control action.
3. If it can be made stable, determine the constraints on the control gains to achieve stability.
4. Using the given command and disturbance input functions (step, ramp, etc.), evaluate the steady-state response. This usually can be done quickly with the final value theorem. Determine the constraints on the gains to satisfy the steady-state specifications.
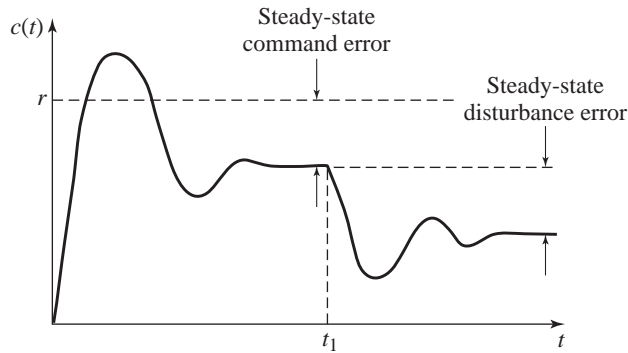
functions, with the command input applied at $t = 0$ and the disturbance acting at time $t = t_1$. We wish the controlled variable to be equal to the command; that is, we want $c = r$. In this figure, it is assumed that the effect of the disturbance is to decrease the controlled variable, whereas in some systems it may increase the controlled variable.

STEP INPUTS AND THE FINAL VALUE THEOREM

Calculation of the steady-state response due to a step input is simple if the system is stable. Let $T(s)$ denote the command transfer function, so that $C(s) = T(s)R(s)$. The steady-state output is obtained from the final value theorem.

$$c_{ss} = \lim_{s \to 0} sC(s) = \lim_{s \to 0} sT(s)R(s) \qquad (10.5.1)$$

If $R(s)$ represents a step of magnitude M, then $R(s) = M/s$ and

$$c_{ss} = \lim_{s \to 0} s \frac{M}{s} T(s) = M \lim_{s \to 0} T(s) \qquad (10.5.2)$$

The theorem applies only if the all the roots of the denominator of $T(s)$ have negative real parts; that is, if $T(s)$ represents a stable system. Stated simply, this result says the steady-state response of a stable system to a step input of magnitude M is $MT(0)$. A similar procedure can be used for the disturbance input. This result is summarized in Table 10.5.1.

RAMP INPUTS

Step inputs are not the only input types to be considered when designing control systems. Ramps, parabolas, and sine functions are also commonly used. Figure 10.5.2 shows some possible responses to a ramp input. For curve (a) the controlled variable continues to deviate from the ramp command, and thus the steady-state command error is infinite. For curve (b) the controlled variable eventually becomes parallel to the ramp command, and thus the steady-state command error is constant but nonzero. For curve (c) the controlled variable eventually becomes equal to the ramp command, and thus the steady-state command error is zero.

Because the output due to a ramp input eventually becomes infinite, the final value theorem gives no useful information when applied to the output equation. Rather, the theorem is applied to the *error* equation to obtain the steady-state error. Let $T(s)$ denote the command transfer function, so that $C(s) = T(s)R(s)$. The transformed error is given by $E(s) = R(s) - C(s)$, and thus the steady-state error is obtained from

$$e_{ss} = \lim_{s \to 0} s[R(s) - C(s)] = \lim_{s \to 0} sR(s)[1 - T(s)] \qquad (10.5.3)$$

5. Evaluate the transient performance in light of the transient specifications, using the given input functions. These specifications often are stated in terms of the desired dominant time constant and damping ratio, but they can be given in terms of overshoot, rise time, settling time, or bandwidth, for example. If the model of the closed-loop system is third order or higher, then we have no formulas to use for time constant, damping ratio, rise time, settling time, or bandwidth, and a trial-and-error approach must be used. This situation is explored in Chapter 11.

6. Evaluate other specifications, such as limits on the maximum available actuator output, and redesign if necessary.

SOME USEFUL RESULTS

Some formulas from earlier chapters particularly useful for computing control system gains are summarized in Table 10.5.2. Table 10.5.3 summarizes a quick way to determine the stability of a model without the need to solve for the roots. These results are obtained with the method developed by Routh and Hurwitz, which is discussed in

Table 10.5.2 Useful results for second-order systems.

1. Model: $m\ddot{x} + c\dot{x} + kx = f(t)$
2. Transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

3. Characteristic equation: $ms^2 + cs + k = 0$
4. Characteristic roots:

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

5. Damping ratio and undamped natural frequency:

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

6. Time constant: If $\zeta \leq 1$,

$$\tau = \frac{2m}{c}$$

If $\zeta > 1$, the dominant (larger) time constant is

$$\tau_1 = \frac{2m}{c - \sqrt{c^2 - 4mk}}$$

and the secondary (smaller) time constant is

$$\tau_2 = \frac{2m}{c + \sqrt{c^2 - 4mk}}$$

7. Maximum percent overshoot and peak time:

$$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

8. The complex root pair $s = -a \pm bj$ corresponds to the characteristic equation

$$(s + a)^2 + b^2 = 0$$

9. The value $\zeta = 0.707$ corresponds to a root pair having equal real and imaginary parts: $s = -a \pm aj$.

Table 10.5.3 Routh-Hurwitz stability conditions.

1. **Second-Order:** $a_2s^2 + a_1s + a_0 = 0$
   Stable if and only if a_2, a_1, and a_0 all have the same sign.
2. **Third-Order:** $a_3s^3 + a_2s^2 + a_1s + a_0 = 0$
   Assuming $a_3 > 0$, stable if and only if a_2, a_1, and a_0 are all positive and $a_2a_1 > a_3a_0$.
3. **Fourth-Order:** $a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0$
   Assuming $a_4 > 0$, stable if and only if a_3, a_2, a_1, and a_0 are all positive, $a_2a_3 > a_1a_4$, and

   $$a_1(a_2a_3 - a_1a_4) - a_0a_3^2 > 0$$

detail in texts devoted to control theory and is applicable to a model of any order. We state the results for second-, third-, and fourth-order models only, because we will not use higher-order models in this text.

10.6 CONTROLLING FIRST-ORDER PLANTS

The success of a chosen control action depends partly on the form of the plant transfer function. In this section and Sections 10.7 and 10.8, we show how to apply the design steps listed in Section 10.5 to various plant models. It is important to realize that the mathematical results of these examples can be used to design a control system for any plant having the same transfer function form. For example, the plant transfer function for the liquid-level system considered in Section 10.1 (Figure 10.1.1c) is

$$G_p(s) = \frac{R}{RAs + g} = \frac{1}{As + g/R}$$

while the plant transfer function for the oven heating system (Figure 10.3.3b) is

$$G_p(s) = \frac{R}{RCs + 1} = \frac{1}{Cs + 1/R}$$

Both transfer functions are first order with no numerator dynamics and have the form

$$G_p(s) = \frac{1}{Is + c}$$

which is the plant transfer function for a speed control system (Figures 10.3.7b and 10.3.10d). Therefore the results of the following example may be applied to design the liquid-level system, the heating system, and the speed control system.

Proportional Control of a First-Order Plant | **EXAMPLE 10.6.1**

■ **Problem**

Consider proportional control of the first-order plant whose transfer function is $1/(Is + c)$. This can represent a rotational system whose model is $I\dot{\omega} + c\omega = T - T_d$, where the controlled variable is the speed ω, the actuator torque is T, and the disturbance torque is T_d. To illustrate the methods simply, we will assume that the actuator has a gain of 1 and has an instantaneous response. This implies that the actuator transfer function is $G_a(s) = 1$. The block diagram is shown in Figure 10.6.1. Discuss the effect of the value of K_P on the system performance when the inputs are step functions.

Figure 10.6.1 P control of a first-order plant.
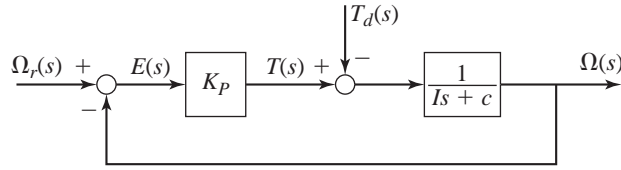
■ Solution

The transfer functions are

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_P}{Is + c + K_P} \tag{1}$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-1}{Is + c + K_P} \tag{2}$$

Steady-State Errors: For a unit-step command, $\Omega_r(s) = 1/s$, the speed approaches the steady-state value

$$\omega_{ss} = \lim_{s \to 0} s \frac{K_P}{Is + c + K_P} \frac{1}{s} = \frac{K_P}{c + K_P} < 1$$

Thus, the steady-state speed is less than the desired value of 1, but it might be close enough if the damping c is small. The time required to reach this value is approximately four time constants, or $4\tau = 4I/(c + K_P)$.

It is impossible for any control system to have a disturbance response that is zero for all time, because it takes time for any system to respond to the disturbance. In many applications, we are satisfied if the disturbance response is zero at steady state. In other applications, we might tolerate a small nonzero disturbance response. The performance specifications should indicate what is required for the design.

The response due to a unit-step disturbance, $T_d(s) = 1/s$, is found from equation (2).

$$\Omega(s) = \frac{-1}{Is + c + K_P} \frac{1}{s}$$

The steady-state response to the disturbance is found with the final value theorem to be $-1/(c + K_P)$. If $(c + K_P)$ is large, this response will be small. Note that this is the speed *change* caused by the disturbance, and is thus the disturbance error.

Frequency Response: The disturbance transfer function has the form of a low-pass filter whose low-frequency gain is $1/(c + K_P)$, and whose bandwidth is $1/\tau = (c + K_P)/I$. Increasing the gain K_P will increase the bandwidth, thus making the system sensitive to higher-frequency disturbances.

Actuator Requirements: It is important to predict what the actuator requirements will be. In the absence of a disturbance, the expression for the motor torque T can be found from the block diagram and equation (1). It is

$$T(s) = K_P[\Omega_r(s) - \Omega(s)] = K_P \left(1 - \frac{K_P}{Is + c + K_P} \right) \Omega_r(s)$$

or

$$T(s) = K_P \frac{Is + c}{Is + c + K_P} \Omega_r(s)$$

If the commanded speed is a unit step, the torque response can be found from a partial fraction expansion to be

$$T(t) = \frac{K_P}{c + K_P} \left[c + K_P e^{-(c + K_P)t/I} \right]$$

Thus the torque decays exponentially to the value $cK_P/(c + K_P)$. The maximum torque occurs at $t = 0$ and is $T(0) = K_P$. Thus the larger the gain K_P, the more torque is required. To speed up the system (by reducing τ) or to reduce the steady-state error, we must increase K_P and therefore must increase the available torque. This has implications for the size and cost of the motor that must be used.

Summary: The performance of proportional control action thus far can be summarized as follows. For the first-order plant $1/(Is + c)$, whose inputs are step functions,

1. The response to the command input never reaches the desired value if there is damping ($c \neq 0$), although it can be made arbitrarily close to the desired value by choosing the gain K_P large enough.
2. The response to the command input approaches its final value without oscillation. The time to reach this value is inversely proportional to K_P.
3. The steady-state disturbance error is inversely proportional to the gain K_P even if there is no damping.
4. For a step command input of magnitude M, the maximum required motor torque occurs at $t = 0$ and equals MK_P. To reduce the time constant or the errors, we must increase K_P and thus the maximum available torque.

The chief disadvantage of proportional control action is that it results in steady-state errors. It can be used only when the gain can be made large enough to reduce errors and reduce the time constant without requiring too large an actuator output. It can be augmented with feedforward command compensation to reduce the command error. An advantage to proportional control action is that the control signal responds to the error instantaneously (in theory at least) and does not cause oscillations.

Response of a Proportional Control System | **EXAMPLE 10.6.2**

■ **Problem**

Suppose the plant shown in Figure 10.6.1 has the parameter values $I = 5$ and $c = 2$ in a consistent set of units in which time is measured in seconds. (a) Determine the smallest value of the gain K_P required so that the steady-state command error will be no greater than 0.05 rad/s if ω_r is a unit-step input. Evaluate the resulting time constant and the steady-state disturbance error for a unit-step disturbance. (b) Obtain the steady-state error due to a unit-ramp command.

■ **Solution**

a. The error equation is

$$E(s) = \Omega_r(s) - \Omega(s)$$

$$= \left(1 - \frac{K_P}{5s + 2 + K_P}\right)\Omega_r(s) + \frac{1}{5s + 2 + K_P}T_d(s)$$

$$= \left(\frac{5s + 2}{5s + 2 + K_P}\right)\Omega_r(s) + \frac{1}{5s + 2 + K_P}T_d(s)$$

The steady-state command error for a unit-step command with no disturbance is

$$e_{ss} = \lim_{s \to 0} s \left(\frac{5s + 2}{5s + 2 + K_P}\right)\frac{1}{s} = \frac{2}{2 + K_P}$$

The error will be 0.05 rad/s if $K_P = 38$. The time constant for this value of K_P is

$$\tau = \frac{I}{c + K_P} = \frac{5}{2 + K_P} = \frac{1}{8} \text{ s}$$

The steady-state error due to a unit-step disturbance is

$$e_{ss} = \lim_{s \to 0} s \left(\frac{1}{5s + 2 + K_P} \right) \frac{1}{s} = \frac{1}{2 + K_P} = \frac{1}{40} = 0.025 \text{ rad/s}$$

If both inputs are applied, the total steady-state error will be

$$e_{ss} = 0.05 + 0.025 = 0.075 \text{ rad/s}$$

and the steady-state speed will be

$$\omega_{ss} = 1 - 0.075 = 0.925 \text{ rad/s}$$

Note that we can make the command error, the disturbance error, and the time constant smaller only by making K_P larger than 38, but this increases the maximum required torque and probably the cost of the system.

b. Using $\Omega_r(s) = 1/s^2$ with the final value theorem, we find the steady-state command error is

$$e_{ss} = \lim_{s \to 0} s \left(\frac{5s + 2}{5s + 40} \right) \frac{1}{s^2} = \infty$$

The speed $\omega(t)$ never catches up with the command input $\omega_r(t)$, and the steady-state error is infinite.

The speed control system with P action is a first-order system and thus cannot oscillate with step inputs. Integral control action often eliminates steady-state error but it always raises the order of the system by one, thus resulting in a higher-order system more likely to oscillate. The logical choice is to combine the two actions and use PI control.

EXAMPLE 10.6.3

Computing PI Gains

■ Problem

A PI control system is shown in Figure 10.6.2. Suppose the plant has the parameter values $I = 5$ and $c = 2$. The dominant time constant is specified to be 0.5 s.

a. Compute the required values for K_P and K_I for each of the following three cases:
    (1) $\zeta = 0.707$, (2) $\zeta = 1$, and (3) The secondary time constant must be 0.05 s.
    Evaluate the steady-state command error and the steady-state disturbance error for each case given that both the command input $\omega_r(t)$ and the disturbance $T_d(t)$ are unit-step functions.

b. Plot the output response $\omega(t)$ and the actuator response $T(t)$ for each case, given that the command input $\omega_r(t)$ is a unit-step function and the disturbance $T_d(t)$ is zero.

Figure 10.6.2 PI control of a first-order plant.
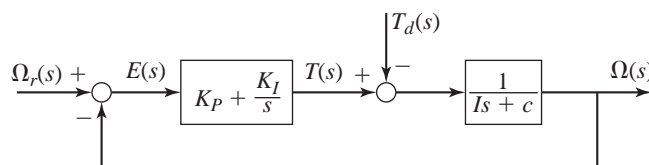
c. Evaluate the steady-state command error for each case given that the command input $\omega_r(t)$ is a unit-ramp function and the disturbance $T_d(t)$ is zero.
d. Evaluate the frequency response characteristics of the disturbance transfer function.

■ Solution

From the block diagram we obtain the following output, error, and actuator equations.

$$\Omega(s) = \frac{K_P s + K_I}{5s^2 + (2 + K_P)s + K_I} \Omega_r(s) - \frac{s}{5s^2 + (2 + K_P)s + K_I} T_d(s) \tag{1}$$

$$E(s) = \frac{5s^2 + 2s}{5s^2 + (2 + K_P)s + K_I} \Omega_r(s) + \frac{s}{5s^2 + (2 + K_P)s + K_I} T_d(s) \tag{2}$$

$$T(s) = \frac{(K_P s + K_I)(5s + 2)}{5s^2 + (2 + K_P)s + K_I} \Omega_r(s) + \frac{K_P s + K_I}{5s^2 + (2 + K_P)s + K_I} T_d(s) \tag{3}$$

The characteristic equation is

$$5s^2 + (2 + K_P)s + K_I = 0 \tag{4}$$

and its roots are

$$s = \frac{-(2 + K_P) \pm \sqrt{(2 + K_P)^2 - 20K_I}}{10}$$

The Routh-Hurwitz condition shows the system is stable if $2 + K_P > 0$ and $K_I > 0$. Applying the final value theorem to the error equation (2) with $\Omega_r(s) = 1/s$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{5s^2 + 2s}{5s^2 + (2 + K_P)s + K_I} \frac{1}{s} = 0$$

if the system is stable. Thus the system has zero command error for a step command.

Applying the final value theorem to the error equation (2) with $\Omega_r(s) = 0$ and $T_d(s) = 1/s$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{s}{5s^2 + (2 + K_P)s + K_I} \frac{1}{s} = 0$$

if the system is stable. Thus the system has zero disturbance error for a step disturbance.

The damping ratio is

$$\zeta = \frac{2 + K_P}{2\sqrt{5K_I}}$$

The presence of K_I enables the damping ratio to be selected without fixing the value of the dominant time constant. For example, if the system is underdamped or critically damped, the time constant is

$$\tau = \frac{10}{2 + K_P} \quad (\text{if } \zeta \leq 1)$$

The gain K_P can be picked to obtain the desired time constant, while K_I is used to set the damping ratio. Complete description of the transient response requires the numerator dynamics present in the transfer functions to be accounted for.

a. *Case 1:* If $\zeta < 1$, the real part of the roots is $-(2 + K_P)/10$, and thus the time constant is

$$\tau = \frac{10}{2 + K_P} = 0.5 \text{ sec}$$

which gives $K_P = 18$. The damping ratio of equation (4) is

$$\zeta = \frac{2 + K_P}{2\sqrt{5K_I}} = 0.707$$

Using $K_P = 18$ and squaring both sides gives

$$\frac{20^2}{4(5K_I)} = (0.707)^2 = 0.5$$

Solve for K_I to obtain $K_I = 40$.

   Case 2:   If $\zeta = 1$, the roots of equation (4) are both equal to $-(2 + K_P)/10$, and thus the time constant is

$$\tau = \frac{10}{2 + K_P} = 0.5$$

which gives $K_P = 18$. Using this value in the damping ratio of equation (4), we have

$$\zeta = \frac{2 + K_P}{2\sqrt{5K_I}} = \frac{10}{\sqrt{5K_I}} = 1$$

Squaring both sides gives

$$\frac{10^2}{5K_I} = 1$$

Solve for K_I to obtain $K_I = 20$.

   Case 3:   Since $\zeta > 1$ in this case, the expressions for the time constants of equation (4) are too complicated to be useful. The easier approach is to express the desired characteristic equation in factored form. The desired time constants are 0.5 s and 0.05 s, which correspond to the roots $s = -2$ and $s = -20$. The desired characteristic equation may thus be expressed as

$$(s + 2)(s + 20) = s^2 + 22s + 40 = 0$$

To compare this with equation (4) we must multiply by 5 to obtain

$$5s^2 + 110s + 200 = 0$$

Comparison of this with equation (4) shows that $K_I = 200$ and $2 + K_P = 110$, which gives $K_P = 108$. These gain values give a damping ratio of $\zeta = 110/2\sqrt{1000} = 1.74$.

b. The plots can be easily obtained with MATLAB using the `tf` and `step` functions applied to equations (1) and (3). The plots are shown in Figures 10.6.3 and 10.6.4. The design having the shortest rise time and shortest settling time is case 3 with $\zeta = 1.74$. It also has the smallest overshoot. That case, however, requires the greatest output from the actuator, as seen in Figure 10.6.4. The maximum torque values for each case are 18, 18, and 108, respectively.

c. Applying the final value theorem to the error equation (2) with $\Omega_r(s) = 1/s^2$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{5s^2 + 2s}{5s^2 + (2 + K_P)s + K_I} \frac{1}{s^2} = \frac{2}{K_I}$$

Thus the steady-state ramp command error is nonzero and is different for each of the three cases. It is the smallest for case 3 because that case has the largest value of K_I.
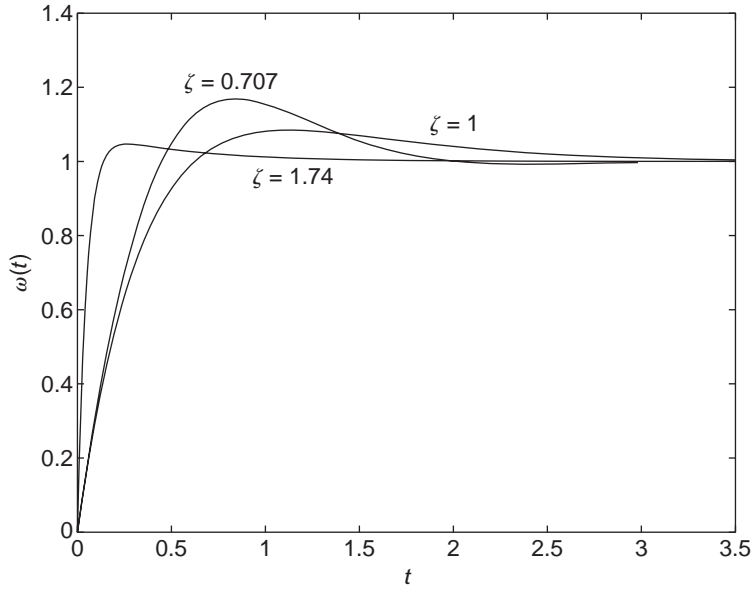
Figure 10.6.3 Step command response of a PI control system.
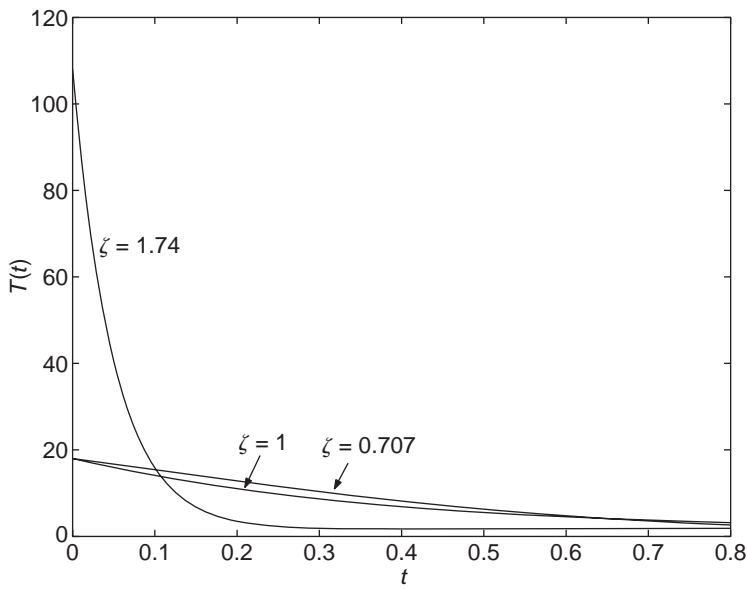
Figure 10.6.4 Actuator response of a PI control system.

The following table summarizes the results.

Case	K_P	K_I	ζ	Maximum torque	Ramp command error
1	18	40	0.707	18	0.05
2	18	20	1	18	0.1
3	108	200	1.74	108	0.005

d. The disturbance frequency response plot for each case is shown in Figure 10.6.5. The bandwidths for each case are slightly different, but the disturbance amplification for case 3 is much smaller than for the other two cases. The peak values are -26.02, -26.02, and -40.85 dB, respectively. These correspond to magnitude ratios of 0.05, 0.05, and 0.0091.

Figure 10.6.5 Frequency response of a PI control system.
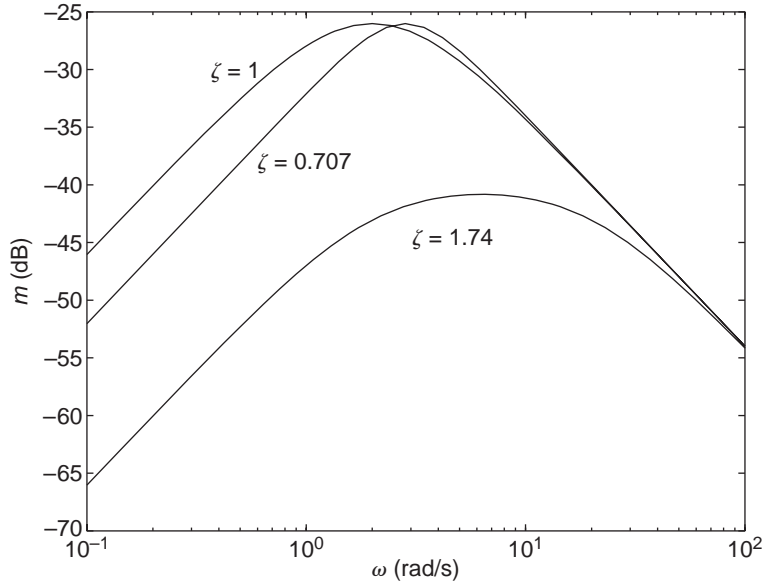
EXAMPLE 10.6.4

Use of Internal Feedback

■ Problem

Suppose the plant to be controlled is

$$G_p(s) = \frac{1}{Is + c}$$

where $I = 5$ and $c = 2$. Figure 10.6.6 shows a proposed alternative to PI control for this plant. It uses an internal feedback loop to adjust the output of the controller. The dominant time constant is specified to be 0.5 s.

a. Compute the required values for K_2 and K_I for each of the following three cases:
   (1) $\zeta = 0.707$, (2) $\zeta = 1$, and (3) the secondary time constant must be 0.05 s.
   Evaluate the steady-state command error and the steady-state disturbance error for each case given that both the command input $\omega_r(t)$ and the disturbance $T_d(t)$ are unit-step functions.
b. Plot the output response $\omega(t)$ and the actuator response $T(t)$ for each case, given that the command input $\omega_r(t)$ is a unit-step function and the disturbance $T_d(t)$ is zero.
c. Evaluate the steady-state command error for each case given that the command input $\omega_r(t)$ is a unit-ramp function and the disturbance $T_d(t)$ is zero.
d. Evaluate the frequency response characteristics of the disturbance transfer function.

Figure 10.6.6 I action with an internal feedback loop.
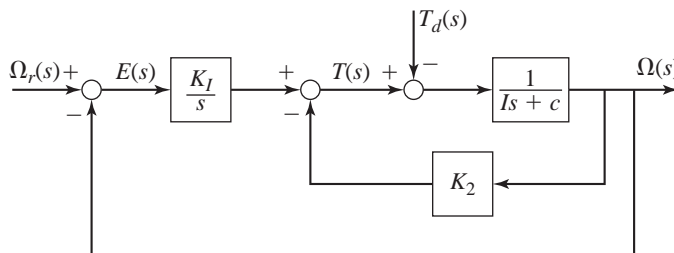
■ Solution

a. From the figure, we obtain the output equation as follows:

$$\Omega(s) = \frac{1}{5s + 2}[T(s) - T_d(s)] \tag{1}$$

$$T(s) = \frac{K_I}{s}E(s) - K_2\Omega(s) \tag{2}$$

$$E(s) = \Omega_r(s) - \Omega(s) \tag{3}$$

Substituting equations (2) and (3) into equation (1) and multiplying both sides by $5s + 2$ gives

$$(5s + 2)\Omega(s) = T(s) - T_d(s) = \frac{K_I}{s}[\Omega_r(s) - \Omega(s)] - K_2\Omega(s) - T_d(s)$$

Solve for $\Omega(s)$:

$$\Omega(s) = \frac{K_I}{5s^2 + (2 + K_2)s + K_I}\Omega_r(s) - \frac{s}{5s^2 + (2 + K_P)s + K_I}T_d(s) \tag{4}$$

For the error equation,

$$E(s) = \Omega_r(s) - \Omega(s)$$

$$= \Omega_r(s) - \frac{K_I}{5s^2 + (2 + K_2)s + K_I}\Omega_r(s) + \frac{s}{5s^2 + (2 + K_P)s + K_I}T_d(s)$$

Solve for $E(s)$:

$$E(s) = \frac{5s^2 + (2 + K_2)s}{5s^2 + (2 + K_2)s + K_I}\Omega_r(s) + \frac{s}{5s^2 + (2 + K_2)s + K_I}T_d(s) \tag{5}$$

For the actuator equation,

$$T(s) = \frac{K_I}{s}E(s) - K_2\Omega(s) = \frac{K_I}{s}[\Omega_r(s) - \Omega(s)] - K_2\Omega(s)$$

Solve for $T(s)$:

$$T(s) = \frac{(5s + 2)K_I}{5s^2 + (2 + K_2)s + K_I}\Omega_r(s) + \frac{K_2s + K_I}{5s^2 + (2 + K_2)s + K_I}T_d(s) \tag{6}$$

The characteristic equation is

$$5s^2 + (2 + K_2)s + K_I = 0 \tag{4}$$

This has the same form as equation (4) of Example 10.6.3 with K_2 replacing K_P. Therefore, the gain values obtained in that example can be used here. This gives the gain values shown in the following table.

Case	K_2	K_I	ζ	Maximum torque	Ramp command error
1	18	40	0.707	7.23	0.5
2	18	20	1	4.29	1
3	108	200	1.74	8.06	5.5

Although the denominators in this example are the same as those in Example 10.6.3, the numerators are different. Applying the final value theorem to the error equation (5)

with $\Omega_r(s) = 1/s$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{5s^2 + (2 + K_2)s}{5s^2 + (2 + K_2)s + K_I} \frac{1}{s} = 0$$

for all three cases. Thus the system has zero command error for a step command.

Applying the final value theorem to the error equation (5) with $\Omega_r(s) = 0$ and $T_d(s) = 1/s$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{s}{5s^2 + (2 + K_2)s + K_I} \frac{1}{s} = 0$$

for all three cases. Thus the system has zero disturbance error for a step disturbance.

b.   The plots can be easily obtained with MATLAB using the `tf` and `step` functions applied to equations (1) and (3). The plots are shown in Figures 10.6.7 and 10.6.8. The rise times

Figure 10.6.7 Step command response of I action with internal feedback.
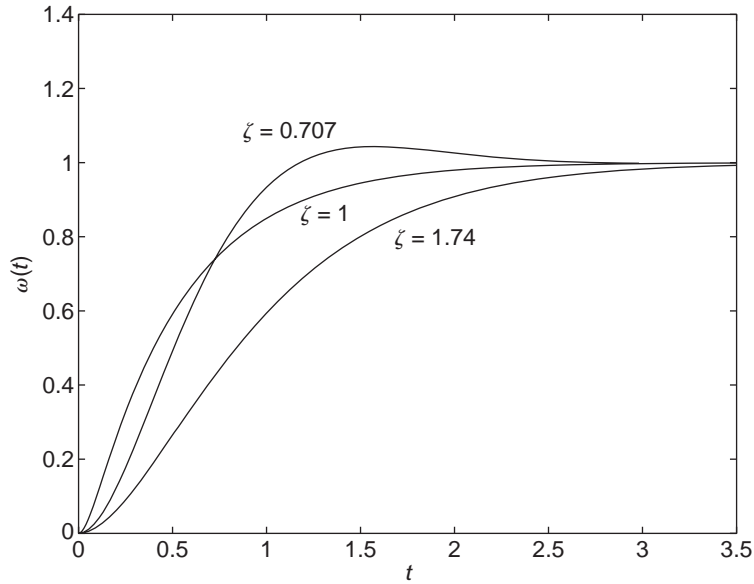
Figure 10.6.8 Actuator response of I action with internal feedback.
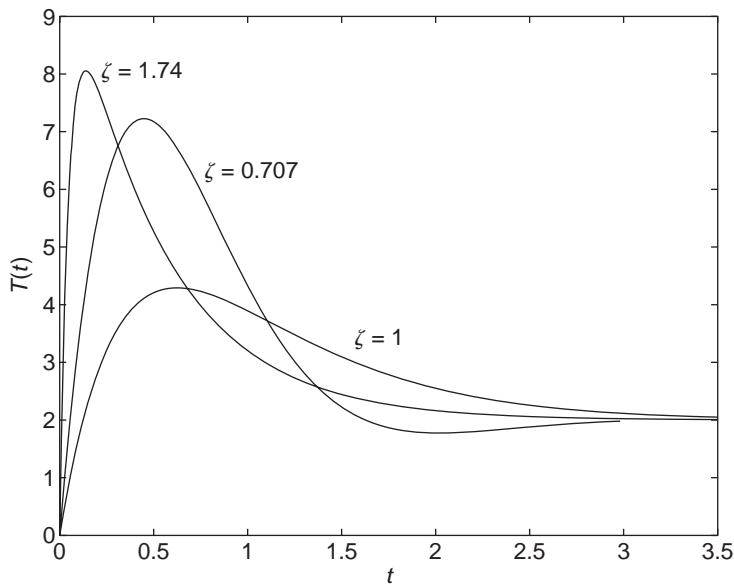
are much longer than for PI control, but the overshoots are much smaller. The peak torque required is 7.23, 4.29, and 8.06 for cases 1, 2, and 3, respectively. These are much less than the peak torque required with PI control.

c. Applying the final value theorem to the error equation (5) with $\Omega_r(s) = 1/s^2$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{5s^2 + (2 + K_2)s}{5s^2 + (2 + K_2)s + K_I} \frac{1}{s^2} = \frac{2 + K_2}{K_I}$$

Thus the ramp command error is different for each of the three cases and much larger than for PI action.

d. The disturbance response of this system is identical to that of the PI controller because they both have the same disturbance transfer function.

10.7 CONTROLLING SECOND-ORDER PLANTS

We now investigate how to control a second-order plant. An example of a general form of a second-order plant without numerator dynamics is given by a rotational system of inertia I, damping c, and torsional elasticity k, whose output is angular position θ and whose inputs are the actuator torque T and a disturbance torque T_d. The equation of motion is

$$I\ddot{\theta} + c\dot{\theta} + k\theta = T(t) - T_d(t)$$

The plant transfer function is

$$G_p(s) = \frac{1}{Is^2 + cs + k}$$

This plant is stable if I, c, and k have the same sign. If there is no elasticity, the plant's transfer function is $1/s(Is + c)$, which is neutrally stable because of the root at $s = 0$.

   The following examples deal with a neutrally stable plant. Applications involving a plant of the form $1/(Is^2 + cs + k)$ are treated in the chapter problems.

PD Control of a Neutrally Stable Second-Order Plant | **EXAMPLE 10.7.1**

■ **Problem**

PD control of a neutrally stable second-order plant is shown in Figure 10.7.1. Investigate its performance for step and ramp inputs for $c \geq 0$.
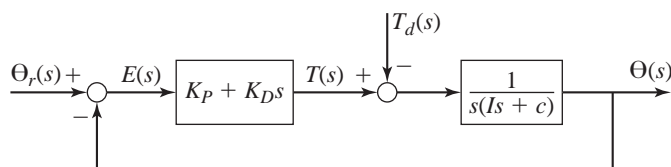
Figure 10.7.1 PD control of a second-order plant.

■ **Solution**

The output equation is

$$\Theta(s) = \frac{K_P + K_D s}{I s^2 + (c + K_D)s + K_P}\Theta_r(s) - \frac{1}{I s^2 + (c + K_D)s + K_P}T_d(s)$$

The characteristic polynomial is $I s^2 + (c + K_D)s + K_P$, and the system is stable if $c + K_D > 0$ and if $K_P > 0$. For unit-step inputs, the steady-state command response is $K_P/K_P = 1$, which is perfect, and the disturbance response is $-1/K_P$. The damping ratio is

$$\zeta = \frac{c + K_D}{2\sqrt{I K_P}}$$

If $\zeta \leq 1$, the time constant is given by

$$\tau = \frac{2I}{c + K_D}$$

For P control (with $K_D = 0$), $\zeta = c/2\sqrt{I K_P}$. Thus, introducing rate action allows the proportional gain K_P to be selected large to reduce the steady-state disturbance response, while K_D can be used to achieve an acceptable damping ratio. The rate action also helps to stabilize the system by adding damping (if $c = 0$, the system with P control is not stable).

EXAMPLE 10.7.2   Design of a PD Control System

■ **Problem**

For the system shown in Figure 10.7.1, we are given that $I = 10$ and $c = 2$. The dominant time constant τ is specified to be 2 s, and the damping ratio is specified to be $\zeta = 1$.

a. Compute the required values for K_P and K_D. Evaluate the steady-state command error and the steady-state disturbance error given that both the command input $\Theta_r(t)$ and the disturbance $T_d(t)$ are unit-step functions.
b. Evaluate the steady-state command error for each case given that the command input $\Theta_r(t)$ is a unit-ramp function and the disturbance $T_d(t)$ is zero.
c. Evaluate the frequency response characteristics of the disturbance transfer function.
d. Discuss the actuator output as a function of time when the command input $\Theta_r(t)$ is a unit-step function and the disturbance $T_d(t)$ is zero.

■ **Solution**

a. From the figure we obtain the following output, error, and actuator equations.

$$\Theta(s) = \frac{K_P + K_D s}{10 s^2 + (2 + K_D)s + K_P}\Theta_r(s) - \frac{1}{10 s^2 + (2 + K_D)s + K_P}T_d(s) \quad (1)$$

$$E(s) = \frac{10 s^2 + 2s}{10 s^2 + (2 + K_D)s + K_P}\Theta_r(s) + \frac{1}{10 s^2 + (2 + K_D)s + K_P}T_d(s) \quad (2)$$

$$T(s) = \frac{(10 s^2 + 2s)(K_P + K_D s)}{10 s^2 + (2 + K_D)s + K_P}\Theta_r(s) + \frac{K_D s + K_P}{10 s^2 + (2 + K_D)s + K_P}T_d(s) \quad (3)$$

The characteristic equation is

$$10 s^2 + (2 + K_D)s + K_P = 0$$

and the damping ratio is

$$\zeta = \frac{2 + K_D}{2\sqrt{10 K_P}} = 1 \quad (4)$$

Since $\zeta = 1$, the expression for the time constant is

$$\tau = \frac{20}{2 + K_D} = 2 \text{ sec}$$

which gives $K_D = 8$. Using this value in equation (4) gives

$$\frac{2 + 8}{2\sqrt{10 K_P}} = 1$$

which gives $K_P = 2.5$.

Applying the final value theorem to the error equation (2) with $\Theta_r(s) = 1/s$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{10s^2 + 2s}{10s^2 + (2 + K_D)s + K_P} \frac{1}{s} = 0$$

Thus the system has zero command error for a step command.

Applying the final value theorem to the error equation (2) with $\Theta_r(s) = 0$ and $T_d(s) = 1/s$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{1}{10s^2 + (2 + K_D)s + K_P} \frac{1}{s} = \frac{1}{K_P} = \frac{1}{2.5} = 0.4 \text{ rad}$$

Thus the system has a nonzero disturbance error for a step disturbance.

b.   Applying the final value theorem to the error equation (2) with $\Theta_r(s) = 1/s^2$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{10s^2 + 2s}{10s^2 + (2 + K_D)s + K_P} \frac{1}{s^2} = \frac{2}{K_P} = 0.8 \text{ rad}$$

Thus the ramp command error is nonzero.

c.   The disturbance transfer function is

$$\frac{\Theta_r(s)}{T_d(s)} = -\frac{1}{10s^2 + (2 + K_D)s + K_P} = -\frac{1}{10s^2 + 10s + 2.5}$$

Its bandwidth is the frequency range $0 \leq \omega \leq 0.32$ rad/s. Its low-frequency gain is $1/2.5 = 0.4$ or $20 \log 0.4 = -7.86$ dB. Thus the system will not respond very much to disturbances that have a frequency content higher than 0.32 rad/s.

d.   Substituting the values of K_P and K_D into the actuator equation (3) and using $\Theta_r(s) = 1/s$ and $T_d(s) = 0$, we obtain

$$T(s) = \frac{(10s^2 + 2s)(8s + 2.5)}{10s^2 + 10s + 2.5} \frac{1}{s} = \frac{80s^2 + 41s + 5}{10s^2 + 10s + 2.5} \tag{5}$$

Because the orders of the numerator and denominator are equal, this can be expressed as

$$T(s) = C_1 + \frac{C_2 s + C_3}{10s^2 + 10s + 2.5}$$

which may be arranged as a single fraction as follows.

$$T(s) = \frac{10 C_1 s^2 + (10 C_1 + C_2)s + 2.5 C_1 + C_3}{10s^2 + 10s + 2.5} \tag{6}$$

Comparing the numerators of equations (5) and (6), we find that $C_1 = 8$, $C_2 = -39$, and $C_3 = -15$, so that

$$T(s) = 8 - \frac{39s + 15}{10s^2 + 10s + 2.5} = 8 - \frac{3.9s + 1.5}{(s + 0.5)^2}$$

This could also have been obtained with synthetic division. Using the inverse transform we obtain

$$T(t) = 8\delta(t) - 3.9e^{-0.5t} + 0.45te^{-0.5t}$$

where $\delta(t)$ is the unit impulse function. Therefore, the actuator output predicted by the model will contain an impulse of strength 8 at $t = 0$. Of course, this is impossible physically, and so we must view these results with skepticism. The impulse is caused by the derivative term $K_D s$ in the control algorithm.

D ACTION AND STEP INPUTS

D action can be used to improve the system's speed of response, but it is best to avoid using it in the main controller if an equivalent effect can be obtained with a compensator elsewhere in the system. When a change in command input occurs, the error signal and therefore the controller are instantaneously affected. If the change is sudden, as with a step input, the physical limitations of the differentiating device mean that an accurate derivative is not computed, and the actual performance of the system will be degraded relative to the ideal performance predicted by the mathematical model.

In addition, as seen in Example 10.7.2, with a step command, the D action calls for an impulse in the actuator output, which is physically impossible. Note that in Example 10.7.2 a step *disturbance* does not produce an impulse in the actuator output because the disturbance passes through the plant's transfer function before entering the control action block via the feedback loop. Because the plant's transfer function contains an integration process, its output will not include an impulse. In conclusion, we must be careful in interpreting the transient response of a system containing a perfect model of a physical or numerical differentiator, whenever a step function is applied to such a differentiator model.

We have seen some devices, such as op-amp circuits, for differentiating the error signal, but it is impossible to construct a differentiating device that performs perfectly. Therefore, the interpretation of the step response must be considered with care. Note that this caution is not necessary in interpreting the step response of a controller with numerator dynamics due to I action, because in that case a step input is not applied to a differentiator.

If D action is required, a good design practice is to place the differentiator at a point in the loop where the signals are more slowly varying. Then the behavior of the differentiator will more closely correspond to its behavior as predicted by the mathematical model. A frequent choice for a differentiator location is in a feedback loop. The output is usually the result of several integrations and therefore will exhibit behavior that is smoothed and slowly varying with respect to the other signals in the system. A common example of this approach is in position control. In addition to the position sensor, a velocity sensor such as a tachometer is used to provide an internal feedback loop. In Example 10.7.3 we will see how this scheme gives the equivalent of derivative action without producing numerator dynamics or an impulse in the actuator output.

EXAMPLE 10.7.3 | Velocity Feedback Compensation

■ Problem
Consider the system shown in Figure 10.7.2 where $I = 10$ and $c = 2$. The output of the feedback sensor, which measures the rate of change $\omega = \dot{\theta}$ of the output, is multiplied by the gain K_2, whose value must be selected. Analyze the system and compare its performance with that of PD control for the specifications $\tau = 2$ s and $\zeta = 1$.
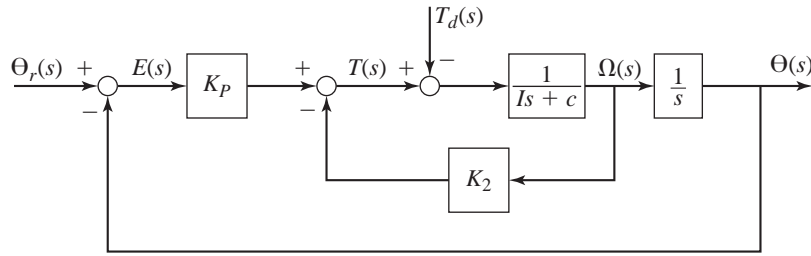
Figure 10.7.2 P action with rate feedback.

■ Solution

The output equation is

$$\Theta(s) = \frac{K_P}{10s^2 + (2 + K_2)s + K_P}\Theta_r(s) - \frac{1}{10s^2 + (2 + K_2)s + K_P}T_d(s)$$

Note that this command transfer function does not possess numerator dynamics, unlike the PD system. The disturbance transfer function is identical to that of the PD system.

The error equation is

$$E(s) = \frac{10s^2 + (2 + K_2)s}{10s^2 + (2 + K_2)s + K_P}\Theta_r(s) + \frac{1}{10s^2 + (2 + K_2)s + K_P}T_d(s)$$

The actuator equation is

$$T(s) = \frac{K_P(10s^2 + 2s)}{10s^2 + (2 + K_2)s + K_P}\Theta_r(s) + \frac{K_2s + K_P}{10s^2 + (2 + K_2)s + K_P}T_d(s)$$

The characteristic equation is the same as that for PD control with K_2 replacing K_D, so we may use the results of Example 10.7.2 to obtain $K_P = 2.5$ and $K_2 = 8$. The numerators of some of the transfer functions differ from those of PD control, however.

Applying the final value theorem to the error equation with $\Theta_r(s) = 1/s$ and $T_d(s) = 0$ gives $e_{ss} = 0$. Thus the system has zero command error for a step command. With $\Theta_r(s) = 0$ and $T_d(s) = 1/s$ we obtain

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\frac{1}{10s^2 + (2 + K_2)s + K_P}\frac{1}{s} = \frac{1}{K_P} = 0.4 \text{ rad}$$

Thus the system has the same nonzero disturbance error as PD control for a step disturbance.

For a ramp command, the error equation with $\Theta_r(s) = 1/s^2$ and $T_d(s) = 0$ gives

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\frac{10s^2 + (2 + K_2)s}{10s^2 + (2 + K_2)s + K_P}\frac{1}{s^2} = \frac{2 + K_2}{K_P} = 4 \text{ rad}$$

Thus the ramp command error is greater than with the PD control system.

The actuator equation with $\Theta_r(s) = 1/s$ and $T_d(s) = 0$ gives

$$T(s) = \frac{K_P(10s + 2)}{10s^2 + (2 + K_2)s + K_P} = \frac{2.5s + 0.5}{(s + 0.5)^2}$$

Following the same procedure used in Example 10.7.2, we obtain

$$T(t) = (2.5 - 0.75t)e^{-0.5t}$$

The actuator output does not contain an impulse, as opposed to the result for PD control.

I action might be included when we need to reduce or eliminate steady-state error. However, this results in a third-order model to analyze. For such models we do not have convenient formulas to use for overshoot, and for the time constant, damping ratio, and natural frequency of the dominant roots.

EXAMPLE 10.7.4

PID Control of a Neutrally Stable Second-Order Plant

■ Problem

The designs in Examples 10.7.1 and 10.7.3 have a non-zero-error for a step disturbance. We now add integral action to the design in Example 10.7.1. The resulting PID control of a neutrally stable second-order plant is shown in Figure 10.7.3. Suppose that $I = 10$ and $c = 2$. The performance specifications require that $\tau = 2$ s and $\zeta = 0.707$. Compute the gain values, and discuss the system's frequency response to a disturbance.

■ Solution

The output equation is

$$\Theta(s) = \frac{K_D s^2 + K_P s + K_I}{10s^3 + (2 + K_D)s^2 + K_P s + K_I}\Theta_r(s) - \frac{s}{10s^3 + (2 + K_D)s^2 + K_P s + K_I}T_d(s)$$

The characteristic polynomial is $10s^3 + (2 + K_D)s^2 + K_P s + K_I$ and system is stable if $2 + K_D > 0$, $K_P > 0$, and $K_I > 0$, and if

$$(2 + K_D)K_P - 10K_I > 0$$

Recall that for PI control, K_P needs to be large to achieve stability. The presence of K_D relaxes this requirement somewhat.

The steady-state errors are zero for step inputs, and the transient response can be improved relative to that of PD action, because three of the coefficients of the characteristic equation can be selected.

Because this system is third order, we do not have formulas to use for the damping ratio and the time constant. In addition, we must interpret the specifications to apply to the *dominant* root or root pair, and must choose the secondary root somewhat arbitrarily. The values $\zeta = 0.707$ and $\tau = 2$ s correspond to the dominant root pair $s = -0.5 \pm 0.5j$. The third root must be less than -0.5 so that it will not be dominant. We arbitrarily select the third root to be $s = -5$. We can later investigate other choices (this is done in one of the chapter problems). These three roots correspond to the polynomial equation

$$(s + 5)\left[(s + 0.5)^2 + (0.5)^2\right] = (s + 5)\left(s^2 + s + 0.5\right) = s^3 + 6s^2 + 5.5s + 2.5 = 0$$

To compare this with the system's characteristic equation, we multiply it by 10.

$$10s^3 + 60s^2 + 55s + 25 = 0$$

Compare this with the system's characteristic equation:

$$10s^3 + (2 + K_D)s^2 + K_P s + K_I$$

Thus, $K_P = 55$, $K_I = 25$, and $2 + K_D = 60$, or $K_D = 58$. The response to a unit-step command is shown in Figure 10.7.4. The overshoot is 10.3% and the 2% settling time is 2.89 s. This is quite less than the settling time of 8 s predicted from the dominant time constant of 2 s. The difference is due to the first- and second-order numerator dynamics of the transfer function.

Figure 10.7.3 PID control of a second-order plant.
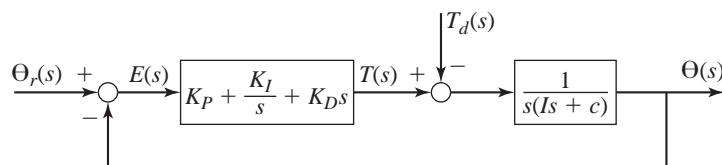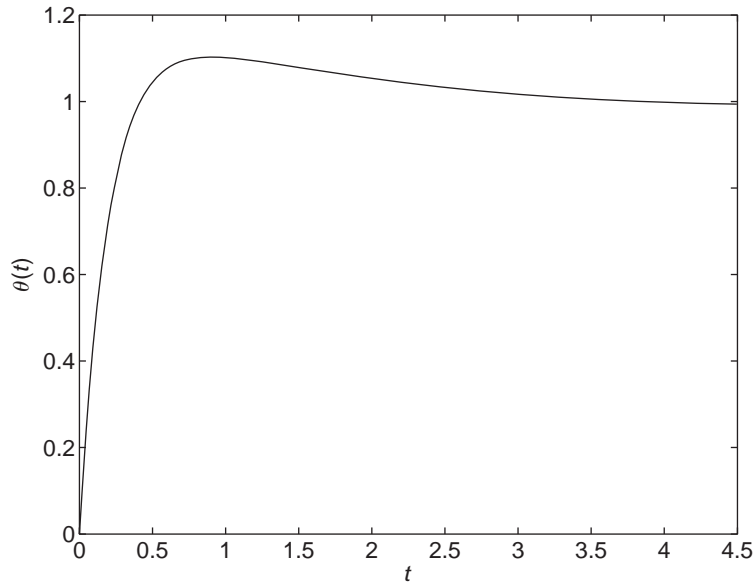
Figure 10.7.4 Step command response of a PID control system.
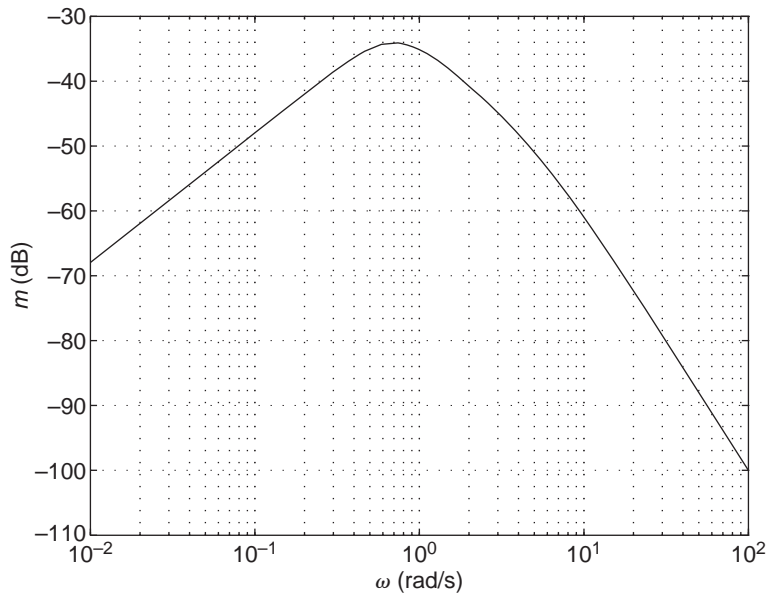
Figure 10.7.5 Frequency response of a PID control system.

The resulting disturbance transfer function for this system is

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{10s^3 + 60s^2 + 55s + 25}$$

The frequency response plot is shown in Figure 10.7.5. It shows that the system attenuates disturbance inputs by a factor of $m = -34.1$ dB or more. This corresponds to an amplitude reduction of $M = 10^{-34.1/20} = 0.02$. The system rejects by an even greater amount any disturbances whose frequencies are outside the bandwidth of $0.363 \leq \omega \leq 1.33$ rad/s. Compare this performance with the PD control system of Example 10.7.2, which does not have as great an attenuation.

Because of the D action, a step command input to a PID controller will cause an impulse in the actuator response, and we must therefore exercise caution in interpreting the transient response due to step commands. The use of an internal feedback loop (velocity feedback compensation) avoids this difficulty.

EXAMPLE 10.7.5

Velocity Feedback with a Second-Order Plant

■ Problem

Note that the gains K_P and K_D produce numerator dynamics in the command transfer function of Example 10.7.4 and that the K_D term produces an unrealistic impulse in the torque $T(t)$ when the command is a step function. We can eliminate the numerator dynamics by measuring the angular velocity ω and using this measurement to modify the output of the controller. This replaces the derivative action. The proportional action is preserved by using an additional internal-feedback loop with the measurement of θ. The diagram is shown in Figure 10.7.6. Compare its performance with the PID system of Example 10.7.4 using the same performance specifications.

■ Solution

Starting from the output $\Theta(s)$ and working to the left, we can write

$$\Theta(s) = \frac{1}{s(10s+2)}[T(s) - T_d(s)] \tag{1}$$

$$T(s) = \frac{K_I}{s}[\Theta_r(s) - \Theta(s)] - K_1\Theta(s) - K_2\Omega(s) \tag{2}$$

Noting that $\Omega(s) = s\Theta(s)$, we can substitute $T(s)$ from equation (2) into equation (1). After some algebra, we obtain

$$\Theta(s) = \frac{K_I}{10s^3 + (2+K_2)s^2 + K_1s + K_I}\Theta_r(s) - \frac{s}{10s^3 + (2+K_2)s^2 + K_1s + K_I}T_d(s)$$

The denominator of the transfer functions is identical to that of Example 10.7.4, with K_1 replacing K_P and K_2 replacing K_D. Thus we can use the results of that example to show that $K_1 = 55$, $K_2 = 58$, and $K_I = 25$. The command transfer function now has no numerator dynamics, and the disturbance transfer function is identical to that of Example 10.7.4.

The response to a unit-step command is shown in Figure 10.7.7. The maximum overshoot is 4% as compared to 10.3% for Example 10.7.4. However, the 2% settling time is now longer ($t = 8.64$ versus $t = 2.89$).

The big difference between the two designs is the maximum torque required. The design of Example 10.7.4 results in a physically unrealistic impulse in $T(t)$ at $t = 0$. It is shown in one of the chapter problems that the peak in $T(t)$ for the present design is only 25.

This design uses an extra sensor, a tachometer to measure the velocity. However, this can be replaced by a differentiation operation acting on the measurement of θ.

Figure 10.7.6 Velocity feedback with integral control for a second-order plant.
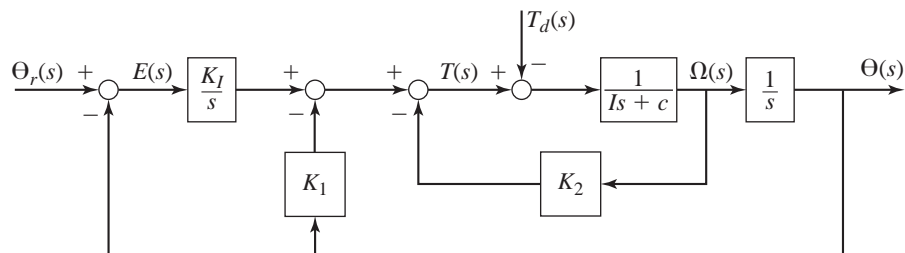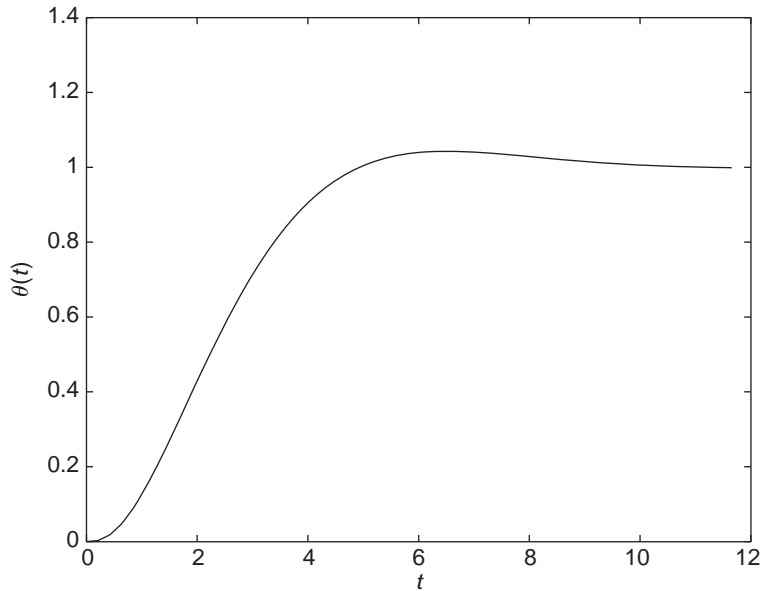
Figure 10.7.7 Unit step command response for Example 10.7.5.

10.8 ADDITIONAL EXAMPLES

PD Control | **EXAMPLE 10.8.1**

■ Problem

Compute the values of the PD control gains for the system shown in Figure 10.8.1 to meet the following specifications for unit-step inputs:

1. The closed-loop time constant τ should be 0.5 s.
2. The damping ratio ζ should be no less than 0.707.
3. The steady-state command error should be zero.
4. The magnitude of the steady-state disturbance error should be as small as possible, as long as the first three specifications are satisfied.

■ Solution

The error equation is

$$E(s) = \frac{4s^2 + 3s}{4s^2 + (3 + K_D)s + K_P} R(s) + \frac{1}{4s^2 + (3 + K_D)s + K_P} D(s)$$

The characteristic equation is $4s^2 + (3 + K_D)s + K_P = 0$. Since $\zeta < 1$ we may express τ as

$$\tau = \frac{8}{3 + K_D} = 0.5 \text{ s}$$
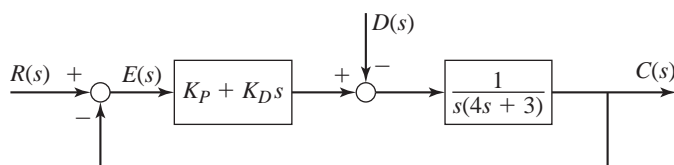
Figure 10.8.1 PD control of a second-order plant.

which gives $K_D = 13$. With this value for K_D, the expression for ζ becomes

$$\zeta = \frac{3 + K_D}{2\sqrt{4K_P}} = \frac{16}{2\sqrt{4K_P}} = \frac{4}{\sqrt{K_P}}$$

From the error equation we see that the steady-state command error is 0, and the steady-state disturbance error is $1/K_P$. Thus to minimize the error, we should make K_P as large as possible, as long as we keep $\zeta \geq 0.707$. Thus we set $\zeta = 0.707$ to obtain

$$\zeta = \frac{4}{\sqrt{K_P}} = 0.707$$

This gives $K_P = 32$. So the solution is $K_D = 13$ and $K_P = 32$.

EXAMPLE 10.8.2

Velocity Feedback

■ Problem
Determine the values of the gains in the system shown in Figure 10.8.2 to satisfy the following specifications for unit-step inputs, and compute the maximum percent overshoot.

1. The closed-loop time constant τ should be no greater than 0.05 s.
2. The steady-state command error should be zero.
3. The steady-state disturbance error should be no greater than 0.001.
4. To minimize cost, the gains should be as small as possible as long as the first three specifications are satisfied.

■ Solution
The error equation is

$$E(s) = \frac{s^2 + (10 + K_2)s}{s^2 + (10 + K_2)s + K_P}R(s) + \frac{1}{s^2 + (10 + K_2)s + K_P}D(s)$$

For unit-step inputs, the steady-state command error is 0 and the steady-state disturbance error is $1/K_P$. Thus to satisfy the third specification, K_P may be no less than 10^3.

Assuming that $\zeta \leq 1$, the expression for the time constant is

$$\tau = \frac{2}{10 + K_2}$$

To minimize K_2 we must set τ equal to its largest permissible value, 0.05 s. This gives $K_2 = 30$. Thus the tentative solution is $K_P = 1000$ and $K_2 = 30$, which is valid if $\zeta \leq 1$. So we must check ζ:

$$\zeta = \frac{10 + K_2}{2\sqrt{K_P}} = 0.632$$

Thus the solution is valid.

Figure 10.8.2 P action with internal feedback.
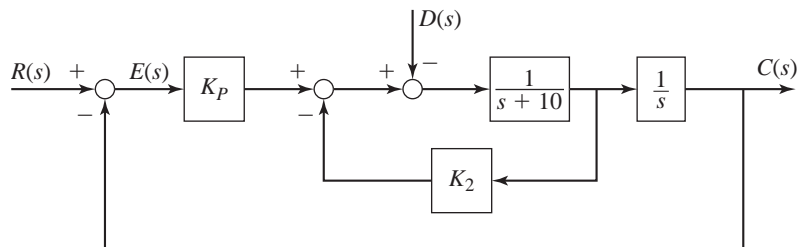
The resulting overshoot is

$$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 7.7\%$$

| Control of an Unstable System | **EXAMPLE 10.8.3** |

■ Problem

It is desired to stabilize the unstable plant shown in Figure 10.8.3 so that the time constant and damping ratio of the dominant root are $\tau = 0.1$ sec and $\zeta = 0.707$.

a. It is desired to have zero steady-state command error for a unit-step input. Determine the required values of the PID gains.
b. Since D action and I action increase the overshoot, can we eliminate them and still satisfy the specifications?

■ Solution

a. With PID action the error equation is

$$E(s) = \frac{s^3 - 4s}{s^3 + K_D s^2 + (K_P - 4)s + K_I} R(s)$$

This shows that if the system is stable, the steady-state command error is zero for a unit-step input. The characteristic equation is

$$s^3 + K_D s^2 + (K_P - 4)s + K_I = 0 \qquad (1)$$

The Routh-Hurwitz criterion shows that the system is stable if and only if $K_D > 0$, $K_I > 0$, $K_P > 4$, and $K_D(K_P - 4) > K_I$. Note that we can place all three roots where we want because the last three coefficients are independent functions of the gains. The specifications require that the dominant roots be $s = -10 \pm 10j$. Choosing the third root to lie to the left, say at $s = -20$, the characteristic equation must be

$$(s + 20)\left[(s + 10)^2 + 10^2\right] = s^3 + 40s^2 + 600s + 4000 = 0$$

Comparing this with equation (1) shows that $K_D = 40$, $K_P = 604$, and $K_I = 4000$.

b. If $K_D = 0$ there will be no s^2 term in the characteristic polynomial, and thus the system will be unstable. So D action is needed for stability.

  If we use P and D action but not I action, then setting $K_I = 0$ gives the error equation

$$E(s) = \frac{s^2 - 4}{s^2 + K_D s + K_P - 4} R(s)$$

The system is stable if and only if $K_D > 0$ and $K_P > 4$. If so, the steady-state command error for a unit-step input is

$$e_{ss} = -\frac{4}{K_P - 4} = \frac{4}{4 - K_P}, \quad K_P > 4$$

The error is nonzero for finite values of K_P, so I action is needed.

R(s) ──+──○──→ $\boxed{K_P + \dfrac{K_I}{s} + K_D s}$ ──→ $\boxed{\dfrac{1}{s^2 - 4}}$ ──→ C(s)
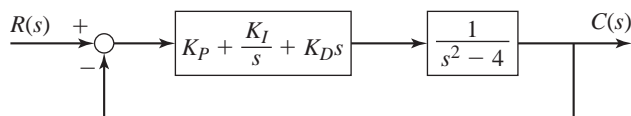
Figure 10.8.3 PID control of an unstable plant.

To achieve the desired roots $s = -10 \pm 10j$ with PD action requires that $K_D = 20$ and $K_P = 204$. The resulting steady-state error is $-1/150$. The negative error means that the steady-state response is $1 + 1/150$, which is *above* the desired value of 1.

Simulation of the PID and PD designs shows that the PID system has an overshoot of 24.6% while the overshoot of the PD system is 20%, but it has nonzero steady-state error.

EXAMPLE 10.8.4

Vehicle Stabilization

■ Problem

Figure 10.8.4 illustrates a vehicle moving through the atmosphere in a horizontal plane. The vehicle attitude θ is inherently unstable because the aerodynamic forces do not act through the mass center G. Instead they act through point P, the center of pressure, thus creating a net moment that tends to rotate the vehicle in the positive θ direction.

We want to control the attitude by controlling the elevator angle ϕ, which exerts a torque $T = B\phi$ about G, where B is a known positive constant. The equation of motion is

$$I\ddot{\theta} - LC_n\theta = T$$

where I is the inertia, C_n is the normal-force coefficient, and L is the distance between G and P.

Develop a control algorithm to control ϕ to stabilize the vehicle.

■ Solution

This problem is different than Example 10.8.3 because the desired value of the output θ is zero. Thus the command transfer function has no use here because the command input is zero. It is typical of many control applications where we simply need to stabilize the system to make the output zero. Instead of using the transfer function, we will work directly with the equation of motion.

Substituting $T = B\phi$ into the equation of motion gives

$$I\ddot{\theta} - LC_n\theta = B\phi \tag{1}$$

Stability requires an equation of the form

$$I\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

where c and k must be positive. From this we see that stability requires that a $\dot{\theta}$ term be present. This suggests that the control algorithm must include D action, $K_D\dot{\theta}$. For the coefficient of θ to be positive, we need P action, $K_P\theta$. Thus the control algorithm should be

$$\phi = -K_P\theta - K_D\dot{\theta}$$

Substituting this into equation (1) and rearranging gives

$$I\ddot{\theta} + BK_D\dot{\theta} + (BK_P - LC_n)\theta = 0$$

which will be stable if $K_D > 0$ and $BK_P > LC_n$. The gains K_P and K_D can be selected to give the desired time constant and damping ratio.

Figure 10.8.4 Vehicle attitude stabilization.
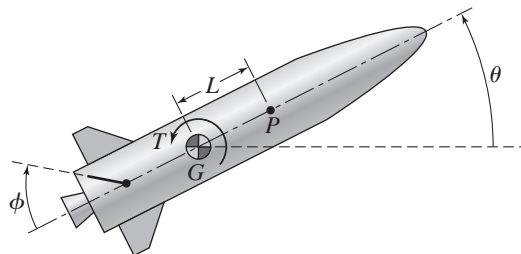
| A Liquid-Level Control System | **EXAMPLE 10.8.5** |

■ Problem

Design a control system to keep the height h_2 in Figure 10.8.5 near a desired value by controlling the volume inflow rate q_1. The volume flow rate q_d is a disturbance. Assume that sensors are available to measure both heights h_1 and h_2. Use the following numerical values:

$$A_1 = 1 \qquad A_2 = 2$$

$$\alpha_1 = \frac{R_1}{g} = 4 \qquad \alpha_2 = \frac{R_2}{g} = 1$$

We require zero steady-state command error for a step input. The dominant time constant must be 2 sec to achieve a 2% settling time of approximately 8 sec, and the damping ratio of the dominant roots must be $\zeta = 0.707$.

Propose a control structure, compute the required gain values, and evaluate the resulting steady-state disturbance error if q_d is a unit step.

■ Solution

Using a method similar to that of Example 7.4.3, we find that the model for tank 1 is

$$\rho A_1 \dot{h}_1 = \rho q_1 - \frac{\rho g}{R_1} h_1$$

The model for tank 2 is

$$\rho A_2 \dot{h}_2 = \rho q_d + \frac{\rho g}{R_1} h_1 - \frac{\rho g}{R_2} h_2$$

The mass density ρ cancels out of the equations, and they can be rearranged as

$$\frac{R_1}{g} A_1 \dot{h}_1 = \frac{R_1}{g} q_1 - h_1$$

$$\frac{R_1}{g} \frac{R_2}{g} A_2 \dot{h}_2 = \frac{R_1}{g} \frac{R_2}{g} q_d + \frac{R_2}{g} h_1 - \frac{R_1}{g} h_2$$

Using the definitions of α_1 and α_2, these equations can be expressed as

$$\alpha_1 A_1 \dot{h}_1 = \alpha_1 q_1 - h_1$$

$$\alpha_1 \alpha_2 A_2 \dot{h}_2 = \alpha_1 \alpha_2 q_d + \alpha_2 h_1 - \alpha_1 h_2$$

Figure 10.8.6a shows the block diagram of these equations. Combining the two inner loops gives the diagram shown in part (b).
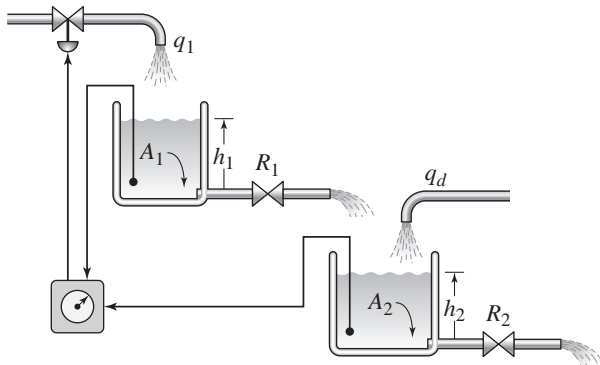
Figure 10.8.5 Liquid-level control.

Figure 10.8.6 Model
simplification for a liquid-level
system.

(a)

(b)
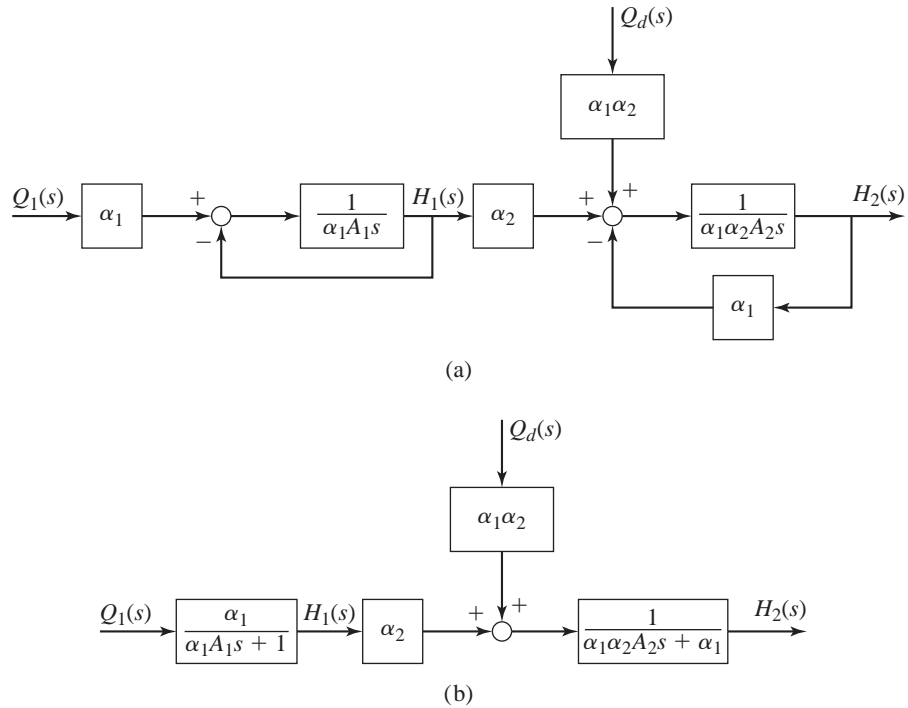
Figure 10.8.7 Block diagram of a liquid-level
control system using two feedback loops.
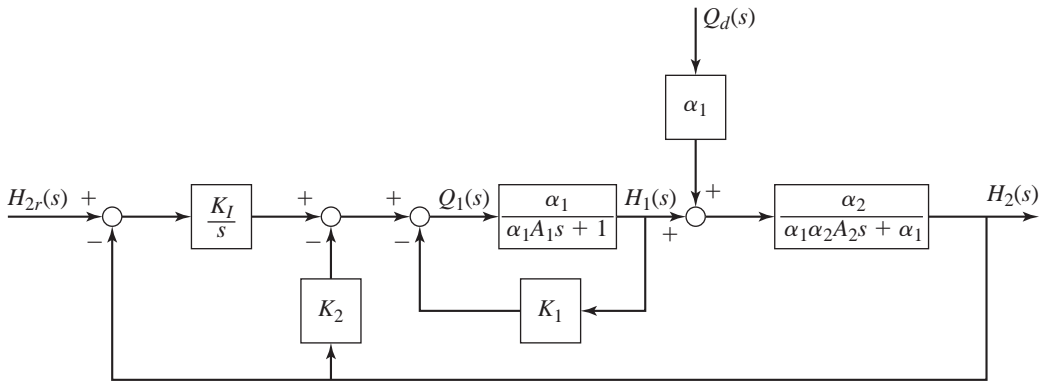
More than one control structure can be used to solve this problem, but because measurements
of h_1 and h_2 are available, we choose to use two feedback loops based on these measurements.
In addition, the requirement for zero steady-state error suggests that the main controller use
integral action. The result is shown in Figure 10.8.7.

Using the given parameter values, the command transfer function is found to be

$$\frac{H_2(s)}{H_{2r}(s)} = \frac{K_I}{8s^3 + (6 + 8K_1)s^2 + (1 + 4K_1 + K_2)s + K_I}$$

For a step command input, the final value theorem shows that $h_{2ss} = h_{2r}$ for any stable values of the gains. To satisfy the transient response requirement, the dominant roots must be $s = -0.5 \pm 0.5j$. The third root should be chosen to lie to the left of these roots. Arbitrarily choosing $s = -5$ we can express the characteristic equation as

$$8\{[(s+0.5)^2 + (0.5)^2](s+5)\} = 8s^3 + 48s^2 + 44s + 20 = 0$$

Comparing the coefficients with those of the denominator of the transfer function, we obtain

$$6 + 8K_1 = 48 \qquad 1 + 4K_1 + K_2 = 44 \qquad K_I = 20$$

which gives $K_I = 20$, $K_1 = 21/4$, and $K_2 = 22$. Simulation shows that the unit-step response has a 2% settling time of 8.64 sec, which is close to the value of 8 sec predicted from the dominant roots. The difference is due to the existence of a third root.

The disturbance transfer function is

$$\frac{H_2(s)}{Q_d(s)} = \frac{4s^2 + (1 + 4K_1)s}{8s^3 + (6 + 8K_1)s^2 + (1 + 4K_1 + K_2)s + K_I}$$

and so the steady-state disturbance error for a unit step is 0.

The Effect of Gain on Current Demand | **EXAMPLE 10.8.6**

■ Problem
The block diagram of a speed control system is shown in Figure 10.8.8. Determine (a) the transfer functions of the current and (b) obtain the current response to a step input of magnitude 104.7 rad/s (1000 rpm), using the parameter values given below.

■ Solution
(a) After some algebra we obtain the equation for the motor torque from the block diagram.

$$K_T = K_b = 0.04 \text{ N} \cdot \text{m/A} \qquad N = 1.5 \qquad K_P = 0.63$$

$$I_e = 1.802 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \qquad c_e = 4.444 \times 10^{-4} \text{ N} \cdot \text{m} \cdot \text{s/rad}$$

$$R_a = 0.6 \ \Omega \qquad L_a = 2 \times 10^{-3} \text{ H}$$
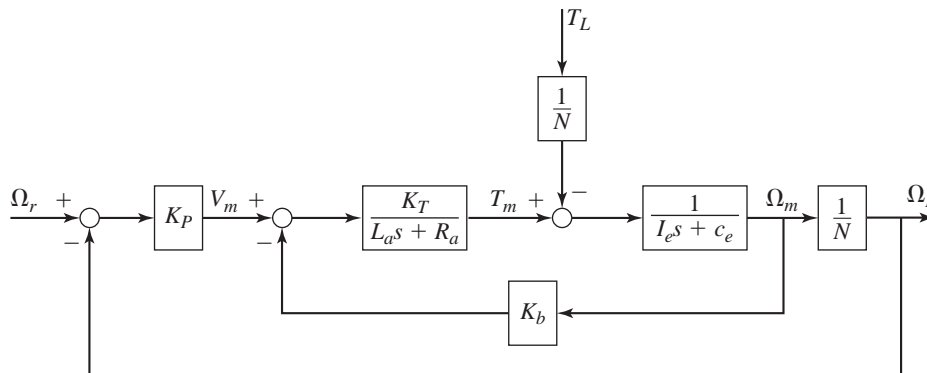
Figure 10.8.8 A speed control system.

$$T_m(s) = NK_P K_T \frac{L_a I_e s^2 + (R_a I_e + c_e L_a)s + R_a c_e}{(L_a s + R_a)D(s)} \Omega_r(s)$$
$$+ \frac{K_T(K_P + NK_b)/N}{D(s)} T_L(s) \tag{1}$$

where

$$D(s) = NL_a I_e s^2 + N(R_a I_e + c_e L_a)s + NR_a c_e + NK_T K_b + K_P K_T$$

The current is found from

$$I_a(s) = \frac{1}{K_T} T_m(s)$$

Equation (1) should raise our suspicions because the denominators of the two transfer functions $T_m(s)/\Omega_r(s)$ and $T_m(s)/T_L(s)$ differ by the factor $L_a s + R_a$. Always keeping in mind the physics of the problem, we observe that there are only two ways energy can be stored in this system (as electromagnetic energy in the inductor L_a and as kinetic energy in the inertia I_e), so the system model should be second order. However, the denominator of the transfer function $T_m(s)/\Omega_r(s)$ appears to be third order. The only way the denominators could be the same is if the term $L_a s + R_a$ is canceled by an identical term in the numerator. If this is the case we could express the numerator as

$$(as + b)(L_a s + R_a) = aL_a s^2 + (bL_a + aR_a)s + bR_a$$

Comparing the coefficients with those of the numerator of equation (1), we see that $a = I_e$ and $b = c_e$. Thus, our suspicion is confirmed, and we may write equation (1) as

$$T_m(s) = \frac{NK_P K_T(I_e s + c_e)}{D(s)} \Omega_r(s) + \frac{K_T(K_P + NK_b)/N}{D(s)} T_L(s)$$

Another way to detect the cancellation of the factor $L_a s + R_a$ is to obtain the partial-fraction expansion for either the free or forced response. The expansion coefficient (the residue) corresponding to the factor $s + R_a/L_a$ would be zero and thus this factor has no influence on the response. A zero residue indicates that a denominator factor has been canceled by a numerator factor.

(b) Use the transfer function $I_a(s)/\Omega_r(s)$ with $\Omega_r(s) = 104.7/s$ to compute the required armature current as a function of time. This can be done analytically, using the methods of Chapter 3, or numerically, using the methods of Chapter 5. The result is shown in Figure 10.8.9. The plot shows that the motor will require 96 A! This is a large value, as most motors with similar parameter values have a demagnetization current of less than 50 A. We can reduce the maximum current by accepting a steady-state speed error greater than 10%, by using a smaller value for K_P. For example, a speed error of 20% with $K_P = 0.28$ results in a maximum current of 45 A.

One way to reduce the maximum required current is to use a speed command that increases slowly, such as a ramp function. The step and impulse functions are the most severe inputs that can be applied to a system, because they change their value instantaneously. In many systems it is physically difficult to apply such an input, but these functions are often used in analysis because they result in simpler mathematics.

Suppose $K_P = 0.63$, which corresponds to a 10% speed error. Consider the command input ω_r that is a ramp for $0 \le t \le t_1$, and for $t > t_1$ is a constant value of 104.7 rad/s (1000 rpm), the desired speed. By experimenting with the value of the time t_1 while examining the resulting maximum current, we can arrive at a value for t_1 that results in a maximum current of less than a desired value. The result for $t_1 = 0.5$ s is shown in Figure 10.8.10, which can be obtained with
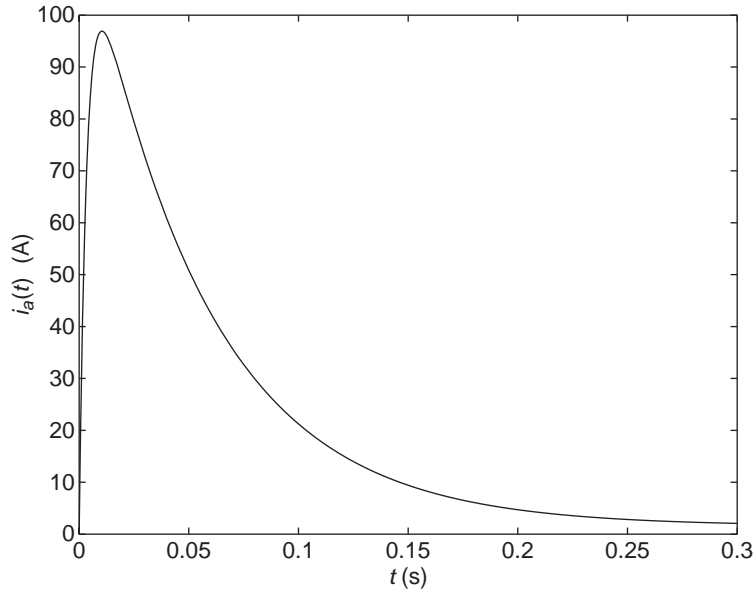
Figure 10.8.9 Current response of a speed control system.
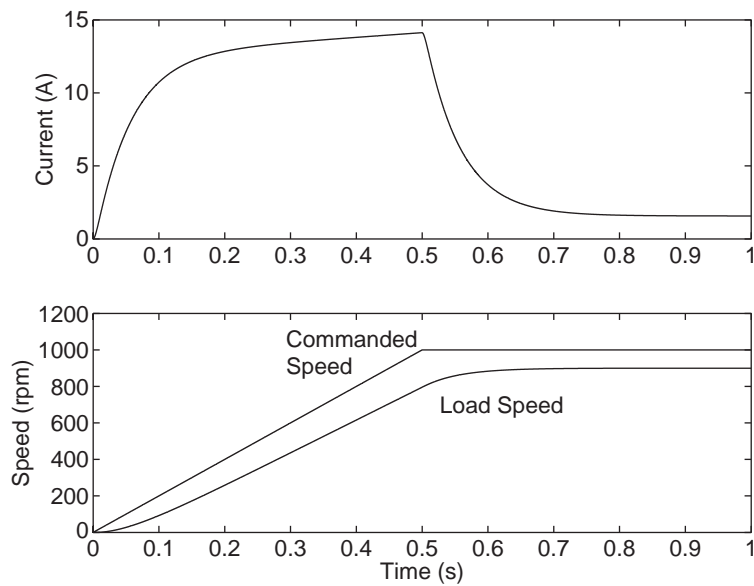
Figure 10.8.10 Control system response with a modified ramp input.

MATLAB. The top graph shows the current and the bottom graph shows the modified-ramp command input and the resulting load speed (assuming that $T_L = 0$). The speed reaches its steady-state value in about 0.6 s, as compared to 0.054 s for the step input. Now, however, the maximum current is 14 A, which is much less than the 96 A required for the step command input.

We mentioned that current limitation can change the dynamics of the system. Figure 10.8.11 was obtained with the Simulink model to be discussed in Section 10.10. The figure shows the effect of a built-in current limiter of 20 A on the step response of
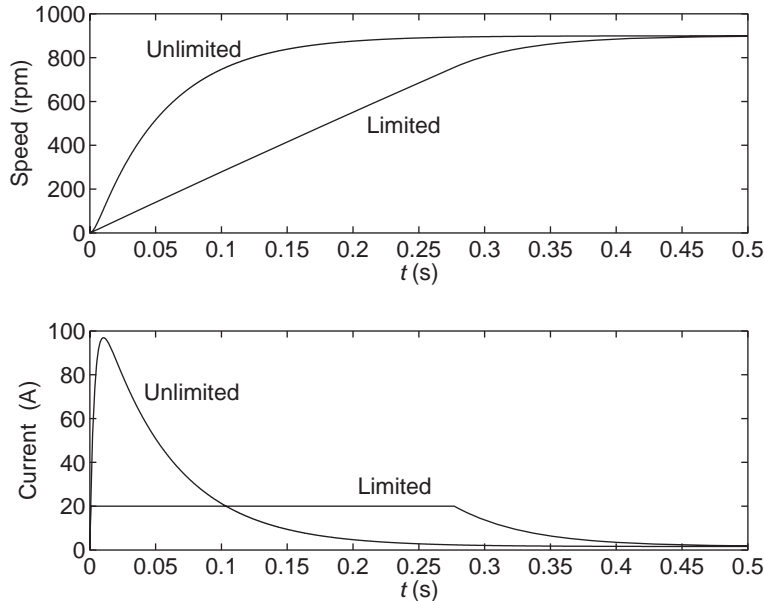
our control system with $K_P = 0.63$. The speed response without a limiter is shown for
comparison. The action of the limiter slows the speed response.

EXAMPLE 10.8.7   Current Control in Motors

■ **Problem**

Controlling the torque of an armature-controlled motor is complicated by the existence of the
back emf. For this reason some amplifiers have a built-in current controller that can accept a
commanded torque from the main controller. The current controller senses the armature current
and adjusts the armature voltage to produce the current required to obtain the desired torque. The
block diagram of such a system is given in Figure 10.8.12, where i_r is the commanded current
and $G(s)$ is the transfer function of the current controller. The current controller uses a feedback
measurement of the current i_a, compares it with the desired current value i_r, and outputs the
armature voltage v_a. Investigate the requirements on $G(s)$ to accomplish this.

■ **Solution**

We obtain the transfer function $\Omega(s)/I_r(s)$ as follows. From Figure 10.8.12 with $T_L(s) = 0$,

$$\Omega(s) = \frac{K_T}{Is + c} I_a(s)$$

Figure 10.8.12 A system for controlling
current in a motor.
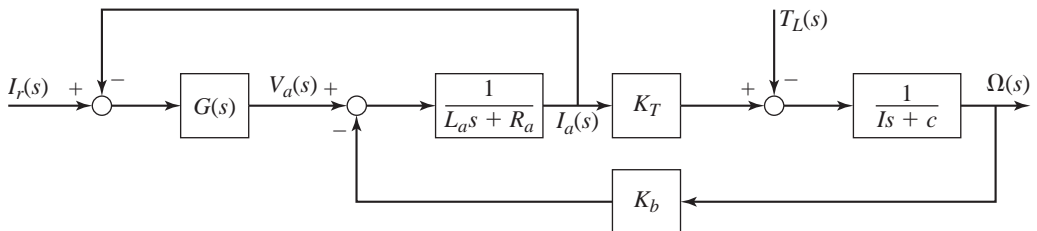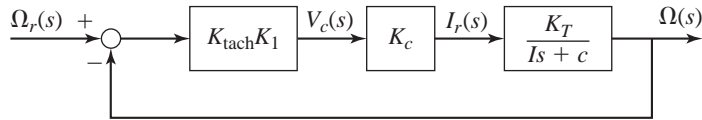
Figure 10.8.13 A proportional control system using current control.

and

$$I_a(s) = \frac{1}{L_a s + R_a} \{G(s)[I_r(s) - I_a(s)] - K_b \Omega(s)\}$$

Eliminating $I_a(s)$ from these equations gives

$$\frac{\Omega(s)}{I_r(s)} = \frac{K_T G(s)}{L_a I s^2 + As + B + G(s)(Is + c)} \tag{1}$$

where $A = cL_a + R_a I$ and $B = R_a c + K_T K_b$.

Suppose that $G(s) = K$, a constant. Then, as $K \to \infty$,

$$\frac{\Omega(s)}{I_r(s)} \to \frac{K_T}{Is + c} \tag{2}$$

In this case, we can draw the diagram shown in Figure 10.8.13, where K_c is a constant that converts the voltage command v_c from the speed controller into a current command for the current controller. The value of the proportional gain in this system is $K_P = K_{\text{tach}} K_1 K_c$ and the value of the product $K_1 K_c$ can be determined from $K_1 K_c = K_P / K_{\text{tach}}$.

In the general case where $G(s)$ is a function of s, if $G(s)$ is "large enough," we obtain the same result. Although the meaning of "large enough" may not be clear for a function of s, you can convince yourself of this result by trying a specific function for $G(s)$. For example, the transfer function for PID action is

$$G(s) = \frac{K_D s^2 + K_P s + K_I}{s}$$

Substituting this into equation (1) and letting the gains K_D, K_P, and K_I approach infinity, we obtain equation (2).

An Active Suspension | **EXAMPLE 10.8.8**

■ Problem

Active suspensions are now appearing more frequently in new cars. An active suspension supplements the car's passive suspension system that consists of springs and shock absorbers. An electric motor, a hydraulic servomotor or pneumatic cylinder is mounted either in parallel or in series with the spring and shock absorber. The actuator is under computer control so that its force can be adjusted to apply more force in response to varying conditions, for example to maintain traction during turns or to compensate for increased passenger and cargo weight.

Figure 10.8.14 illustrates the principle for a single-mass, quarter-car suspension model. The servomotor is shown in parallel with the spring and damper. Assume that the servomotor force is proportional to the chassis displacement x and the chassis velocity $\dot{x}$, so that $f = -K_P x - K_D \dot{x}$. The gains can be adjusted by the computer to achieve desirable response under different conditions.

Determine the values of the gains to achieve a damping ratio of $\zeta = 0.707$ and an undamped natural frequency of $\omega_n = 5$ rad/s. Use the nominal values $m = 275$ kg, $c = 1768$ N · s/m, and $k = 6250$ N/m.

Figure 10.8.14 An active suspension system.
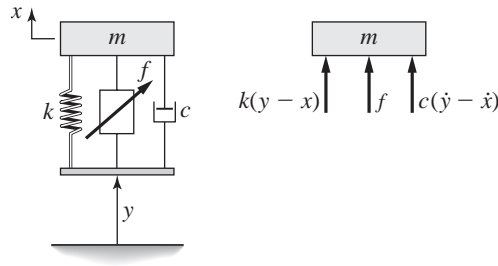
■ Solution
The equation of motion is

$$m\ddot{x} = k(y - x) + c(\dot{y} - \dot{x}) + f = k(y - x) + c(\dot{y} - \dot{x}) - K_P x - K_D \dot{x}$$

or

$$m\ddot{x} + (c + K_D)\dot{x} + (k + K_P)x = c\dot{y} + ky$$

From this equation we see that this active suspension, which implements PD control action, supplements the damping and stiffness of the passive suspension.

Using the given parameter values, we have

$$\zeta = \frac{c + K_D}{2\sqrt{m(k + K_P)}} = \frac{1768 + K_D}{2\sqrt{275(6250 + K_P)}} = 0.707$$

$$\omega_n = \sqrt{\frac{k + K_P}{m}} = \sqrt{\frac{6250 + K_P}{275}} = 5 \text{ rad/s}$$

Solving the latter equation for K_P gives $K_P = 625$ N/m. Substitute this value into the equation for ζ and solve to obtain $K_D = 176$ N · s/m.

EXAMPLE 10.8.9

Feedforward Compensation with Proportional Control

■ Problem
Consider the liquid-level system shown in Figure 10.1.1a, with the controller diagram given by Figure 10.1.3. Suppose we make the feedback control adjustment proportional to the error $e = h_r - h$, so that $G_c(s) = K_P$, where K_P is the proportional gain. This scheme is called feedforward compensation with proportional control. The resulting diagram is shown in Figure 10.8.15. The controlled flow rate is

$$q_c = A_e \dot{h}_r + \frac{g}{R_e} h_r + K_P(h_r - h)$$

Investigate the stability, speed of response, and error response of this system, assuming that h_r and q_d are step functions. Determine the value of K_P required to achieve a specified 2% settling time t_s.

■ Solution
First obtain the equation for the output $H(s)$. From the block diagram,

$$H(s) = \frac{R}{RAs + g} \left\{ Q_d(s) + \frac{R_e A_e s + g}{R_e} H_r(s) + K_P[H_r(s) - H(s)] \right\}$$
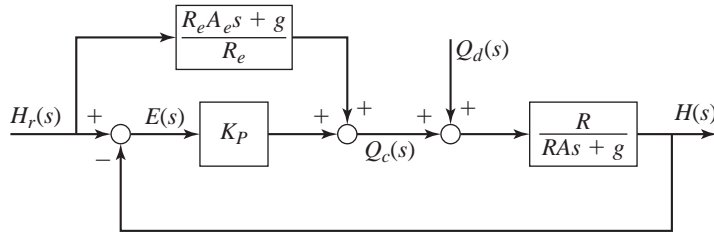
The solution for $H(s)$ is

$$H(s) = \frac{(R/R_e)(R_e A_e s + g) + R K_P}{RAs + g + R K_P} H_r(s) + \frac{R}{RAs + g + R K_P} Q_d(s) \qquad (1)$$

Note that if $R_e = R$ and $A_e = A$,

$$H(s) = H_r(s) + \frac{R}{RAs + g + R K_P} Q_d(s)$$

which means the system will have zero error if there is no disturbance. Thus the good feature of open-loop control has been preserved.

The characteristic equation, given by the denominators, is $RAs + g + R K_P = 0$, which has the root

$$s = -\frac{g + R K_P}{RA}$$

Since $RA > 0$ the system is stable as long as $g + R K_P > 0$; that is, if $K_P > -g/R$. The time constant of the closed-loop system is

$$\tau = \frac{RA}{g + R K_P}$$

and the 2% settling time is 4τ. So the required value for K_P is

$$K_P = \frac{4A}{t_s} - \frac{g}{R}$$

We note that accurate calculation of the derivative $\dot{h}_r$ is often difficult to achieve. This can limit the applicability of the feedforward method.

GENERAL FORM OF FEEDFORWARD COMMAND COMPENSATION

The concept of feedforward command compensation can be generalized as follows. The system is shown in Figure 10.8.16. The open-loop control transfer function is $G_f(s)$, which is called the *compensator* transfer function. The output equation is

$$C(s) = G_p(s)[G_a(s)\{G_f(s)R(s) + G_c(s)[R(s) - C(s)]\} + D(s)]$$

$$= G_p(s)G_a(s)G_f(s)R(s) + G_p(s)G_a(s)G_c(s)R(s)$$

$$- G_p(s)G_a(s)G_c(s)C(s) + G_p(s)D(s)$$

Solve for $C(s)$:

$$C(s) = \frac{G_p(s)G_a(s)G_f(s) + G_p(s)G_a(s)G_c(s)}{1 + G_p(s)G_a(s)G_c(s)} R(s) + \frac{G_p(s)}{1 + G_p(s)G_a(s)G_c(s)} D(s)$$

In the absence of a disturbance, the output $C(s)$ will exactly equal the command input $R(s)$ if $C(s)/R(s) = 1$. This will be true if

$$\frac{C(s)}{R(s)} = \frac{G_p(s)G_a(s)G_f(s) + G_p(s)G_a(s)G_c(s)}{1 + G_p(s)G_a(s)G_c(s)} = 1$$

This equation will be satisfied if

$$G_f(s) = \frac{1}{G_p(s)G_a(s)} \tag{10.8.1}$$

This expresses the general principle of feedforward compensation of the command; namely, in the absence of a disturbance, the output will follow the command *exactly* if $G_f(s) = 1/G_p(s)G_a(s)$. Note that this says that the compensator transfer function should be the reciprocal of the transfer function encountered as the compensated signal flows from the compensator to the output $C(s)$.

Of course, this startling principle is a mathematical result only. In practice we do not have exact models of the plant and the actuator, and we cannot build a compensator to have the exact transfer function specified by $G_f(s)$. If the compensator is implemented in analog form, the hardware (electronic, pneumatic, or hydraulic) cannot be designed to give the desired expression for $G_f(s)$ exactly. If implemented in digital form as a computer algorithm, the algorithm can reproduce the transfer function $G_f(s)$ only approximately because of quantization and sampling effects. Finally, the command compensator cannot deal with the effects of a disturbance. For that we always need the main controller block $G_c(s)$ and feedback loop shown in Figure 10.8.16. The compensator is used merely to augment the feedback controller.

DISTURBANCE COMPENSATION

Suppose that we can measure the disturbance $d(t)$. If so, we can use the measurement to improve the performance of a control system. This technique is called *disturbance compensation*. The arrangement is shown in Figure 10.8.17, in which the feedforward compensation has been omitted to simplify the discussion. The transfer function $G_d(s)$ represents a mathematical operation acting on the measurement of $d(t)$. We now determine what $G_d(s)$ should be.

Figure 10.8.16 General structure of feedforward compensation.
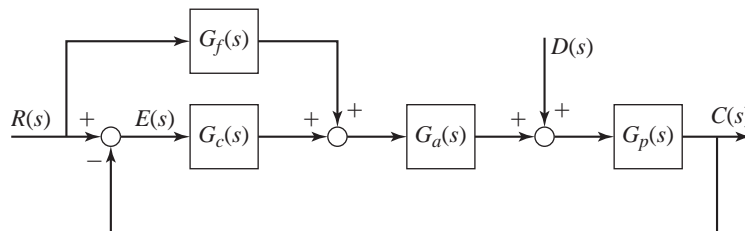
Figure 10.8.17 General structure of disturbance compensation.
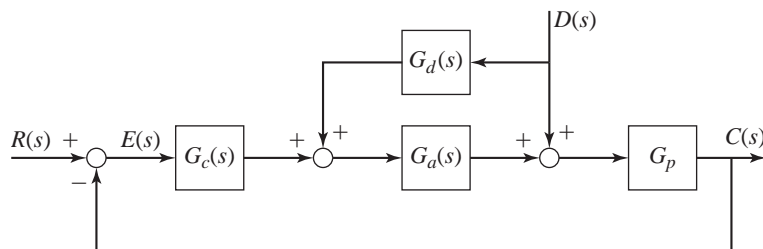
From the block diagram we can write

$$C(s) = G_p(s)[G_a(s)\{G_d(s)D(s) + G_c(s)[R(s) - C(s)]\} + D(s)]$$

Collecting terms gives

$$C(s) = G_p(s)\{[G_a(s)G_d(s) + 1]D(s) + G_c(s)[R(s) - C(s)]\}$$

Note that $D(s)$ will not appear in the equation if $G_a(s)G_d(s) + 1 = 0$. Thus, if

$$G_d(s) = -\frac{1}{G_a(s)} \tag{10.8.2}$$

the disturbance will be canceled by the compensator $G_d(s)$ and will not affect the output $c(t)$. To see this, follow the flow of $D(s)$ to the output $C(s)$ in Figure 10.8.17.

Of course, this tidy result is difficult to achieve in practice because (1) the disturbance might not be measurable or the measurement might be noisy, and (2) the operation specified by $G_d(s)$ might be difficult to implement in hardware or software.

In many of the examples to follow in this chapter and the next two, we will not use feedforward compensation or disturbance compensation to simplify the discussion, so that we can concentrate on the design of the main feedback controller. They can, however, be included if the limitations discussed earlier are not too severe. As illustrated by Figure 10.1.4, the calculations required to implement feedforward and disturbance compensation are easier to do if a digital computer is used as the controller.

10.9 MATLAB APPLICATIONS

The MATLAB functions `tf`, `step`, `lsim`, `bode`, and `bodemag` provide a powerful set of tools for analyzing the performance of control systems. All of these functions have extended syntax that was covered in earlier chapters. Here we will focus on their application to control system analysis. Recall that when these functions display a plot, you can right-click on the plot to determine characteristics such as maximum overshoot, peak time, settling time, and peak response. Left-clicking on the curve lets you move the cursor along the curve to identify coordinates.

THE `tf` AND `step` FUNCTIONS

The `tf` function creates an LTI object in transfer function form, using the numerator and denominator expressed as arrays. The `step` function displays the unit-step response. The command transfer function in Example 10.7.4 is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_D s^2 + K_P s + K_I}{10s^3 + (2 + K_D)s^2 + K_P s + K_I}$$

Using the gain values $K_P = 55$, $K_I = 25$, and $K_D = 58$, the unit-step response shown in Figure 10.7.4 was obtained with the following M-file:

```
KP = 55; KI = 25; KD = 58;
sys1 = tf([KD, KP, KI],[10, 2+KD, KP, KI]);
step(sys1)
```

The resulting plot was then edited with the Plot Editor to produce the figure.

THE `bode` AND `bodemag` FUNCTIONS

The `bode` function displays the plots of the magnitude ratio in decibels and the phase angle in degrees. The `bodemag` function does not display the phase plot. The disturbance transfer function in Example 10.7.4 is

$$\frac{\Theta(s)}{T_d(s)} = -\frac{s}{10s^3 + (2 + K_D)s^2 + K_P s + K_I}$$

Using the gain values $K_P = 55$, $K_I = 25$, and $K_D = 58$, the frequency response plot shown in Figure 10.7.5 was obtained with the following M-file:

```
KP = 55; KI = 25; KD = 58;
sys2 = tf([-1, 0],[10, 2+KD, KP, KI]);
bodemag(sys2)
```

The resulting plot was then edited with the Plot Editor to produce the figure.

THE `lsim` FUNCTION

The `lsim` function displays the response of a linear model to a user-defined input. This function is useful for computing the ramp response. For example, the following M-file plots the error response to a ramp command of slope 6, for a PI control system whose gains are $K_P = 18$ and $K_I = 40$. This system was analyzed in Example 10.6.3. The error transfer function is

$$\frac{E(s)}{\Omega_r(s)} = \frac{5s^2 + 2s}{5s^2 + (2 + K_P)s + K_I}$$

The M-file is

```
KP = 18; KI = 40;
sys3 = tf([5, 2, 0],[5, 2+KP, KI]);
t = (0:0.01:2);
omr = 6*t;
lsim(sys3, omr, t)
```

The `lsim` function plots the input `omr` as well as the response.

TRAPEZOIDAL RESPONSE

The response of a control system to a trapezoidal command input is easily found with MATLAB. If the profile and the transfer functions are defined in their own programs, we can use a modular approach that gives us the ability to examine different command inputs and different systems easily. The following example illustrates this approach.

EXAMPLE 10.9.1

Simulation with a Trapezoidal Profile

■ Problem

Consider the P, PI, and modified I control systems discussed in Examples 10.6.2, 10.6.3, and 10.6.4. The plant transfer function is $1/(Is+c)$, where $I = 5$ and $c = 2$. This plant can represent an inertia whose rotational velocity is the output and whose input is a torque. So the control systems are examples of speed controllers.

Investigate the performance of these systems for the trapezoidal command input shown in Figure 10.9.1. Use the parameter values $I = 5$ and $c = 2$.
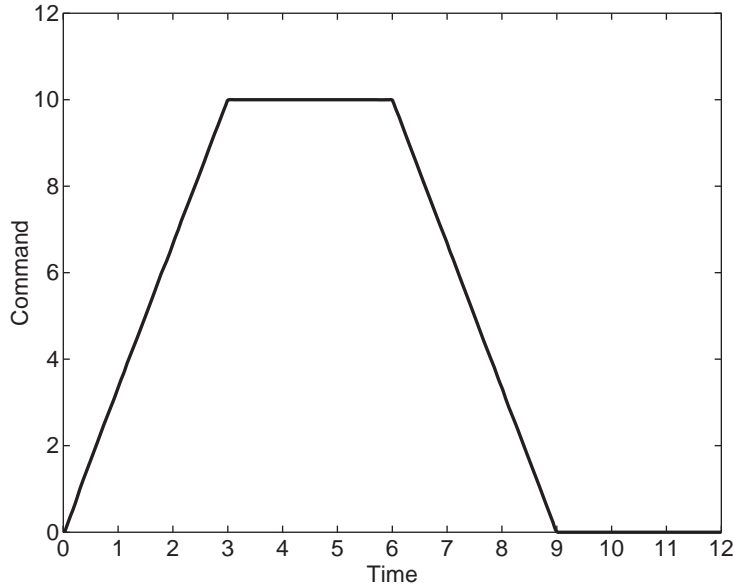
Figure 10.9.1 A trapezoidal command profile.

■ Solution

The transfer functions and parameters for these three control systems are shown in Table 10.9.1.

We can use the expressions in Table 10.9.1 to compute the gain values required to achieve a time constant of $\tau = 1$ and a steady-state error $e_{ss} = 0.2$ for a *unit*-ramp command. Because the slope of the profile is $10/3$, this will give a steady-state error of $0.2(10/3) = 5/3$. The following table shows the results.

P	$K_P = 3$	—
PI	$K_P = 8$	$K_I = 10$
Modified I	$K_2 = 8$	$K_I = 50$

Table 10.9.1 P, PI, and Modified I Control Systems

	Reference: Examples 10.6.2, 10.6.3, and 10.6.4		
	P	**PI**	**Modified I**
Command transfer function $\Omega(s)/\Omega_r(s)$	$\dfrac{K_P}{5s + 2 + K_P}$	$\dfrac{K_P s + K_I}{5s^2 + (2 + K_P)s + K_I}$	$\dfrac{K_I}{5s^2 + (2 + K_2)s + K_I}$
Disturbance transfer function $\Omega(s)/T_d(s)$	$-\dfrac{1}{5s + 2 + K_P}$	$-\dfrac{s}{5s^2 + (2 + K_P)s + K_I}$	$-\dfrac{s}{5s^2 + (2 + K_2)s + K_I}$
Actuator transfer function $T(s)/T_d(s)$	$K_P \dfrac{5s + 2}{5s + 2 + K_P}$	$\dfrac{(K_P s + K_I)(5s + 2)}{5s^2 + (2 + K_P)s + K_I}$	$\dfrac{(5s + 2)K_I}{5s^2 + (2 + K_2)s + K_I}$
Steady-state unit-ramp error e_{ss}	$\dfrac{1}{2 + K_P}$	$\dfrac{2}{K_I}$	$\dfrac{2 + K_2}{K_I}$
Time constant τ	$\dfrac{5}{2 + K_P}$	$\dfrac{10}{2 + K_P}$ if $\zeta \leq 1$	$\dfrac{10}{2 + K_2}$ if $\zeta \leq 1$
Damping ratio ζ	—	$\dfrac{2 + K_P}{2\sqrt{5K_I}}$	$\dfrac{2 + K_2}{2\sqrt{5K_I}}$

Table 10.9.2 Transfer function files

```
% command_tf.m
% Create the command transfer functions.
sysa = tf(KPa,[5, 2+KPa]);
sysb = tf([KPb, KIb],[5, 2+KPb, KIb]);
sysc = tf(KIc,[5, 2+K2, KIc]);
```

```
% actuator_tf.m
% Create the actuator transfer functions.
sysaACT = tf(KPa*[5, 2],[5, 2+KPa]);
sysbACT = tf(conv([KPb, KIb],[5, 2]),[5, 2+KPb, KIb]);
syscACT = tf(KIc*[5, 2],[5, 2+K2, KIc]);
```

```
% dist_tf.m
% Create the disturbance transfer functions.
sysaDIS = tf(-1, [5, 2+KPa]);
sysbDIS = tf(-[1, 0],[5, 2+KPb, KIb]);
syscDIS = tf(-[1, 0],[5, 2+K2, KIc]);
```

The following file sets the gain values for the three systems. System *a* is the P control system, system *b* is the PI system, and system *c* is the modified I system.

```
% gains.m
% Sets the gains.
% P control (system a):
KPa = 3;
% PI control (system b):
KPb = 8;KIb = 10;
% I action modified with velocity feedback (system c):
K2 = 8;KIc = 50;
```

After running the program gains.m, the files shown in Table 10.9.2 are used to create the command, disturbance, and actuator transfer functions given in Table 10.9.1.

The trapezoidal command input shown in Figure 10.9.1 can be created with the following script file.

```
% trap.m
% A specific trapezoidal profile
t = (0:0.01:12);
for k = 1:length(t)
   if t(k) <= 3
      r(k) = (10/3)*t(k);
   elseif t(k) <= 6
      r(k) = 10;
   elseif t(k) <= 9
      r(k) = 30-(10/3)*t(k);
   else
      r(k) = 0;
   end
end
```

After running trap.m and one of the transfer function programs given in Table 10.9.2, the files shown in Table 10.9.3 are used to compute and plot the command, disturbance, and actuator responses.

Table 10.9.3 Solution and plotting files

```
% plot_command.m
% Obtain and plot the command response.
ya = lsim(sysa,r,t);
yb = lsim(sysb,r,t);
yc = lsim(sysc,r,t);
plot(t,ya,t,yb,t,yc,'--',t,r)
```

```
% plot_actuator.m
% Obtain and plot the actuator repsonse.
yaACT = lsim(sysaACT,r,t);
ybACT = lsim(sysbACT,r,t);
ycACT = lsim(syscACT,r,t);
plot(t,yaACT,t,ybACT,t,ycACT,'--')
```

```
% plot_dist.m
% Add the disturbance response to the command response
% and plot the total response.
ya = lsim(sysa,r,t);
yb = lsim(sysb,r,t);
yc = lsim(sysc,r,t);
disturbance = 20*t;
yaDIS = lsim(sysaDIS,disturbance,t);
ybDIS = lsim(sysbDIS,disturbance,t);
ycDIS = lsim(syscDIS,disturbance,t);
plot(t,ya+yaDIS,t,yb+ybDIS,t,yc+ycDIS,'--',t,r)
```

For example, to create the plot shown in Figure 10.9.2, use the M-file:

```
gains
command_tf
trap
plot_command
```

Notice that the P control system does not follow the trapezoidal profile very well. (Figure 10.9.2 was enhanced using the Plot Editor.)
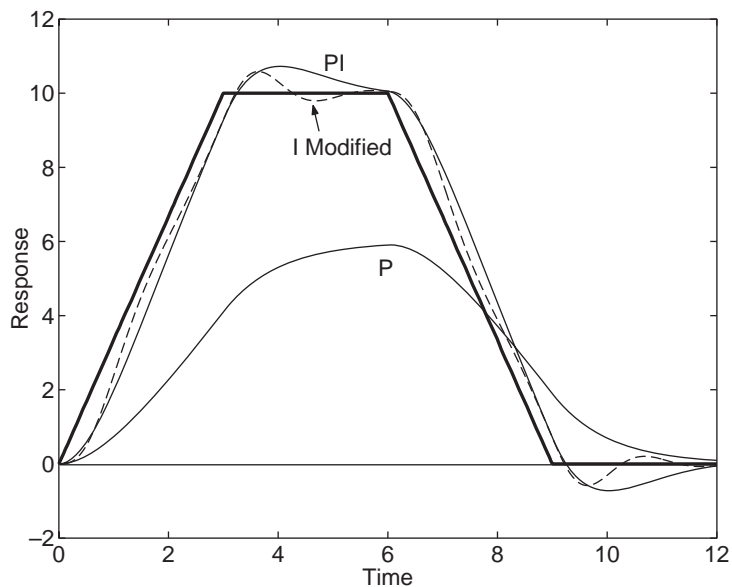
Figure 10.9.2 System response to a trapezoidal command.

Figure 10.9.3 System response to a trapezoidal command. Here the gain of the P control system has been increased.
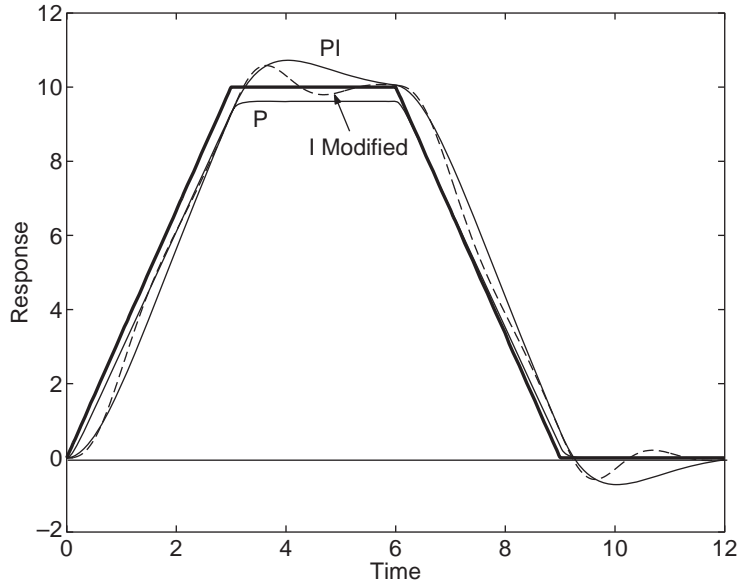
Figure 10.9.4 Actuator response to a trapezoidal command.
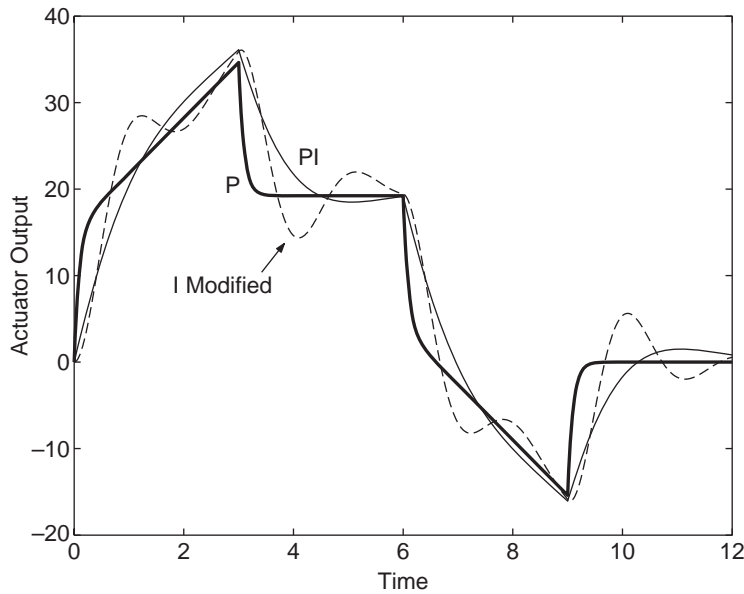
The performance of the P control system can be improved by increasing the gain. For example, to use $K_P = 50$, change KPa to 50 in gains.m and run the following M-file. Note that we need not run trap.m again unless we cleared the variables.

```
gains
command_tf
plot_command
```

The result is shown in Figure 10.9.3. Now the P control follows the trapezoidal profile much better.
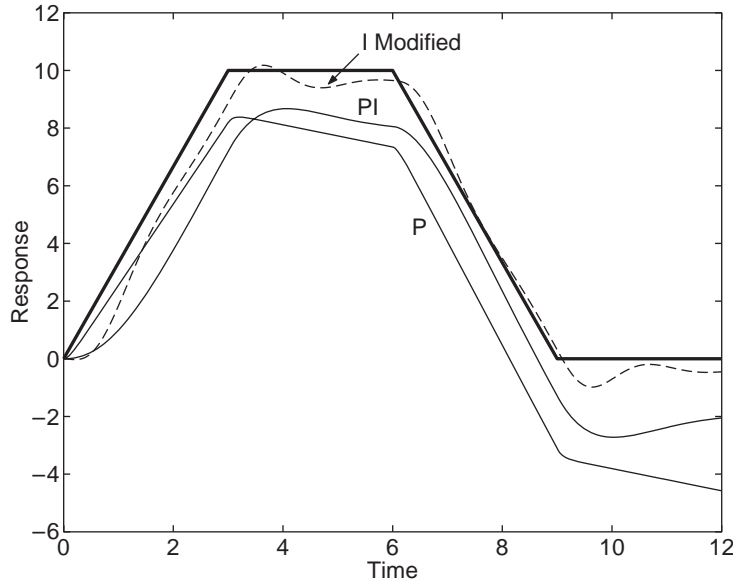
Figure 10.9.5 Response to a ramp disturbance.

The actuator response is found by running the M-file:

```
actuator_tf
plot_actuator
```

The result is shown in Figure 10.9.4.

Finally, let us see how well the systems do with a ramp disturbance $T_d = 20t$. Run the following M-file.

```
dist_tf
plot_dist
```

The total response is shown in Figure 10.9.5. Both PI and P control have difficulty following the profile, but the modified I system does rather well.

Using the approach outlined in Section 6.7 of Chapter 6, we can also use MATLAB to evaluate the performance specifications discussed in Section 6.5 for a trapezoidal command input. These specifications include the maximum torque, rms torque, and maximum velocity required by the control system to follow the profile.

10.10 SIMULINK APPLICATIONS

Simulink cannot be used to investigate control system performance until a preliminary analysis has developed a system model, selected a control algorithm, and computed a set of gain values. Once this has been done, however, Simulink can be used to investigate aspects of the design that are not easily analyzed with closed-form solution methods. These aspects include

1. The effects of unmodeled sensor, actuator, and plant dynamics. Often we neglect actuator time constants and smaller plant time constants in order to obtain a low-order model of the system.
2. Nonlinear dynamics, such as discontinuities (e.g., two-position control) and actuator saturation.

3. Input functions more complicated than simple steps, ramps, and sinusoids. These include trapezoidal commands and random disturbances.

EXAMPLE 10.10.1

Control of a Liquid-Level System

■ Problem

As an example, consider the liquid-level control system developed in Example 10.8.5, whose block diagram is shown in Figure 10.8.7. The analysis in that example ignored the obvious fact that the control flow rate q_1 can never be negative! For some initial heights, a simulation would show that $q_1(t)$ takes on negative values. Create a Simulink model using the numerical parameters and control gains given in that example, except for the integral gain K_I. The model should prevent q_1 from becoming negative. Investigate the response with this limitation.

■ Solution

We can use the Saturation block to prevent the control flow rate q_1 from becoming negative in the simulation. The Simulink model is shown in Figure 10.10.1. It can be easily constructed from the block diagram, with some rearrangement to allow for placement of the Mux. Note that we have not used a Saturation block to prevent the liquid heights from being negative, because the physics used to develop the differential equation models prevents this from happening.

We have introduced a new block in this model. In Simulink this block is called the "Transfer Fcn (with initial outputs)," to distinguish it from the Transfer Fcn block we have used earlier. With this new block we can set the initial value of the block output. In our model, this corresponds to the initial liquid heights in the tanks. This feature thus provides a useful improvement over traditional transfer function analysis, in which initial conditions are set to zero.

Figure 10.10.1 Simulink model of a liquid-level control system.
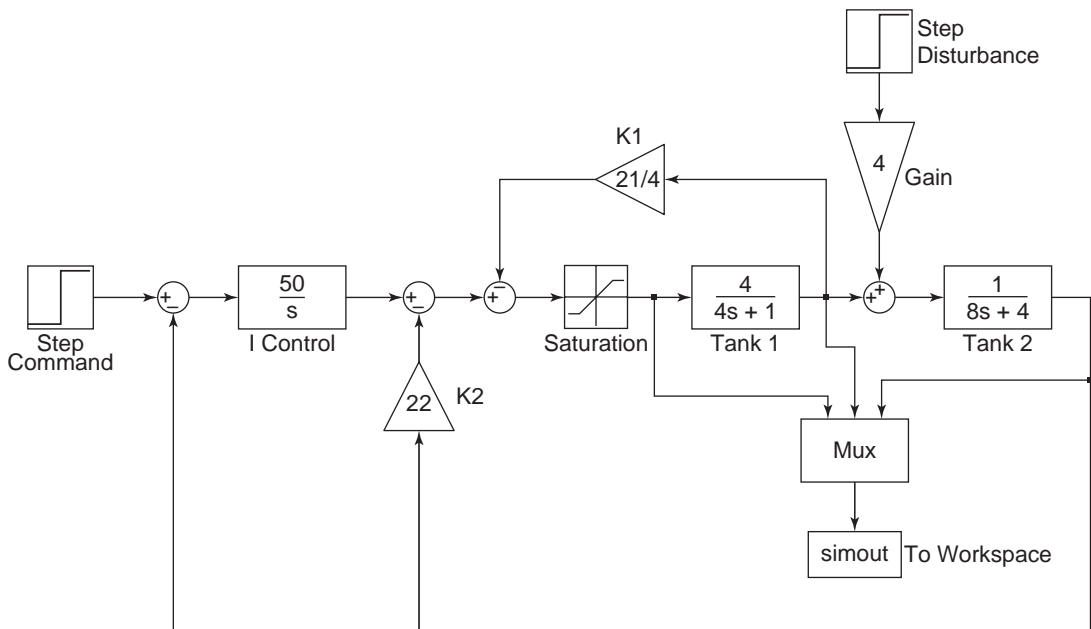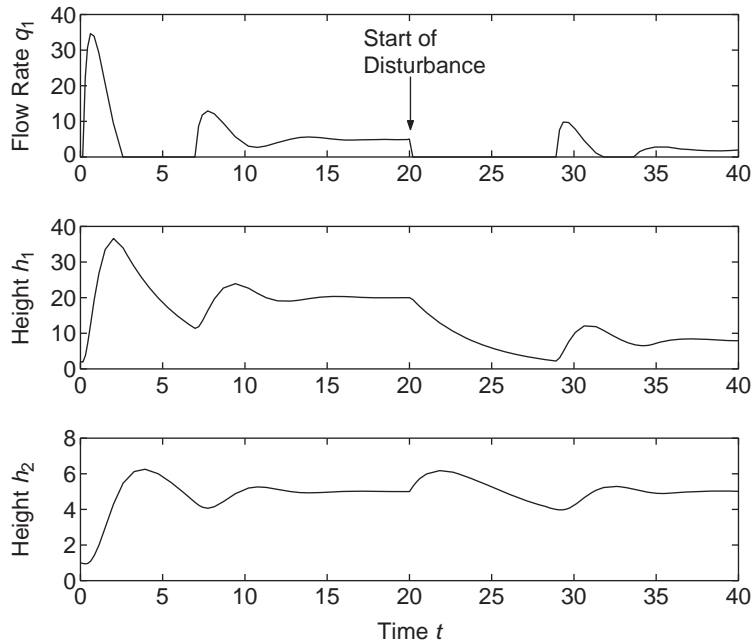
The new block is equivalent to adding the free response to the block output, with all the block's state variables set to zero except for the output variable. The new block also lets you assign an initial value to the block input, but we will not use this feature and so will leave the initial input set to 0 in the Block parameters window.

The steady-state error due to a ramp command for this system was shown to be $e_{ss} = 44/K_I$, so to decrease this error, in our simulation we will use a larger integral gain, say, $K_I = 50$. This makes the system faster and more oscillatory, and this means that the commanded control flow q_1 will more likely be computed to be negative.

The simulation used a step command of magnitude 5 starting at $t = 0$ and a step disturbance of magnitude 3 starting at $t = 20$. The saturation limits were 0 and 100. The initial heights were set to $h_1 = 2$ and $h_2 = 1$.

The results are shown in Figure 10.10.2. Note that sometimes the control flow q_1 is zero. Were it not for the Saturation block, the flow would become negative. The MATLAB M-file used to produce this plot is the following. The plot was then modified with the Plot Editor.

```
subplot(3,1,1),plot(tout,simout(:,1))
subplot(3,1,2),plot(tout,simout(:,2))
subplot(3,1,3),plot(tout,simout(:,3))
```

Current Saturation in a Motor Control System | **EXAMPLE 10.10.2**

■ **Problem**

The block diagram of a proportional control system was given in Figure 10.3.9 and is shown here in modified form as Figure 10.10.3. Create a Simulink model to investigate the effects of current saturation, using the parameter values given in Example 10.8.6 and a step-command input of magnitude 104.7 rad/s (which corresponds to 1000 rpm). Assume that the current is limited to ± 20 A.

Figure 10.10.3 Block diagram of a speed control system.
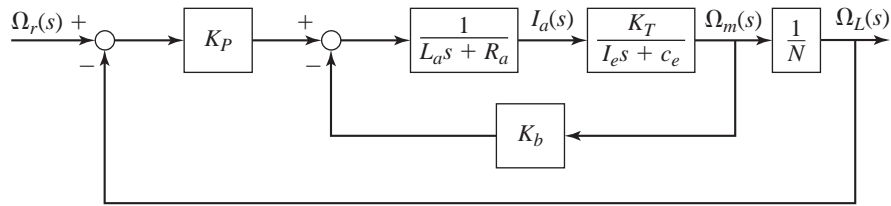
Figure 10.10.4 Simulink model of a speed control system with a current limiter.
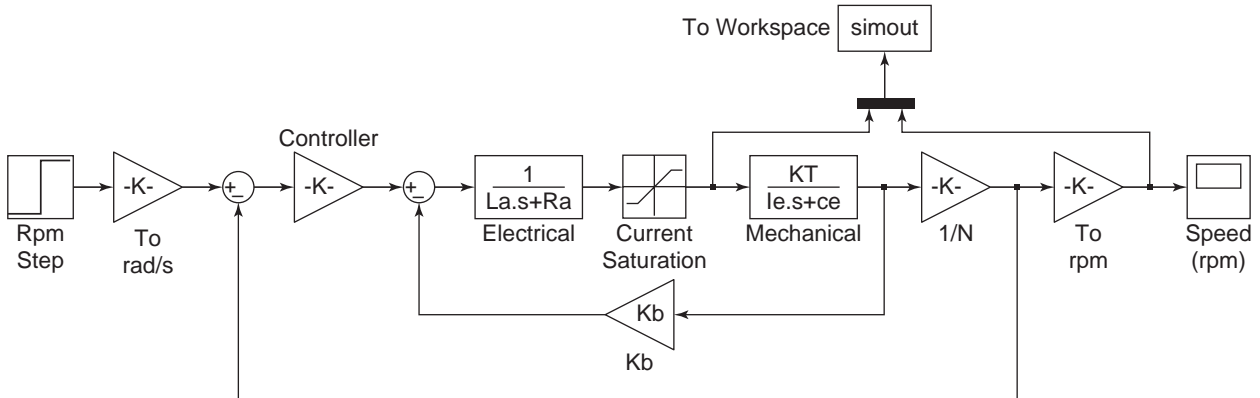
■ Solution

To investigate the effects of current saturation we need to place a Saturation block between the circuit transfer function $1/(L_a s + R_a)$ and the mechanical subsystem that includes the torque constant K_T. The limits on the Saturation block were set to -20 and 20. The result is the Simulink model shown in Figure 10.10.4. Because we are not interested in the load torque T_L here, we have omitted it from the diagram for simplicity. Gains of $2\pi/60$ and $60/2\pi$ are used to convert from rpm to rad/s for the input, and from rad/s to rpm for the output. Set the Save Format to Array in the Block Parameters window of the To Workspace block. The data fed to the To Workspace block are used in MATLAB to create the plots shown in Figure 10.8.11.

We will use variables for the parameters in the simulation. In the Electrical block, set the numerator to 1 and the denominator to [La Ra]. In the Mechanical block, set the numerator to [KT] and the denominator to [Ie ce]. Set the gain to KP in the Controller block. In the block labeled Kb, set the gain to Kb, and in the block labeled 1/N set the gain to 1/N. Before running the simulation, type the following in the Command window to set the values of the parameters.

```
≫KT = 0.04; Kb = KT; Ra = 0.6; La = 2e-3;
≫Ie = 1.802e-3; ce = 4.444e-4;
≫N = 1.5; KP = 0.63;
```

The To Workspace block puts the time variable `tout` and the array `simout` in the MATLAB workspace. The first column of `simout` contains the current, and the second column contains the speed in rpm. To plot the current response, in the Command window type `plot(tout,simout(:,1))`. Type `plot(tout,simout(:,2))` to obtain a plot of the speed. The plots are shown in Figure 10.8.11.

The model in Example 10.10.1 implements proportional control. To examine PID control in general, replace the Controller gain block in Figure 10.10.4 with the PID Controller block. This block enables you to specify the proportional, integral, and derivative gains.

10.11 CHAPTER REVIEW

This chapter introduced the basic concepts of feedback control. It showed how to model control system components and analyze control system performance.

When designing a control system, the control systems engineer is usually given the plant, the actuator, and the physical type of controller (electronic, pneumatic, hydraulic, or digital), and often is expected to develop a model of these components. The command and disturbance inputs might be specified, or the engineer might be expected to develop suitable test inputs based on the application. Step functions are the principal test inputs, because they are the most common and perhaps represent the severest test of system performance. Ramp, trapezoidal, and sinusoidal test inputs are also employed. The type to use should be made clear in the design specifications.

The engineer then proceeds to design the control system. Based on the system model and the performance specifications, a control action is chosen, and the output, error, and actuator equations are derived. These are then analyzed for stability. If the system cannot be made stable with a gain change, a different control action is tried. Using the given command and disturbance input functions (step, ramp, etc.), the steady-state response is evaluated with the final value theorem. Any constraints on the gain values required to satisfy the steady-state specifications are then determined.

The transient performance is then evaluated in light of the transient specifications, using the given input functions. These specifications often are stated in terms of the desired dominant time constant and damping ratio, but they can also be given in terms of overshoot, rise time, settling time, or bandwidth, for example. Other specifications, such as limits on the maximum available actuator output, are evaluated, and the system redesigned if necessary.

Now that you have finished this chapter you should be able to do the following:

1. Model common control system components.
2. Select an appropriate control algorithm of the PID type or one of its variations, for a given application and for given steady-state and transient performance specifications.
3. Analyze the performance of a control algorithm using transfer functions, block diagrams, and computer methods.
4. Compute the gain values to meet the specifications.
5. Use MATLAB and Simulink to analyze and simulate control systems.

REFERENCE

[Cannon, 1967] R. H. Cannon, Jr., *Dynamics of Physical Systems,* McGraw-Hill, New York, 1967.

PROBLEMS

Section 10.1 Closed-Loop Control

10.1 Discuss whether or not the following devices and processes are open-loop or closed-loop. If they are closed-loop, identify the sensing mechanism.

   a. A traffic light.
   b. A washing machine.
   c. A toaster.
   d. Cruise control.
   e. An aircraft autopilot.
   f. Temperature regulation in the human body.

10.2 Draw the block diagram of a system using proportional control and feedforward command compensation, for the plant $1/(4s^2 + 6s + 3)$. Determine the transfer function of the compensator. Discuss any practical limitations to its use.

10.3 Investigate the performance of proportional control using feedforward command compensation with a constant gain K_f and disturbance compensation with a constant gain K_d, applied to the plant $10/s$. Set the gains to achieve a closed-loop time constant of $\tau = 2$ and zero steady-state error for a step command and a step disturbance.

Section 10.2 Control System Terminology

10.4 Derive the output $C(s)$, error $E(s)$, and actuator $M(s)$ equations for the diagram in Figure P10.4, and obtain the characteristic polynomial.

Figure P10.4
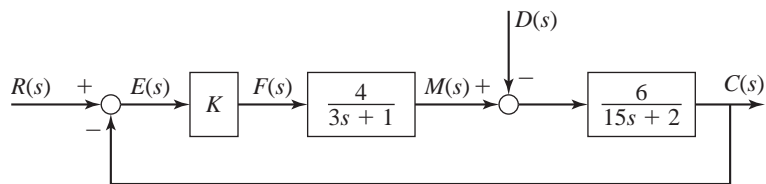
Section 10.3 Modeling Control Systems

10.5 For the system shown in Figure P10.5, the plant time constant is 5 and the nominal value of the actuator time constant is $\tau_a = 0.05$. Investigate the effects of neglecting this time constant as the gain K_P is increased.

Figure P10.5
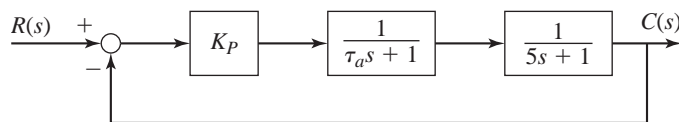
10.6 In Figure P10.6, the block is pulled up the incline by the tension force f in the inextensible cable. The motor torque T is controlled to regulate the speed v of the block to obtain some desired speed v_r. The precise value of the friction coefficient μ is unknown, as is the slope angle α, so we model them as a disturbance. Neglect all masses and inertias in the system except for the block mass m. Also neglect the field time constant of the field-controlled motor. Feedback of the block speed v is provided by a sensor that measures the pulley rotational speed ω, which is directly related to v by $v = R\omega$.

   a. Obtain the equation of motion of the block speed v, with the voltage v_m, friction force $F = \mu mg \cos \alpha$, and the weight component $W_x = mg \sin \alpha$ as the inputs.
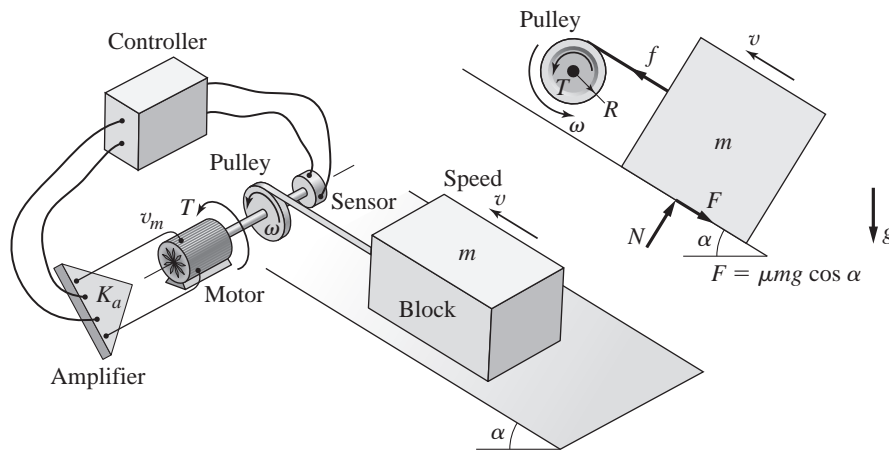
b. Draw a block diagram representing the control system, with the command input v_r, the output v, and the disturbance $D = F + W_x$. Model the speed sensor as directly sensing the speed ω. Show the necessary transfer functions for each block in the diagram.

c. Obtain the output, error, and torque transfer functions from the block diagram.

10.7 The diagram in Figure P10.7 shows a system for controlling the angular position of a load, such as an antenna. There is no disturbance.

a. Draw the block diagram of a system using proportional control, similar to that shown in Figure 10.3.9 except that the command and the output are angular positions. Assume that the motor is armature-controlled and that

$$\theta_2 = \frac{1}{2}\theta_1$$

$$\theta = \frac{1}{3}\theta_2$$
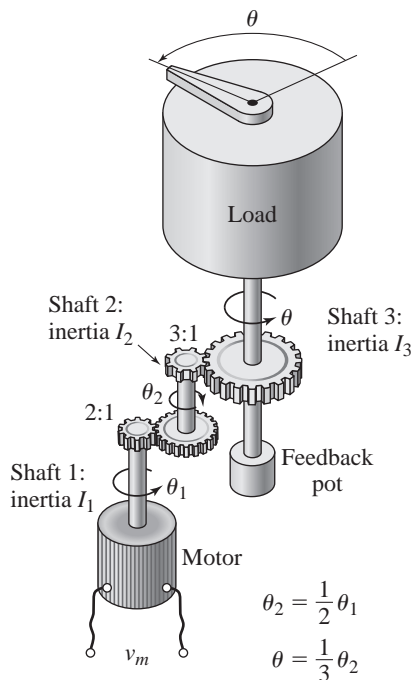
its armature time constant is negligible. Let the inertia I_e be the equivalent inertia of the entire system, as felt on the motor shaft, and let N_e be the equivalent gear ratio of the entire system, as felt on the motor shaft. Show the necessary transfer functions for each block.

b. Determine the value for N_e, and determine I_e as a function of the inertias I_1, I_2, and I_3.

Section 10.4 The PID Control Algorithm

10.8 In the following controller transfer function, identify the values of K_P, K_I, K_P, T_I, and T_D.

$$G_c(s) = \frac{F(s)}{E(s)} = \frac{15s^2 + 6s + 4}{s}$$

10.9 Determine the resistance values required to obtain an op-amp PI controller with $K_P = 4$ and $K_I = 0.08$. Use a 1-μF capacitor.

10.10 a. Determine the resistance values to obtain an op-amp PD controller with $K_P = 2$, $T_D = 2$ s. The circuit should limit frequencies above 5 rad/s. Use a 1-μF capacitor.

   b. Plot the frequency response of the circuit.

10.11 a. Determine the resistance values to obtain an op-amp PID controller with $K_P = 10$, $K_I = 1.4$, and $K_D = 4$. The circuit should limit frequencies above 100 rad/s. Take one capacitance to be 1 μF.

   b. Plot the frequency response of the circuit.

Section 10.5 Control System Analysis

10.12 Obtain the steady-state response, if any, of the following models for the given input. If it is not possible to determine the response, state the reason.

   a. $$\frac{Y(s)}{F(s)} = \frac{6}{7s + 3} \qquad\qquad f(t) = 14u_s(t)$$

   b. $$\frac{Y(s)}{F(s)} = \frac{7s - 3}{10s^2 + 6s + 9} \qquad f(t) = 5u_s(t)$$

   c. $$\frac{Y(s)}{F(s)} = \frac{3s + 5}{s^2 - 9} \qquad\qquad f(t) = 12u_s(t)$$

   d. $$\frac{Y(s)}{F(s)} = \frac{4s + 3}{s^2 + 2s - 7} \qquad f(t) = 8u_s(t)$$

10.13 For the following models, the error signal is defined as $e(t) = r(t) - c(t)$. Obtain the steady-state error, if any, for the given input. If it is not possible to determine the response, state the reason.

   a. $$\frac{C(s)}{R(s)} = \frac{1}{3s + 1} \qquad\qquad r(t) = 6t$$

   b. $$\frac{C(s)}{R(s)} = \frac{5}{3s + 1} \qquad\qquad r(t) = 6t$$

c.
$$\frac{C(s)}{R(s)} = \frac{4}{3s^2 + 5s + 4} \qquad r(t) = 12t$$

d.
$$\frac{C(s)}{R(s)} = \frac{10}{2s^2 + 4s + 5} \qquad f(t) = 8t$$

10.14 Given the model

$$3\ddot{x} - (3b + 6)\dot{x} + (6b + 15)x = 0$$

a. Find the values of the parameter b for which the system is
   1. Stable.
   2. Neutrally stable.
   3. Unstable.
b. For the stable case, for what values of b is the system
   1. Underdamped?
   2. Overdamped?

10.15 A certain system has the characteristic equation $s^3 + 9s^2 + 26s + K = 0$. Find the range of K values for which the system will be stable.

10.16 For the characteristic equation $s^3 + 9s^2 + 26s + K = 0$, use the Routh-Hurwitz criterion to compute the range of K values required so that the dominant time constant is no larger than $1/2$.

10.17 For the following characteristic equations, use the Routh-Hurwitz criterion to determine the range of K values for which the system is stable, where a and b are assumed to be known.
   a. $2s^3 + 2as^2 + Ks + b = 0$
   b. $5s^3 + 5as^2 + bs + 5K = 0$
   c. $4s^3 + 12s^2 + 12s + 4 + K = 0$

10.18 The parameter values for a certain armature-controlled motor, load, and tachometer are

$$K_T = K_b = 0.2 \text{ N} \cdot \text{m/A}$$
$$c_m = 5 \times 10^{-4} \text{ N} \cdot \text{m} \cdot \text{s/rad} \qquad c_L = 2 \times 10^{-3}$$
$$R_a = 0.8 \ \Omega \qquad\qquad\qquad L_a = 4 \times 10^{-3} \text{ H}$$
$$I_m = 5 \times 10^{-4} \qquad\qquad\quad I_t = 10^{-4} \qquad I_L = 5 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$
$$N = 2 \qquad\qquad\qquad\qquad K_a = 10 \text{ V/V}$$
$$K_{\text{tach}} = 20 \text{ V} \cdot \text{s/rad} \qquad K_{\text{pot}} = 10 \text{ V/rad} \qquad K_d = 2 \text{ rad/(rad/s)}$$

For the control system whose block diagram is given by Figure 10.3.9, determine the value of the proportional gain K_P required for the load speed to be within 10% of the desired speed of 2000 rpm at steady state, and use the characteristic roots to evaluate the resulting transient response. For this value of K_P, evaluate the resulting steady-state deviation of the load speed caused by a load torque $T_L = 0.2 \text{ N} \cdot \text{m}$.

Section 10.6 Controlling First-Order Plants

10.19 Suppose the plant shown in Figure 10.6.1 has the parameter values $I = 2$ and $c = 3$. Find the smallest value of the gain K_P required so that the steady-state offset error will be no greater than 0.2 if ω_r is a unit-step input. Evaluate the

resulting time constant and steady-state response due to the disturbance if T_d is also a unit step.

10.20 Suppose the plant shown in Figure 10.6.1 has the parameter values $I = 2$ and $c = 3$. The command input and the disturbance are unit-ramp functions. Evaluate the response of the proportional controller with $K_P = 12$.

10.21 For the control system shown in Figure 10.6.2, $I = 20$, and suppose that only I action is used, so that $K_P = 0$. The performance specifications require the steady-state errors due to step command and disturbance inputs to be zero. Find the required gain value K_I so that $\zeta = 1$. Evaluate the resulting time constant. Do this for each of the following values of c:

  a. $c = 10$
  b. $c = 0.2$

10.22 Suppose that $I = c = 4$ for the PI controller shown in Figure 10.6.2. The performance specifications require that $\tau = 0.2$. (a) Compute the required gain values for each of the following cases.

  1. $\zeta = 0.707$
  2. $\zeta = 1$
  3.   A root separation factor of 10

  (b) Use a computer method to plot the unit-step command responses for each of the cases in part (a). Compare the performance of each case.

10.23 For the designs obtained in part (a) of Problem 10.22, use a computer method to plot the actuator torque versus time. Compare the peak torque values for each case.

10.24 For the designs found in part (a) of Problem 10.22, evaluate the steady-state error due to a unit-ramp command and due to a unit-ramp disturbance.

10.25 Consider the PI speed control system shown in Figure 10.6.2, where $I = c = 2$. The desired time constant is $\tau = 0.1$. (a) Compute the required values of the gains for the following three sets of root locations.

  1. $s = -10, -8$ (root separation factor is $10/8 = 1.25$)
  2. $s = -10, -20$ (root separation factor is 2)
  3. $s = -10, -50$ (root separation factor is 5)

  (b) Use a computer method to plot the response of the speed $\omega(t)$ for a unit-step command for each of the cases in part (a). Discuss the effects of the root separation factor on the rise time, the overshoot, and the maximum required torque.

10.26 Suppose that $I = c = 4$ for the I controller with internal feedback shown in Figure 10.6.6. The performance specifications require that $\tau = 0.2$.
(a) Compute the required gain values for each of the following cases.

  1. $\zeta = 0.707$
  2. $\zeta = 1$
  3.   A root separation factor of 10

  (b) Use a computer method to plot the unit-step command responses for each of the cases in part (a). Compare the performance of each case.

10.27 For the designs obtained in part (a) of Problem 10.26, use a computer method to plot the actuator torque versus time. Compare the peak torque values for each case.

10.28 For the designs found in part (a) of Problem 10.26, evaluate the steady-state error due to a unit-ramp command and due to a unit-ramp disturbance.

10.29 Consider the speed control system using I control with internal feedback shown in Figure 10.6.6, where $I = c = 2$. The desired time constant is $\tau = 0.1$.

    a.   Compute the required values of the gains for the following three sets of root locations.

        1. $s = -10, -8$ (root separation factor is $10/8 = 1.25$)

        2. $s = -10, -20$ (root separation factor is 2)

        3. $s = -10, -50$ (root separation factor is 5)

    b.   Use a computer method to plot the response of the speed $\omega(t)$ for a unit-step command for each of the cases in part (a). Discuss the effects of the root separation factor on the rise time, the overshoot, and the maximum required torque.

10.30 Modify the diagram shown in Figure P10.30 to include feedforward command compensation with a constant compensator gain K_f. Determine whether such compensation can eliminate steady-state error for step and ramp commands.
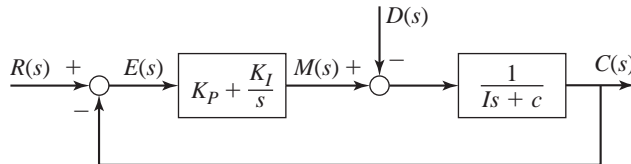
Figure P10.30

Section 10.7 Controlling Second-Order Plants

10.31 Consider the PD control system shown in Figure 10.7.1. Suppose that $I = 20$ and $c = 10$. The specifications require the steady-state error due to a unit-step command to be zero and the steady-state error due to a unit-step disturbance to be no greater than 0.1 in magnitude. In addition, we require that $\zeta = 0.707$.

    Compute the required values of the gains, and evaluate the resulting time constant.

10.32 Suppose that $I = 10$ and $c = 3$ in the PD control system shown in Figure 10.7.1. The performance specifications require that $\tau = 1$ and $\zeta = 0.707$. Compute the required gain values.

10.33 Figure 10.7.2 shows a system using proportional control with velocity feedback. Suppose that $I = 20$ and $c = 10$. The specifications require the steady-state error due to a unit-step command to be zero and the steady-state error due to a unit-step disturbance to be no greater than 0.1 in magnitude. In addition, we require that the time constant be $\tau = 0.1$. Compute the required values of the gains and evaluate the resulting damping ratio.

10.34 For the system discussed in Problem 10.32,

    a.   Use a computer method to plot the output $\theta(t)$ and the actuator response $T(t)$ for a unit-ramp command input.

    b.   Use a computer method to plot the disturbance frequency response. Determine the peak response and the bandwidth.

10.35 Suppose that $I = 10$ and $c = 3$ for the PID control system shown in Figure 10.7.3. The performance specifications require that $\tau = 1$ and $\zeta = 0.707$.

    a.   Compute the required gain values.

    b.   Use a computer method to plot the disturbance frequency response. Determine the peak response and the bandwidth.

10.36 Consider the PD control system shown in Figure 10.7.1. Suppose that $I = 20$ and $c = 10$. The specifications require the steady-state error due to a unit-step command to be zero and the steady-state error due to a unit-step disturbance to be no greater than 0.1 in magnitude. In addition, we require that the time constant be $\tau = 0.1$.

    Compute the required values of the gains, and evaluate the resulting damping ratio.

10.37 Modify the PD system diagram shown in Figure 10.7.1 to include feedforward compensation with a compensator gain of K_f. Determine whether such compensation can reduce the steady-state error for step and ramp commands.

10.38 Consider a plant whose transfer function is $1/(20s + 0.2)$. The performance specifications are

    1.   The magnitude of the steady-state command error must be no more than 0.01 for a unit-ramp command.

    2.   The damping ratio must be unity.

    3.   The dominant time constant must be no greater than 0.1.

    a.   Select a control algorithm to meet the first specification, and compute the required values of its gains. What is the damping ratio that results? What is the time constant?

    b.   Select a control algorithm to meet the first two specifications, and compute the required values of its gains. Evaluate the resulting time constant.

    c.   Select a control algorithm to meet all three specifications.

10.39 For the system shown in Figure 10.7.1, $I = c = 1$. Derive the expressions for the steady-state errors due to a unit-ramp command and to a unit-ramp disturbance.

10.40 For the PD control system shown in Figure 10.7.1, $I = c = 2$. Compute the values of the gains K_P and K_D to meet all of the following specifications:

    1.   No steady-state error with a step input

    2.   A damping ratio of 0.9

    3.   A dominant time constant of 1

10.41 Consider the PID position control system shown in Figure 10.7.3, where $I = 10$ and $c = 2$. The desired time constant is $\tau = 2$.

    a.   Compute the required values of the gains for the following two sets of root locations.

        1. $s = -0.5$, $s = -5 \pm 5j$

        2. $s = -0.5$, $s = -1$, $s = -2$.

    b.   For both cases, use a computer method to plot the command response to a unit step. Discuss the effects of the root separation factor on the response. Compare the results with those of Example 10.7.4, where the roots are $s = -5$ and $s = -0.5 \pm 0.5j$.

    c.   For both cases, use a computer method to plot the disturbance frequency response. Discuss the effects of the root separation factor on the frequency response. Compare the results with those of Example 10.7.4.

10.42 Derive the expression for $T(s)$ in Figure 10.7.6. Using the values given and computed in Example 10.7.5, use MATLAB to plot $T(t)$ for a unit-step command input. Determine the maximum value of $T(t)$.

10.43 Integral control of the plant

$$G_p(s) = \frac{3}{5s + 1}$$

results in a system that is too oscillatory. Will D action improve this situation?

10.44 Modify the system diagram shown in Figure P10.44 to include feedforward compensation with a compensator gain K_f. Determine whether such compensation can reduce the steady-state error for step and ramp commands.
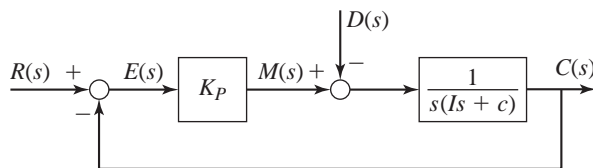
Section 10.8 Additional Examples

10.45 We need to stabilize the plant $3/(s^2 - 4)$ with a feedback controller. The closed-loop system should have a damping ratio of $\zeta = 0.707$ and a dominant time constant $\tau = 0.1$.
  a. Use PD control and compute the required values of the gains.
  b. Use P control with rate feedback and compute the required values of the gains.
  c. Compare the unit-step command responses of the two designs.

10.46 The system shown in Figure P10.46 represents the problem of stabilizing the attitude of a rocket during takeoff or controlling the balance of a personal transporter. The applied force f represents that from the side thrusters of the rocket or the tangential force on the transporter wheels. For small angles, Newton's law for the system reduces to

$$ML\ddot{\theta} - (M + m)g\theta = f$$

where f is the control variable. Design a control law to maintain θ near zero. The specifications are $\zeta = 0.707$ and a 2% settling time of 8 sec. The parameter values are $M = 40$ slugs, $m = 8$ slugs, $L = 20$ ft, and $g = 32.2$ ft/sec^2.

10.47 Figure P10.47 shows PD control applied to an unstable plant. The gains have been computed so that the damping ratio is $\zeta = 0.707$ and the time constant is 2.5 sec, assuming that the transfer functions of the actuator and the feedback sensor are unity. Suppose that the actuator has the transfer function

$$G_a(s) = \frac{1}{\tau s + 1}$$

This lag in the response of the actuator might affect the system's stability, its overshoot in response to a step input, or its 2% settling time.

Figure P10.47
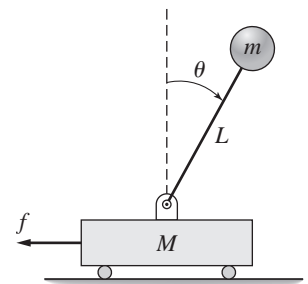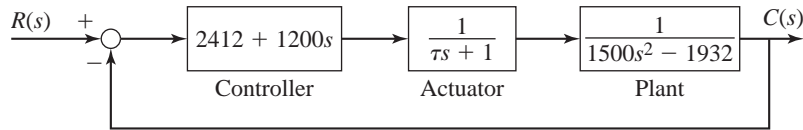
a.   What effect does this lag have on the system's performance if $\tau = 0.1$ sec?

b.   What effect does this lag have on the system's performance if $\tau = 1$ sec?

10.48 Figure P10.48 shows PD control applied to an unstable plant. The gains have been computed so that the damping ratio is $\zeta = 0.707$ and the time constant is 2.5 sec, assuming that the transfer functions of the actuator and the feedback sensor are unity. Suppose that the feedback sensor has the transfer function

$$G_s(s) = \frac{1}{\tau s + 1}$$

This lag in the response of the feedback elements might affect the system's stability, its overshoot in response to a step input, or its 2% settling time.

a.   What effect does this lag have on the system's performance if $\tau = 0.1$ sec?

b.   What effect does this have on the system's performance if $\tau = 1$ sec?

Figure P10.48
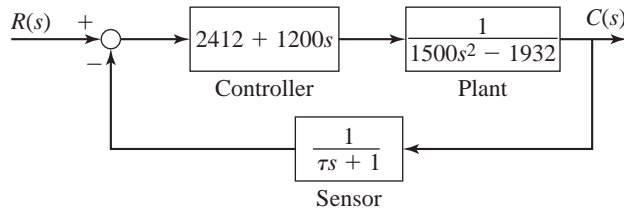
10.49 Figure P10.49 shows a proposed scheme for controlling the position of a mechanical system such as a link in a robot arm. It uses two feedback loops—one for position and one for velocity—and a feedforward compensator transfer function s^2.

a.   Suppose that the estimates of the mass, damping, and stiffness are accurate so that $m_e = m$, $c_e = c$, and $k_e = k$. Derive the expressions for the transfer functions $X(s)/X_r(s)$ and $X(s)/F_d(s)$. How well does this control scheme work? Obtain the expressions for the PID gains to achieve a damping ratio of $\zeta = 1$ and a closed-loop time constant of specified value τ_d.

b.   Discuss any practical limitations to this scheme.

10.50 Refer to Figure 10.3.9, which shows a speed control system using an armature-controlled dc motor. The motor has the following parameter values.

$K_b = 0.199$ V-sec/rad $R_a = 0.43 \ \Omega$

$K_T = 0.14$ lb-ft/A $c_e = 3.6 \times 10^{-4}$ lb-ft-sec/rad

$I_e = 2.08 \times 10^{-3}$ slug-ft^2 $L_a = 2.1 \times 10^{-3}$ H

$N = 1$

a.   Compute the time constants of the plant transfer function $\Omega_L(s)/V_m(s)$.

b.   Modify Figure 10.3.9 to use PI control instead of P control. Compute the PI control gains required to give a response having a dominant time constant of no less than 0.05 sec and a dominant damping ratio in the range $0.5 \leq \zeta \leq 1$.

Figure P10.49

Control algorithm
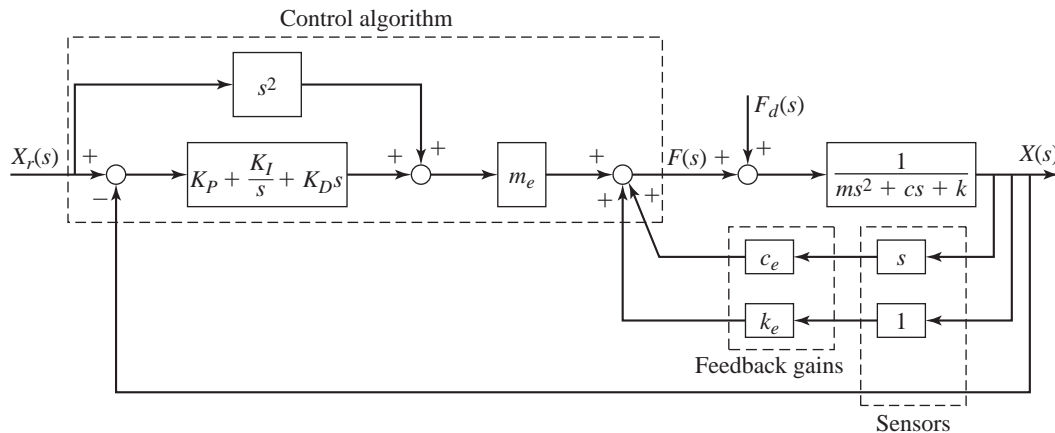
Section 10.9 MATLAB Applications

10.51 Using the value of K_P computed in Problem 10.18, obtain a plot of the current versus time for a step-command input of 209.4 rad/s (2000 rpm).

10.52 Consider Example 10.6.3. Modify the diagram in Figure 10.6.2 to show an actuator transfer function $T(s)/M(s) = 1/(0.1s + 1)$. Use the same gain values computed for the three cases in that example.

    a. Use MATLAB to plot the command response and the actuator response to a unit-step command. Identify the peak actuator values for each case.

    b. Use MATLAB to plot the disturbance frequency response.

    c. Compare the results in parts (a) and (b) with those of Example 10.6.3.

10.53 Consider Example 10.6.3. Use the same gain values computed for the three cases in that example.

    a. Use MATLAB to plot the command response and the actuator response to the modified unit-step command $r(t) = 1 - e^{-20t}$. Identify the peak actuator values for each case.

    b. Compare the results in part (a) with those of Example 10.6.3.

10.54 Consider Example 10.6.4. Modify the diagram in Figure 10.6.6 to show an actuator transfer function $T(s)/M(s) = 1/(0.1s + 1)$. Use the same gain values computed for the three cases in that example.

    a. Use MATLAB to plot the command response and the actuator response to a unit-step command. Identify the peak actuator values for each case.

    b. Use MATLAB to plot the disturbance frequency response.

    c. Compare the results in parts (a) and (b) with those of Example 10.6.4.

10.55 Figure P10.7 shows a system for controlling the angular position of a load, such as an antenna. Figure P10.55 shows the block diagram for PD control of this system using a field-controlled motor. Use the following values:

$$K_a = 1 \ \text{V/V} \qquad R = 0.3 \ \Omega \qquad K_T = 0.6 \ \text{N} \cdot \text{m/A}$$
$$K_{\text{pot}} = 2 \ \text{V/rad} \qquad\qquad I_1 = 0.01 \ \text{kg} \cdot \text{m}^2$$
$$I_2 = 5 \times 10^{-4} \ \text{kg} \cdot \text{m}^2 \qquad I_3 = 0.2 \ \text{kg} \cdot \text{m}^2$$

The inertia I_e in the block diagram is the equivalent inertia of the entire system, as felt on the *motor* shaft.

Figure P10.55
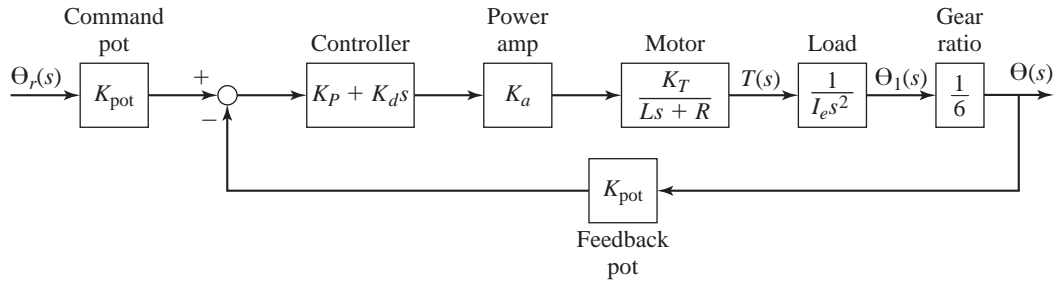
a. Assume that the motor inductance is very small and set $L = 0$. Compute I_e, obtain the transfer function $\Theta(s)/\Theta_r(s)$, and compute the values of the control gains K_P and K_d to meet the following specifications: $\zeta = 1$ and $\tau = 0.5$ s.

b. Use the MATLAB `tf` function to create the model `sys1` from this transfer function.

c. Using the values of K_P and K_d computed in part (a), and the value $L = 0.015$ H, obtain the transfer function $\Theta(s)/\Theta_r(s)$ and use the MATLAB `tf` function to create the model `sys2` from this transfer function.

d. Use the MATLAB `step(sys1,sys2)` function to plot the unit step response of both transfer functions. Right-click on the plots to obtain the maximum percent overshoot and settling time for each. How close are the two responses? What is the effect of neglecting the inductance?

10.56 Consider the P, PI, and modified I control systems discussed in Examples 10.6.2, 10.6.3, and 10.6.4. The plant transfer function is $1/(Is + c)$, where $I = 10$ and $c = 3$. Investigate the performance of these systems for a trapezoidal command input having a slew speed of 1 rad/s, an acceleration time of 4 s, a slew time of 6 s, a deceleration time of 4 s, and a rest time of 5 s.

10.57 A speed control system using an armature-controlled motor with proportional control action was discussed in Section 10.3. Its block diagram is shown in Figure 10.3.8 with a simplified version given in Figure 10.3.9. The given parameter values for a certain motor, load, and tachometer are

$$K_T = K_b = 0.04 \text{ N} \cdot \text{m/A}$$

$c_m = 0$	$c_L = 10^{-3} \text{ N} \cdot \text{m} \cdot \text{s/rad}$	
$R_a = 0.6 \ \Omega$	$L_a = 2 \times 10^{-3} \text{ H}$	
$I_m = 2 \times 10^{-5}$	$I_t = 10^{-5}$	$I_L = 4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$
$N = 1.5$	$K_a = 5 \text{ V/V}$	
$K_{\text{tach}} = 10 \text{ V/(rad/s)}$	$K_{\text{pot}} = 5 \text{ V/rad}$	$K_d = 2 \text{ rad/(rad/s)}$

where the subscript m refers to the motor, L refers to the load, and t refers to the tachometer. (a) Determine the value of the proportional gain K_P required for the load speed to be within 10% of the desired speed of 1000 rpm at steady-state, and plot the resulting transient response. (b) For this value of K_P, plot the resulting deviation of the load speed caused by a load torque $T_L = 1 \text{ N} \cdot \text{m}$.

10.58 Consider the control system of Problem 10.57. Use MATLAB to evaluate the following performance measures: energy consumption, maximum current, maximum speed error, rms current, and rms speed error.

Section 10.10 Simulink Applications

10.59 Consider Example 10.6.3. Use the diagram in Figure 10.6.2 to create a Simulink model. Modify the model to use an actuator saturation with the limits 0 and 20. Use the same gain values computed in that example for the three cases.
   a. Plot the command response and the actuator response to a unit-step command.
   b. Compare the results in part (a) with those of Example 10.6.3.

10.60 Consider Example 10.7.4. Use the diagram in Figure 10.7.3 to create a Simulink model using the same gain values computed in that example. Set the initial position to 3. Plot the command response to a unit-step command and compare the results with those of Example 10.7.4.

10.61 Consider Example 10.7.4. Use the diagram in Figure 10.7.3 to create a Simulink model. Modify the model to use an actuator saturation with the limits 0 and 20. Use the same gain values computed in that example.
   a. Plot the command response and the actuator response to a unit-step command.
   b. Compare the results in part (a) with those of Example 10.7.4.

10.62 Consider Example 10.7.4. Use the diagram in Figure 10.7.3 to create a Simulink model. Modify the model to use an actuator transfer function $G_a(s) = 1/(0.2s + 1)$. Use the same gain values computed in that example.
   a. Plot the command response and the actuator response to a unit-step command.
   b. Compare the command response with that of Example 10.7.4.

10.63 Refer to Figure 10.3.9, which shows a speed control system using an armature-controlled dc motor. The motor has the following parameter values. Create a Simulink model by modifying Figure 10.3.9 to use PI control instead of P control. Use the PI control gains computed in Problem 10.50 part (b).

$$K_b = 0.199 \text{ V-sec/rad} \qquad R_a = 0.43 \ \Omega$$
$$K_T = 0.14 \text{ lb-ft/A} \qquad c_e = 3.6 \times 10^{-4} \text{ lb-ft-sec/rad}$$
$$I_e = 2.08 \times 10^{-3} \text{ slug-ft}^2 \qquad L_a = 2.1 \times 10^{-3} \text{ H}$$
$$N = 1$$

   a. Run the simulation using a unit-step command starting at $t = 0$ and a unit-step disturbance starting at $t = 4$ sec. Plot the speed and the motor current versus time.
   b. Run the simulation for $0 \le t \le 2$ sec using a unit-ramp command. Plot the speed error and the motor current versus time.

10.64 For the system in Problem 10.63 part (a), create a Simulink model that has a current limiter of ± 10 A. Run the simulation for a step-command input of 104.7 rad/s (1000 rpm). Plot the current and the speed.

Control System Design and the Root Locus Plot

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Sketch the root locus plot for lower-order models, and use MATLAB to obtain the plot for higher-order models.

2. Interpret and use the root locus plot to determine the location of dominant roots and roots having desired properties such as damping ratio and time constant.

3. Determine the major features of a root locus plot, and use it to assess the effectiveness of a proposed control scheme.

4. Design a controller incorporating state-variable feedback.

5. Use the Ziegler-Nichols methods to design controllers.

6. Design a control system to avoid actuator saturation.

7. Apply MATLAB and Simulink to analyze and design control systems using the concepts presented in this chapter.

Chapter 10 introduced the basic concepts of feedback control. It showed how to choose an appropriate control action for a first- or second-order plant, and how to compute the control gains required to meet a simple set of performance specifications. This chapter shows how the root locus plot can be used to develop a more systematic approach to designing a control system. Such an approach is usually needed when the plant order is greater than two or where it is not clear how to select the gains to meet the performance specifications.

Section 11.1 introduces the root locus plot. Section 11.2 illustrates this approach using the root locus plot to design PID controllers. When PID control action fails to yield the desired performance, it must be either modified or replaced by an entirely different control scheme. Inserting an additional control element in one or more feedback loops within the main controller is called *feedback compensation*.

In Chapter 10 we illustrated the use of P action with rate feedback to replace PD control. This eliminated the numerator dynamics and the resulting overshoot. *State-variable feedback* is a generalization of that technique and uses some or all of the system's state variables to modify the control signal. With state-variable feedback, we have a better chance of placing the characteristic roots of the closed-loop system in locations that will give desirable performance. This topic is treated in Section 11.3.

In many applications, especially in process control involving thermodynamic, fluid, or chemical processes, a transfer function model of the plant is not available, and the gains must be computed from experimentally determined response data or from computer simulations of the plant. Often the gain values computed in a preliminary analysis do not yield the desired performance, and their values must be adjusted either in simulation or with the actual controller hardware. This process is called *tuning*. Tuning is discussed in Section 11.4.

Assumptions made to obtain a linear model can have a significant effect on the controller's performance. Using high gain values tends to drive the control elements to such an extent that they overload or "saturate" and thus exhibit nonlinear behavior. Design concepts that avoid these unwanted effects are covered in Section 11.5.

Root locus plots for simple systems can be sketched by hand with the aid of a calculator. If these plots are to be used for higher-order systems, however, the use of computer methods is highly recommended. The MATLAB `rlocus`, `rltool`, and `sisotool` functions make it easy to view the root locus and transient response plots to see the effect of changing parameter values. These methods are discussed in Section 11.6.

Simulink provides a quick and easy way of simulating systems having nonlinear or discontinuous elements such as the saturation nonlinearity. The Simulink features relevant to the topics of this chapter are summarized in Section 11.7.

Section 11.8 provides a review of the chapter's main concepts. ■

11.1 ROOT LOCUS PLOTS

As shown in Figure 11.1.1, a graphical display of the characteristic root locations gives insight into the system response. A *root locus plot* is a plot of the location of the characteristic roots as a parameter value is varied. Such a plot gives an overview of how the response will change if the parameter value is changed. Root locus plots were used widely in engineering design well before digital computers became available, but the usefulness of the root locus has been enhanced by programs such as MATLAB, which can quickly generate the plots. In this section, we will present some simple second-order system plots that can be sketched by hand to illustrate the concept. This section shows how to sketch plots for higher-order systems. In Section 11.6 we will show how to use MATLAB to generate the plots.

VARYING THE SPRING CONSTANT

In this section we will consider several examples where the characteristic equation has the form

$$ms^2 + cs + k = 0 \qquad\qquad (11.1.1)$$

Figure 11.1.1 Graphical interpretation of the parameters ζ, τ, ω_n, and ω_d.

$$\zeta = \cos\theta = \cos\left[\tan^{-1}(\tau\omega_d)\right]$$
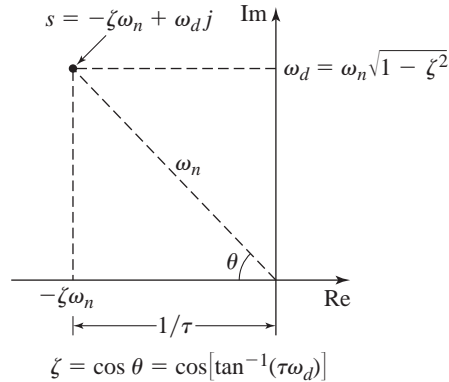
Figure 11.1.2 Plot of the roots of the equation $2s^2 + 8s + k = 0$ as k varies through positive values.
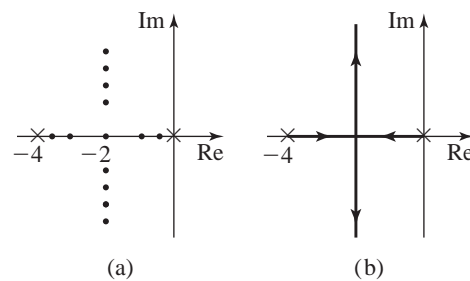
We will consider, in order, the effects of varying first the spring constant k, then the damping constant c, and finally the mass m.

Suppose that $m = 2$ and $c = 8$ and that we wish to display the root locations as k varies. In this case, the characteristic equation becomes $2s^2 + 8s + k = 0$. The roots are found from the quadratic formula

$$s = \frac{-8 \pm \sqrt{64 - 8k}}{4} \tag{11.1.2}$$

It is easily seen that if $k < 8$, the roots are real and distinct. They are repeated if $k = 8$ and complex conjugates if $k > 8$. By repeatedly evaluating (11.1.2) for various values of $k \geq 0$, the root locations can be plotted with dots in the s plane as in Figure 11.1.2a. Noting the general trend of the dots, we can connect them with solid lines to produce the plot in part (b) of the figure. The spacing of the dots is uneven but corresponds to an even spacing in the k values. By convention, the root locations corresponding to $k = 0$ are denoted by a cross ($\times$). These locations are $s = 0$ and $s = -4$. The arrows on the plot indicate the direction of root movement as k increases.

The plot shows that if $k \geq 0$, we cannot achieve a dominant time constant any smaller than $1/2$ by changing k. If $k < 8$, the dominant root lies between $s = 0$ and $s = -2$, and thus the dominant time constant is never less than $1/2$. If $k \geq 8$, the time constant is always $1/2$. This illustrates the type of insight that can be obtained from the root locus plot.

The root locus plot gives us a picture of the roots' behavior as one parameter is varied. It thus enables us to obtain a more general understanding of how the system's response changes as a result of a change in that parameter. The plot is useful in system design where we need to select a value of the parameter to obtain a desired response.

VARYING THE DAMPING CONSTANT

Suppose that $m = 2$ and $k = 8$, and that we wish to display the root locations as c varies. In this case, the characteristic equation becomes $2s^2 + cs + 8 = 0$. The roots are found from the quadratic formula

$$s = \frac{-c \pm \sqrt{c^2 - 64}}{4} \tag{11.1.3}$$

When $c = 0$, the roots are $s = \pm j2$, and we mark these locations on the plot with a ×. By evaluating and plotting the roots for many values of c, and connecting the points with a solid line, we obtain the plot shown in Figure 11.1.3. We find that as c is increased through large values, one root moves to the left, while the other root gradually approaches the origin $s = 0$. This can be proved analytically by taking the limit of (11.1.3) as $c \to \infty$; we find that one root approaches 0 and the other root approaches $s = -\infty$. By convention, the root location corresponding to an infinite value of the parameter is denoted by a circle ($\bigcirc$).

When $c = 8$, both roots are $s = -2$. Recall that ω_n is the radius of a circle centered at the origin. From the plot it is easily seen that for $c < 8$, the roots are complex and that $\omega_n = 2$ (because the plot is a circle of radius 2 centered at the origin). For $c > 8$, the roots are real and distinct, and the plot shows that the dominant root is always no less than -2, and thus the dominant time constant is always $\geq 1/2$. Thus, as long as $c \geq 0$, we cannot reduce the dominant time constant below $1/2$ by changing c.

In this example, the characteristic equation can be expressed as

$$s^2 + 4 + \frac{c}{2}s = 0$$

which is a special case of the more general form in terms of the variable parameter μ.

$$s^2 + \beta s + \gamma + \mu(s + \alpha) = 0 \tag{11.1.4}$$

where $\beta = 0$, $\gamma = 4$, $\alpha = 0$, and $\mu = c/2$. When $\mu = 0$, $s^2 + \beta s + \gamma = 0$, and thus the starting points of the root locus, denoted by crosses (×), are given by

$$s = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

These can be real or complex numbers, depending on the values of β and γ. It can be shown that as $\mu \to \infty$, one root of (11.1.4) approaches $s = -\alpha$ and the other root approaches $s = -\infty$.

A simple geometric analysis will show that off the real axis the root locus of (11.1.4) in terms of the variable parameter μ is a circle centered at $s = -\alpha$ and having a

(a)

(b)
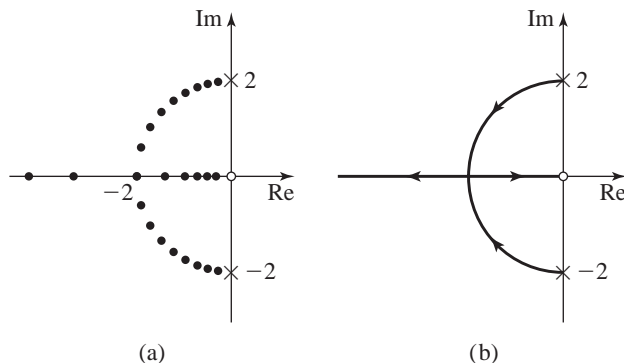
Figure 11.1.3 Plot of the roots of the equation $2s^2 + cs + 8 = 0$ as c varies through positive values.

radius of

$$R = \sqrt{\alpha^2 + \gamma - \alpha\beta} \qquad (11.1.5)$$

To prove this, substitute $s = x + jy$ into (11.1.4), and separate the real and imaginary parts to obtain

$$x^2 - y^2 + \beta x + \gamma + \mu(x + \alpha) + jy(2x + \beta + \mu) = 0 + j0$$

Thus, for the real part,

$$x^2 - y^2 + \beta x + \gamma + \mu(x + \alpha) = 0$$

and for the imaginary part,

$$y(2x + \beta + \mu) = 0$$

Since $y \neq 0$ in general, the second equation gives $2x + \beta + \mu = 0$, or $\mu = -2x - \beta$. When this is substituted into the first equation we obtain

$$(x + \alpha)^2 + y^2 = \alpha^2 + \gamma - \alpha\beta \qquad (11.1.6)$$

which is the equation of the circle just described.

EXAMPLE 11.1.1

Root Locus of a Motor Model

■ Problem
Sketch the root locus of the armature-controlled dc motor model in terms of the damping constant c, and evaluate the effect on the motor time constant. The characteristic equation is

$$L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T = 0$$

Use the following parameter values:

$$K_b = K_T = 0.05 \text{ N} \cdot \text{m/A} \qquad I = 9 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

$$R_a = 0.5 \ \Omega \qquad\qquad L_a = 2 \times 10^{-3} \text{ H}$$

■ Solution
Substituting the given values gives

$$1.8 \times 10^{-7}s^2 + (4.5 \times 10^{-5} + 2 \times 10^{-3}c)s + 0.5c + 2.5 \times 10^{-3} = 0$$

Dividing by the highest coefficient and factoring out c gives the standard form (11.1.4):

$$s^2 + 250s + \frac{5 \times 10^5}{36} + \frac{10^5 c}{9}(s + 250) = 0$$

where $\alpha = \beta = 250$, $\gamma = (5/36) \times 10^5$, and $\mu = 10^5 c/9$. From the results of (11.1.5) and (11.1.6) we see that the root locus is a circle centered at $s = -250$ with a radius of 118. See Figure 11.1.4. The starting points with $c = 0$ are at $s = -83$ and $s = -167$. As $c \to \infty$ one root approaches $s = -250$ and the other root approaches $s = -\infty$.

The plot illustrates the sometimes counterintuitive behavior of dynamic systems. One would think that increasing the damping c would increase the time constant and thus slow the response. However, as c is increased up to the value where the two roots meet at $s = -368$, the dominant time constant decreases, and we see that the smallest possible dominant time constant is $\tau = 1/368$.
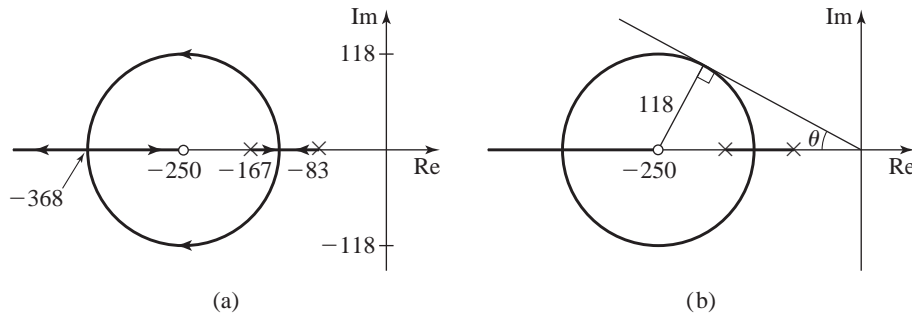
Figure 11.1.4 (a) Root locus plot of the motor model discussed in Example 11.1.1. (b) Line of minimum damping ratio.

It is instructive to see how this plot can be used to determine the minimum damping ratio the motor can have. The line corresponding to the minimum damping ratio is tangent to the circle, as shown in Figure 11.1.4b. From the triangle shown, we see that

$$\theta = \sin^{-1} \frac{118}{250} = 0.49 \text{ rad}$$

Thus the minimum damping ratio is $\zeta = \cos \theta = 0.88$.

VARYING THE MASS

Now suppose that $c = 8$ and $k = 6$ and we want to investigate the effects of varying the mass m. The characteristic equation becomes $ms^2 + 8s + 6 = 0$, and the quadratic formula gives

$$s = \frac{-8 \pm \sqrt{64 - 24m}}{2m}$$

We immediately see that a problem arises if we try to examine the effects of varying the mass starting with $m = 0$, because m is in the denominator. In fact, if $m = 0$, the equation is no longer second order and the quadratic formula no longer applies. So we conclude that we must be careful when varying the leading coefficient in the characteristic equation, because the equation order will change.

Instead, suppose we examine the effects of varying the mass for $m \geq 2$. For this case the equation always remains second order. When $m = 2$, the two roots are $s = -1$ and $s = -3$, and these starting points are marked in Figure 11.1.5. As $m \to \infty$, the quadratic formula shows that both roots approach $s = 0$, and we mark this point with a small circle. By repeated evaluation of the quadratic formula for different m values, we obtain the plot shown in Figure 11.1.5. The locus off the real axis is a circle.

From the plot we see that as m is increased from $m = 2$, the two roots are real and approach one another, meeting at $s = -1.5$ when $m = 8/3$. For $m > 8/3$, the roots
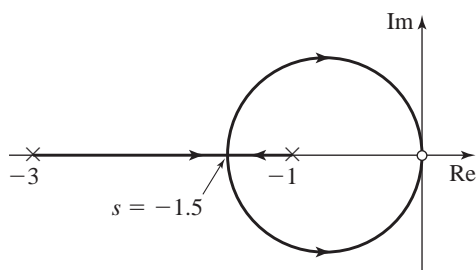
Figure 11.1.5 Plot of the roots of the equation $ms^2 + 8s + 6 = 0$ for $m \geq 2$.

become complex and move to the right. Thus the smallest dominant time constant the model can have is $\tau = 1/1.5 = 2/3$ and the largest possible undamped natural frequency is $\omega_n = 1.5$, which occurs when $m = 8/3$.

Let μ be the relative deviation of m from its nominal value of 2, that is let $\mu = (m - 2)/2$. Make the substitution $m = 2\mu + 2$ to obtain $(2\mu + 2)s^2 + 8s + 6 = 0$ or

$$s^2 + 4s + 3 + \mu s^2 = 0$$

where $\mu \geq 0$. This equation is a special case of the more general form

$$s^2 + \beta s + \gamma + \mu(s + \alpha)(s + \delta) = 0 \tag{11.1.7}$$

The root locus properties of this form can be developed as done with (11.1.4).

The examples in this section were second-order equations, for which a closed-form solution for the roots is available. For higher-order equations, however, a computer method is needed to obtain the root locus plot, and a MATLAB method is presented in Section 11.6.

11.2 DESIGN USING THE ROOT LOCUS PLOT

We now illustrate how to use the root locus plot to design control systems. The root locus plot is a plot of the location of the characteristic roots in terms of some system parameter, such as the proportional gain. Each path on the plot corresponds to one root of the characteristic equation. These paths are referred to as the *loci* or *branches*.

Although formulas exist for computing the roots of third- and fourth-order polynomials, their complexity is such that they do not generate much insight into the system's behavior. The Routh-Hurwitz criterion is useful for stability analysis, but it cannot give a complete picture of the transient response. The theory of polynomial equations, however, is sufficiently well developed to enable us to develop guides for sketching the general behavior of the roots without actually solving for them in many cases. These guides are useful and often even sufficient to make design decisions, and they also should be kept in mind when checking and interpreting a computer-generated plot.

The root locus design method is widely used, but when it was first proposed by Walter Evans in the 1940s [Evans, 1948] however, his paper was rejected for publication because the reviewers thought it not useful enough to merit publication![1] For aeronautical applications, see [Abzug, 1997].

TERMINOLOGY

The general form of an equation whose roots can be studied by the root locus method is

$$D(s) + KN(s) = 0 \tag{11.2.1}$$

where K is the parameter to be varied, and $D(s)$ and $N(s)$ are polynomials in s with constant coefficients. We consider the case $K \geq 0$ and later extend the results to $K \leq 0$. The guides to be developed for plotting the root locus require that the coefficients of the highest powers of s in both $N(s)$ and $D(s)$ are normalized to unity. The multipliers required to do this are absorbed into the parameter to be varied. The result is K.

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[1] The author is grateful to Professors Thomas Kurfess and Mark Nagurka for this information.

For example, consider the variation of the parameter a in the equation $6s^2 + 13as + 5 = 0$. This can be written as

$$s^2 + \frac{5}{6} + \frac{13a}{6}s = 0$$

From (11.2.1), we see that $D(s) = s^2 + 5/6$, $N(s) = s$, and $K = 13a/6$. The root locus plot would be made in terms of the parameter K and the values for a recovered from $a = 6K/13$.

Another standard form of the problem is obtained by rewriting (11.2.1) as

$$1 + KP(s) = 0 \tag{11.2.2}$$

where

$$P(s) = \frac{N(s)}{D(s)} \tag{11.2.3}$$

The roots of $N(s) = 0$ are called the *zeros* of the problem. The name refers to the fact that they are the finite values of s that make $P(s)$ zero. The roots of $D(s) = 0$ are the *poles*. They are the finite values of s that make $P(s)$ become infinite. We see that when $K = 0$, the roots of (11.2.1) are the poles. To see what happens as $K \to \infty$, write (11.2.2) as

$$\frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

When $K \to \infty$, $1/K \to 0$, and the equation becomes $N(s)/D(s) = 0$. This is equivalent to $N(s) = 0$, because we assume there are no poles at infinity. This shows that the roots of (11.2.1) approach the zeros as $K \to \infty$.

A single-loop control system is shown in Figure 11.2.1. The transfer function is

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{11.2.4}$$

The *open-loop transfer function* of this system is $G(s)H(s)$ and so named because it is the transfer function relating the feedback signal $B(s)$ to the input $R(s)$ if the loop is "opened" or broken at $B(s)$. That is, $B(s) = G(s)H(s)R(s)$. The *closed-loop transfer function* is $T(s)$ given by (11.2.4).

The root locus form of the characteristic equation for a single-loop system is easily obtained from the equation $1 + G(s)H(s) = 0$ if the variable parameter of interest is a multiplicative factor in $G(s)H(s)$. The poles and zeros of the problem are also the poles and zeros of $G(s)H(s)$. Thus, the terms *open-loop poles* and *open-loop zeros* are often used for the poles and zeros on the root locus plot. The *closed-loop poles* are the characteristic roots, because they are the finite values of s that make the closed-loop transfer function become infinite.
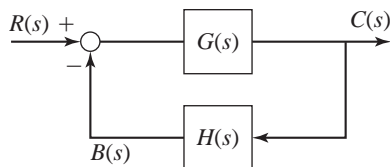
Figure 11.2.1 A single-loop control system.

EXAMPLE 11.2.1   Obtaining the Root Locus Equation

■ Problem

Obtain the root locus equations for the systems shown in Figure 11.2.2, where the variable parameter of interest is the proportional gain K_P. In parts (b) and (c) assume that T_I and T_D are known constants.

Figure 11.2.2 Block diagrams for Example 11.2.1.

(a)

(b)

(c)
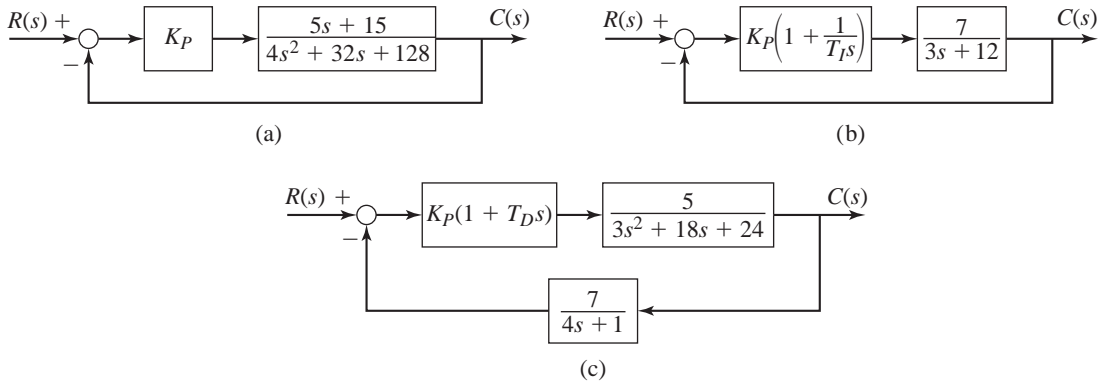
■ Solution

a.   From the diagram in part (a) of Figure 11.2.2,

$$1 + G(s)H(s) = 1 + \frac{K_P(5s + 15)}{4s^2 + 32s + 128} = 0$$

Factor out $5K_P$ from the numerator and 4 from the denominator so that the highest coefficients will be 1. This gives

$$1 + \frac{5K_P}{4}\frac{s + 3}{s^2 + 8s + 32} = 0$$

So the root locus parameter K is $K = 5K_P/4$. The zero is $s = -3$ and the poles are $s = -4 \pm 4j$.

b.   From the diagram in part (b) of Figure 11.2.2,

$$1 + G(s)H(s) = 1 + \frac{7K_P(1 + 1/T_I s)}{3s + 12} = 0$$

Multiply top and bottom by s, and factor out $7K_P$ from the numerator and 3 from the denominator. This gives

$$1 + \frac{7K_P}{3}\frac{s + 1/T_I}{s(s + 4)} = 0$$

The root locus parameter K is $K = 7K_P/3$. The zero is $s = -1/T_I$ and the poles are $s = 0$ and $s = -4$.

c.   From the diagram in part (c) of Figure 11.2.2,

$$1 + G(s)H(s) = 1 + \frac{7(5)K_P(1 + T_D s)}{(3s^2 + 18s + 24)(4s + 1)} = 0$$

Factor out $7(5)K_P T_D$ from the numerator and $3(4)$ from the denominator so that the highest coefficients will be 1. This gives

$$1 + \frac{35 K_P T_D}{12} \frac{s + 1/T_D}{(s^2 + 6s + 8)(s + 0.25)} = 0$$

So the root locus parameter K is $K = 35 K_P T_D/12$. The zero is $s = -1/T_D$ and the poles are $s = -0.25$, $s = -2$, and $s = -4$.

From these examples we conclude that

1. P action never produces an open-loop pole or zero.
2. PD action produces an open-loop zero at $s = -1/T_D$.
3. I action produces an open-loop pole at $s = 0$.
4. PI action produces an open-loop pole at $s = 0$ and an open-loop zero at $s = -1/T_I$.
5. PID action produces an open-loop pole at $s = 0$ and two open-loop zeros, which may be real or complex, depending on the values of T_I and T_D.

If the variable parameter of interest is *not* a multiplicative factor in $G(s)H(s)$, you must obtain the characteristic equation from the denominator of the closed-loop transfer function and isolate the parameter to express the equation in the standard form. For example, with $H(s) = 1$, PI control of the plant $1/(4s + c)$ leads to the characteristic equation $4s^2 + (c + K_P)s + K_I = 0$. If $c = 4$, the gain values required to give critical damping with a time constant of 0.2 are $K_P = 36$ and $K_I = 100$. Now if the value of the damping c is uncertain, we may use the root locus to assess the effects of this uncertainty on the control system performance. The characteristic equation can be expressed as $4s^2 + (c + 36)s + 100 = 0$, which in standard form becomes

$$1 + \frac{c}{4} \frac{s}{s^2 + 9s + 25} = 0$$

The root locus parameter K is $K = c/4$. The zero is $s = 0$, and the poles are $s = -4.5 \pm 2.179 j$.

ANGLE AND MAGNITUDE CRITERIA

The general problem in the form (11.1.2) can be written as

$$K P(s) = -1 \tag{11.2.5}$$

From this, we see that two requirements must be met if s is to be a root of this equation, which is a statement of equality between two complex numbers, $K P(s)$ and -1. Recall that a complex number may be expressed as a magnitude and an angle. Thus, for two complex numbers to be equal, their magnitudes must be equal and their angles must also be equal. Applying this insight to (11.2.5) we obtain the following magnitude and angle criteria, respectively.

$$|K P(s)| = 1 \tag{11.2.6}$$

$$\angle K P(s) = \angle(-1) = (2n + 1)180°, \qquad n = 0, 1, 2, 3, \ldots \tag{11.2.7}$$

Figure 11.2.3 Vector
addition and vector
representation of complex
numbers.
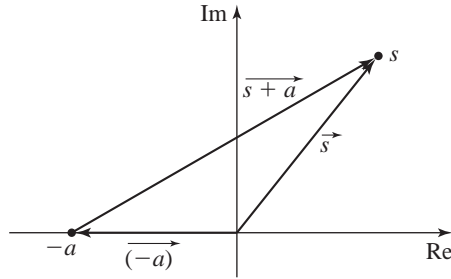
For now, we consider only $K \geq 0$. Thus, $|K| = K$ and $\angle K = 0°$, so that (11.2.6) becomes

$$K = \frac{1}{|P(s)|} = \frac{|D(s)|}{|N(s)|} \tag{11.2.8}$$

From Figure 11.2.3, we see that the property of vector addition gives

$$\vec{s} = (\overrightarrow{-a}) + (\overrightarrow{s + a})$$

From this we deduce that the vector $(\overrightarrow{s + a})$, which represents the complex number $s + a$, has its head at the point s and its tail at $-a$. This information is useful for applying (11.2.8), which can be stated as

$$K = \frac{\text{product of the magnitudes of vectors from poles to point } s}{\text{product of the magnitudes of vectors from zeros to point } s} \tag{11.2.9}$$

because the magnitude of a complex number is the length of its corresponding vector, and the magnitude of a product of complex numbers is the product of the magnitudes. Note that the point s must be on the root locus.

For example, consider the equation

$$1 + K \frac{s + 4}{s\left[(s + 3)^2 + 2^2\right]} = 0$$

The zero is $s = -4$ and the poles are $s = 0$ and $s = -3 \pm 2j$. The root locus plot is shown in Figure 11.2.4a. Suppose we want to determine the value of K that puts one root at $s = -2$. Part (b) of the figure shows the vectors drawn from the poles and zero to the point $s = -2$. Then

$$K = \frac{|\vec{s}||\overrightarrow{s + 3 - 2j}||\overrightarrow{s + 3 + 2j}|}{|\overrightarrow{s + 4}|}$$

From trigonometry we can compute the length of each vector to be

$$|\vec{s}| = 2 \qquad |\overrightarrow{s + 4}| = 2$$

$$|\overrightarrow{s + 3 - 2j}| = |\overrightarrow{s + 3 + 2j}| = \sqrt{5}$$

Thus

$$K = \frac{2(\sqrt{5})(\sqrt{5})}{2} = 5$$

That is, if $K = 5$, one root is $s = -2$. Note that we could have obtained this result more easily by substituting $s = -2$ into the characteristic equation, but we will see that the vector interpretation is also useful.

Figure 11.2.4 (a) Root locus for the open-loop transfer function $(s + 4)/s\left[(s + 3)^2 + 2^2\right]$. (b) Vector calculation of the value of K corresponding to $s = -2$.

(a)

(b)
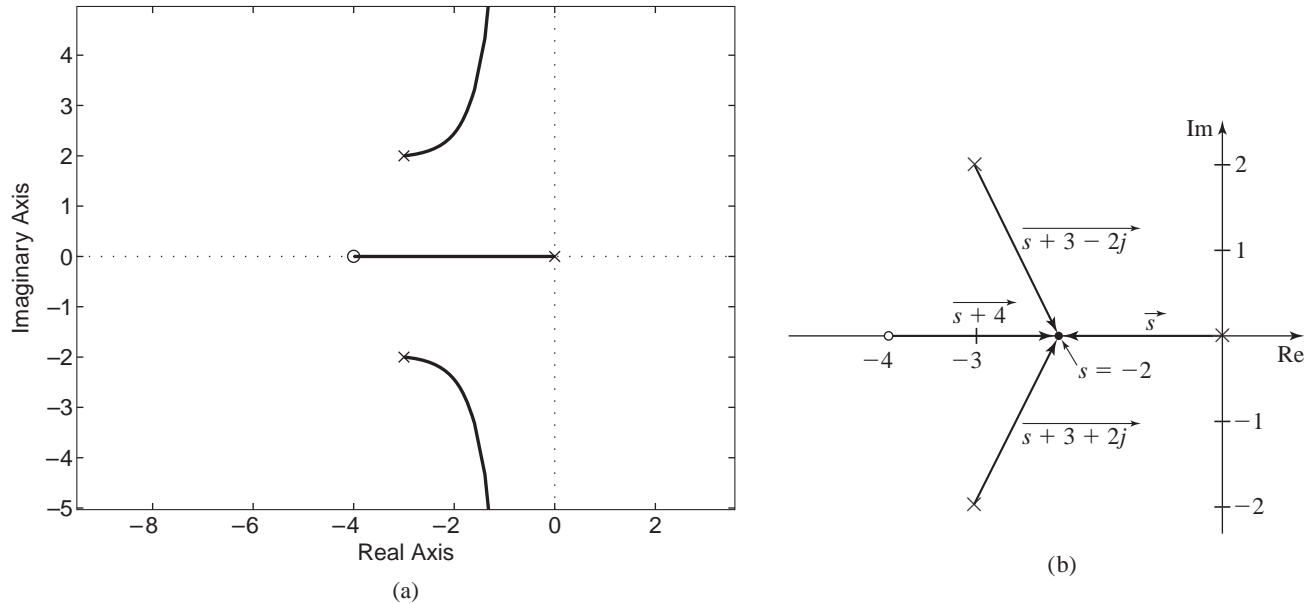
Because the angle of a product of complex numbers is the sum of their angles, we have, because K is real and positive,

$$\angle KP(s) = \angle K + \angle P(s) = 0° + \angle P(s) = \angle P(s)$$

Equation (11.2.7) implies that

$$\angle P(s) = \angle N(s) - \angle D(s) = (2n + 1)180°, \qquad n = 0, 1, 2, 3, \ldots \qquad (11.2.10)$$

Expressed in terms of vectors, this equation states that the sum of the angles of the vectors from the zeros to the point s minus the sum of the angles of the vectors from the poles to the point s must equal $(2n + 1)180°$ if the point s lies on the root locus. This interpretation is useful for checking to see if a specific point is on the locus. For example, referring to Figure 11.2.4b,

$$\angle N(s) - \angle D(s) = 0° - 180° - \left[360° - \tan^{-1}\left(\frac{2}{1}\right)\right] - \tan^{-1}\left(\frac{2}{1}\right)$$

$$= -540°$$

which is equivalent to 180°. So the point $s = -2$ lies on the root locus.

Equations (11.2.8) and (11.2.10) are the *only* conditions that every point on the root locus must satisfy. It is interesting to note that the angle criterion (11.2.10) does not contain K. Thus the shape of the root locus plot is determined entirely by the angle criterion (11.2.10). All of the plotting guides to follow, except two, are the result of this single condition. The magnitude criterion (11.2.8) is used only to obtain the associated value of K for a designated point s on the root locus.

ROOT LOCUS SKETCHING GUIDES

Let us collect the insights generated thus far and formalize them as guides for sketching the root locus where $K \geq 0$. First note that the root locus plot is *symmetric* about the real axis because complex roots occur only in conjugate pairs if the polynomial coefficients are real.

If you will be using the root locus plot for graphical calculations, such as measuring an angle to compute a damping ratio, the real and imaginary axes must have the same scale so that the angles will not be distorted. You can force MATLAB to use equal scales with the `axis equal` command.

We next note some facts that are obvious when we consider the general root locus equation

$$1 + KP(s) = 1 + K\frac{N(s)}{D(s)} = 0 \qquad (11.2.11)$$

Assumption: We assume that the order of $N(s)$ is no greater than the order of $D(s)$.

The reason for this assumption will become clear shortly.

Guide 1: The number of paths equals the number of poles of $P(s)$.

This tells us how many paths we must account for. The proof uses the form $1 + KN(s)/D(s) = 0$. If the order of $D(s)$ is greater than or equal to that of $N(s)$, then the order of the equation, and thus the number of roots, is determined by $D(s)$. The order of $D(s)$, however, equals the number of poles of $P(s)$.

Guide 2: The paths start at the poles of $P(s)$ with $K = 0$.

When $K = 0$, (11.2.11) shows that the roots are given by $D(s) = 0$; in other words, by the poles.

Guide 3: The paths terminate with $K = \infty$ either at the zeros of $P(s)$ or by leaving the plot.

When $K \to \infty$, there are only two ways that (11.2.11) can be satisfied. The first requires that $N(s) \to 0$; that is, that s approach a zero of $P(s)$. If $N(s)$ does not approach zero as $K \to \infty$, then $D(s)$ must approach infinity, which can occur only if $s \to \infty$.

Variation of the Leading Coefficient   Be careful when using the root locus where the variable parameter is the coefficient of the highest power of s in the characteristic equation. For example, consider the second-order equation $ms^2 + s + 8 = 0$. When $m = 0$, the equation is $s + 8 = 0$, which is *first* order. The sketching guides to follow assume that the order of the equation remains the same as the parameter varies from 0 through positive values, and so we cannot use these guides to obtain the root locus plot of this second-order equation with m as the parameter.

If we put $ms^2 + s + 8 = 0$ into standard form we obtain

$$1 + m\frac{s^2}{s + 8} = 0 \qquad (11.2.12)$$

Thus, $K = m$. Note that the order of the numerator is greater than the order of the denominator. This indicates that the order of the equation will be different for $K = 0$ than for $K > 0$, and so the guides will be invalid.

Thoughtless application of guides 1, 2, and 3 results in a contradiction when applied to (11.2.12). Guides 1 and 2 state that there is one path and it starts at $s = -8$, but guide 3 says that two termination points exist because there are two zeros. The contradiction arises because the order of the numerator is greater than that of the denominator. When $m > 0$, there are two roots; when $m = 0$, there is only one.

Order of $N(s)$ and $D(s)$ We therefore assume that the order of $N(s)$, the numerator of $P(s)$, is no greater than the order of $D(s)$, the denominator of $P(s)$. Even if you do not use the following guides to obtain a root locus plot, but instead use a computer program, such as MATLAB, you should avoid equations where the order of the numerator is greater than that of the denominator.

Variation of the leading coefficient can be studied if the order of the equation remains the same throughout the variation. For example, with (11.2.12), if the nominal value of m is 0.5, let $\Delta = m - 0.5$, and express the equation as $(0.5 + \Delta)s^2 + s + 8 = 0$ or

$$1 + \frac{\Delta}{0.5} \frac{s^2}{s^2 + 2s + 16} = 0$$

Then $K = \Delta/0.5$. The guides for $K \geq 0$ can now be applied without any difficulty if we wish to study the effects of m varying *above* the nominal value of 0.5.

Root Locus Paths on the Real Axis   The next guide is one of the most useful. Its proof is lengthy so we will omit it. For proof of this and other guides, see [Cannon, 1967].

> **Guide 4:** The root locus exists on the real axis only to the left of an odd number of real poles and/or real zeros. The numbering system is as follows. The real pole or real zero that lies the farthest to the right is number 1; the next real pole or zero is number 2; etc. If real poles or real zeros are identical, number each one.

Note that this guide implies that the locus cannot exist on the real axis to the *right* of an odd numbered real pole or real zero.

Some examples of the application of this guide are shown in Figure 11.2.5, which does not show the root locus off the real axis.
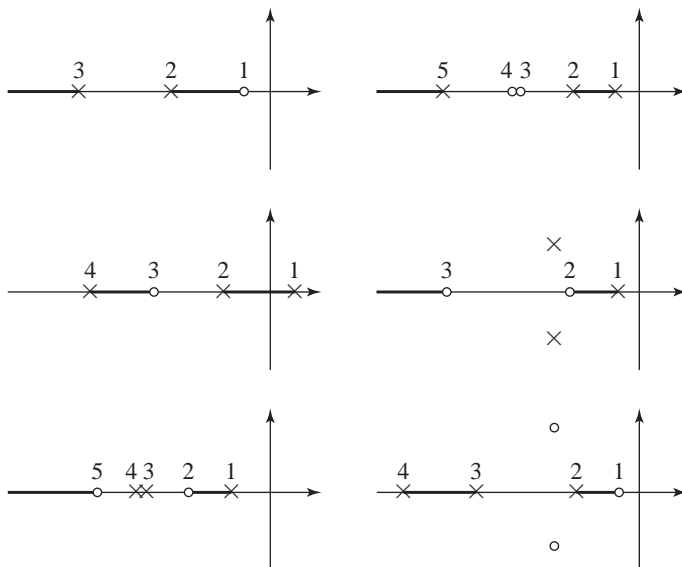
Figure 11.2.5 Examples of the application of guide 4. The root loci off the real axis are not shown.

It is impossible for two root paths to coincide over any finite length (but they can cross). So two roots approaching each other on the real axis must leave the real axis at the point where they collide (called the *breakaway point*), and they must do this in a way that is symmetric about the real axis. This means that if one path breaks *up* from the real axis, the other path must break *down*.

Look for breakaway points where the root locus lies between two real poles, but they may occur at other locations. To compute the location of a breakaway point, note that the paths approach each other as K increases. Thus, at the breakaway point, K attains the relative *maximum* value it has on the *real* axis in the vicinity of the breakaway point. The value of s corresponding to the breakaway point can be found by computing dK/ds from the characteristic equation, setting $dK/ds = 0$, and solving for the value of s. In general, multiple solutions will occur. The extraneous ones can be discarded with a knowledge of the location of the locus on the real axis from guide 4. Thus the second derivative often need not be computed to distinguish between a minimum and a maximum.

A path can also enter the real axis, and the point at which this occurs is called the *breakin point*. If we increase K after the path has entered the real axis, the path continues along the real axis. Thus, at a breakin point, K attains the relative *minimum* value it has on the real axis in the vicinity of the breakin point. Look for breakin points where the root locus lies between two real zeros or between a real zero and $s = \pm\infty$, but they may occur at other locations. The location of the breakin point is determined from $dK/ds = 0$ in exactly the same manner as for a breakaway point. Although the method is the same, no difficulty is encountered in identifying the type of point, because this is usually obvious once the first four guides have been applied.

It is possible to have multiple breakaway or breakin points. This is why we speak of K attaining a *relative* or *local* maximum or minimum value only in the vicinity of each such point. Thus we have guide 5.

> **Guide 5:** The locations of breakaway and breakin points are found by determining where the parameter K attains a local maximum or minimum.

Example 11.2.2 illustrates how to find breakaway and breakin points.

Root Locus Paths Off the Real Axis   The vector properties of complex numbers can be used to help understand how the root locus behaves off the real axis.

As $K \to \infty$, the paths approach a zero or the roots become infinite, which means that the paths exit the plot. To see what happens in this case, consider Figure 11.2.6, which is sketched for a case having three poles and one zero. The vectors from the

Figure 11.2.6 Illustration of guide 6.
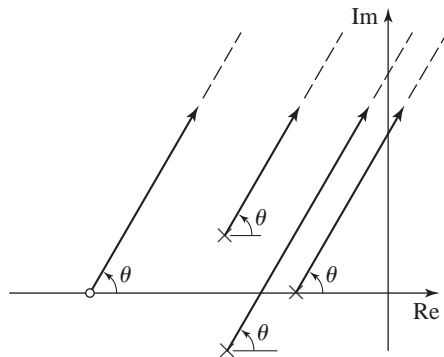
poles and zero to a point on the locus at infinity are parallel and make the same angle θ with the real axis. If we let p and z denote the number of poles and zeros, respectively, the angle criterion states that

$$P(s) = z\theta - p\theta = (2n + 1)180°, \qquad n = 0, 1, 2, 3, \ldots$$

Solving for θ, we obtain

$$\theta = \frac{(2n + 1)180°}{z - p} \qquad n = 0, 1, 2, 3, \ldots$$

This is equivalent to

$$\theta = \frac{(2n + 1)180°}{p - z} \qquad n = 0, 1, 2, 3, \ldots$$

This leads to guide 6.

> **Guide 6:** The paths that do not terminate at a zero approach infinity along asymptotes. The angles that the asymptotes make with the real axis are found from
>
> $$\theta = \frac{(2n + 1)180°}{p - z} \qquad n = 0, 1, 2, 3, \ldots \qquad (11.2.13)$$
>
> where n is chosen successively as $n = 0, 1, 2, 3, \ldots$, until enough angles have been found.

Figure 11.2.7 shows the most commonly found patterns for asymptotes.

Unless θ is 0° or 180°, the asymptotes cannot be drawn until we know where they intersect the real axis. This is given by guide 7. The proof follows from the angle criterion, but is detailed (see [Cannon, 1967]).

> **Guide 7:** The asymptotes intersect the real axis at the common point $s = \sigma$ given by
>
> $$\sigma = \frac{\sum s_p - \sum s_z}{p - z} \qquad (11.2.14)$$
>
> where $\sum s_p$ and $\sum s_z$ are the algebraic sums of the values of the poles and zeros.

> Note that this guide states that all the asymptotes intersect at the same point.

Determination of Instability Some systems become unstable for certain values of the parameter K. This occurs if any path crosses the imaginary axis into the right half-plane. The value of K at which this happens can be determined from the Routh-Hurwitz criterion. Often, however, the substitution $s = j\omega$ into the polynomial equation of interest is quicker and gives the crossing location as well as the value of K. This leads to guide 8.

$p - z = 1 \qquad p - z = 2 \qquad p - z = 3 \qquad p - z = 4$
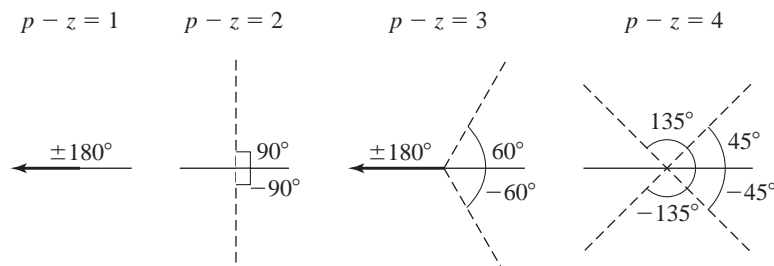
Figure 11.2.7 Common patterns for asymptotes.

Guide 8: The points at which the paths cross the imaginary axis and the associated values of K can be found by substituting $s = j\omega$ into the characteristic equation.

The solution for the frequency ω is called the *crossover frequency,* and the points $s = \pm j\omega$ are called the *crossover points*. Examples 11.2.2 and 11.2.4 illustrate the procedure for computing the crossover points.

In many applications, it is not necessary to determine the precise location of the locus off the real axis, and the first eight guides are often quite sufficient for obtaining ample information about the system's behavior. When more accuracy is required, a computer program such as MATLAB can be used.

Locus Behavior Near Complex Poles and Zeros   Sometimes it is helpful to know the direction in which the locus leaves a complex pole (called the *angle of departure*) and the direction in which it terminates at a complex zero (called the *angle of arrival*). To determine these angles, we again call on the angle criterion.

Guide 9: Angles of departure and angles of arrival are determined by choosing an arbitrary point infinitesimally close to the pole or zero in question and applying the angle criterion

$$\angle N(s) - \angle D(s) = (2n + 1)180° \tag{11.2.15}$$

Locating All the Roots   The following guide comes from the theory of polynomial equations.

Guide 10: For the polynomial equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{11.2.16}$$

the sum of the roots $r_1, r_2, \ldots, r_n$ is

$$r_1 + r_2 + \cdots + r_n = -a_{n-1} \tag{11.2.17}$$

Note that the coefficient of s^n is unity.

This guide is useful for determining the location of the remaining roots once some roots have been found.

Determining the Value of K   Once the root locus is drawn, the magnitude criterion can be used to compute the value of K associated with a particular point on the locus. If the polynomial is written as $D(s) + KN(s) = 0$, the magnitude criterion for $K \geq 0$ states that

$$K = \frac{|D(s)|}{|N(s)|} \tag{11.2.18}$$

Table 11.2.1 summarizes the procedure for sketching a root locus plot, and for obtaining information from it. Figures 11.2.8 through 11.2.12 display a number of possible root locus plots for second- and third-order equations. Figure 11.2.8 shows six of the possible plots for second-order equations. Note that you can recover the characteristic equation from the values of the poles and zeros.

Most of the root locus plots in this chapter were generated with MATLAB. At this time, MATLAB users may wish to study Section 11.6. Users of other software should consult the appropriate software documentation.

Table 11.2.1 Root locus sketching procedure for $K \geq 0$.

Procedure step	Appropriate guide or equation
1. Express the characteristic equation in the standard form.	■ $1 + KP(s) = 0$ where $P(s) = N(s)/D(s)$, $K \geq 0$ ■ Highest coefficients of $N(s)$ and $D(s)$ must be 1. ■ The order of $N(s)$ must be no greater than the order of $D(s)$.
2. Determine the poles and zeros.	■ Solve $D(s) = 0$ for the poles. ■ Solve $N(s) = 0$ for the zeros. ■ $p =$ number of poles. $z =$ number of zeros.
3. Plot the poles with $\times$ and the zeros with $\bigcirc$. Use same scale on both axes. Locus is symmetric about the real axis.	■ The number of paths equals the number of poles. ■ The zeros are termination points for z paths. ■ The remaining $p - z$ paths leave the plot.
4. Sketch the root locus on the real axis.	■ The locus lies only to the left of odd-numbered real poles and/or real zeros.
5. Determine any breakaway and breakin points.	■ Solve $\dfrac{dK}{ds} = -\dfrac{dP(s)}{ds} = 0$ for s.
6. If $p > z$, compute the angles and intersection point of the asymptotes.	■ $\theta = \dfrac{(2n+1)180°}{p - z} \quad n = 0, 1, 2, 3, \ldots$ ■ $\sigma = \dfrac{\sum s_p - \sum s_z}{p - z}$
7. Determine any crossover points.	■ Substitute $s = j\omega$ into the characteristic equation, separate the real and imaginary parts, and solve for ω and K, or ■ use the Routh-Hurwitz criterion to find K, then solve for the characteristic roots.
8. Determine any angles of departure from complex poles, and any angles of arrival at complex zeros.	■ Apply $\angle N(s) - \angle D(s) = (2n+1)180°$ $n = 0, 1, 2, 3, \ldots$, where s is the pole or zero location.
9. Determine the value of K at any desired root location s_d.	■ Use the magnitude criterion: $$K_d = \frac{|D(s_d)|}{|N(s_d)|}$$
10. Locate remaining roots corresponding to K_d.	■ Use the fact that $-a_{n-1}$ is the sum of the roots of the equation $s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$

Figure 11.2.9 shows four of the possible plots for third-order equations having no zeros. Note that part (d) has a breakin point even though it does not have a zero, and a breakaway point even though there is only one real pole.

Figure 11.2.10 shows four of the possible plots for third-order equations having one zero. Note that part (d) has both a breakin and a breakaway point between the zero and the pole.

Figure 11.2.11 shows two of the possible plots for third-order equations having complex poles and one zero. Note that part (b) has both a breakin and a breakaway point between the zero and the real pole.

Figure 11.2.12 shows three of the possible plots for third-order equations having two zeros. Note that part (a) has multiple crossover points. Note that part (c) has both a breakin and a breakaway point between the real pole and $s = -\infty$.

Note that these figures do not show all of the possible root locus plots for second- and third-order equations.

Figure 11.2.8 Root locus plots for some second-order equations.
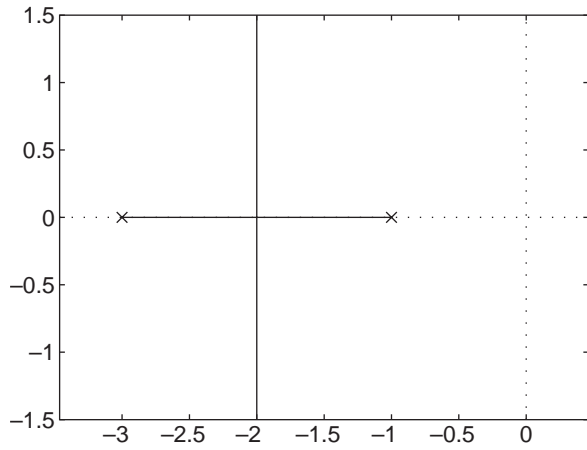
(a)
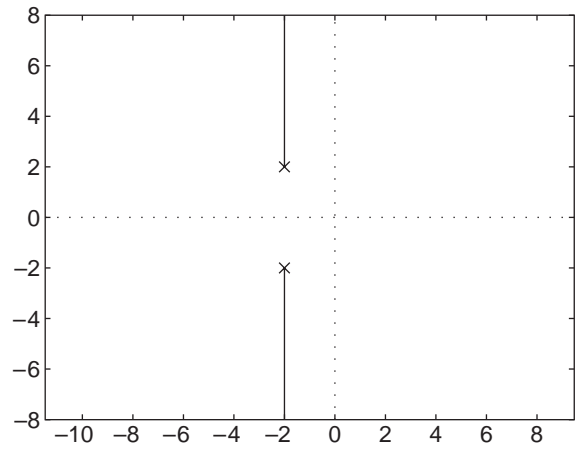
(b)
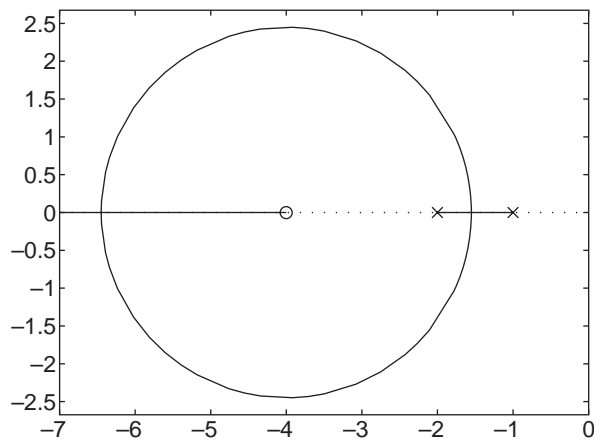
(c)
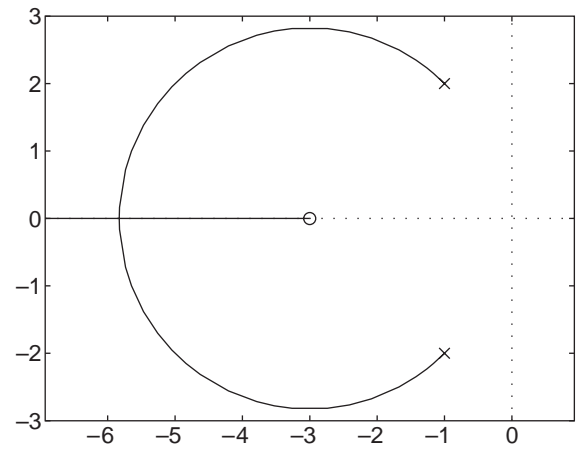
(d)
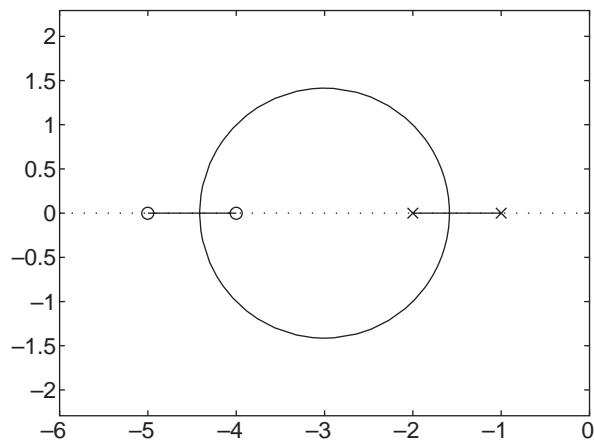
(e)
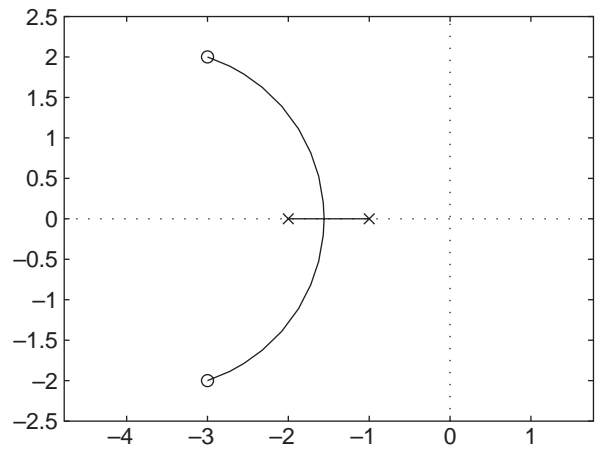
(f)

Figure 11.2.9 Root locus plots for some third-order equations with no zeros.

(a)

(b)

(c)

(d)
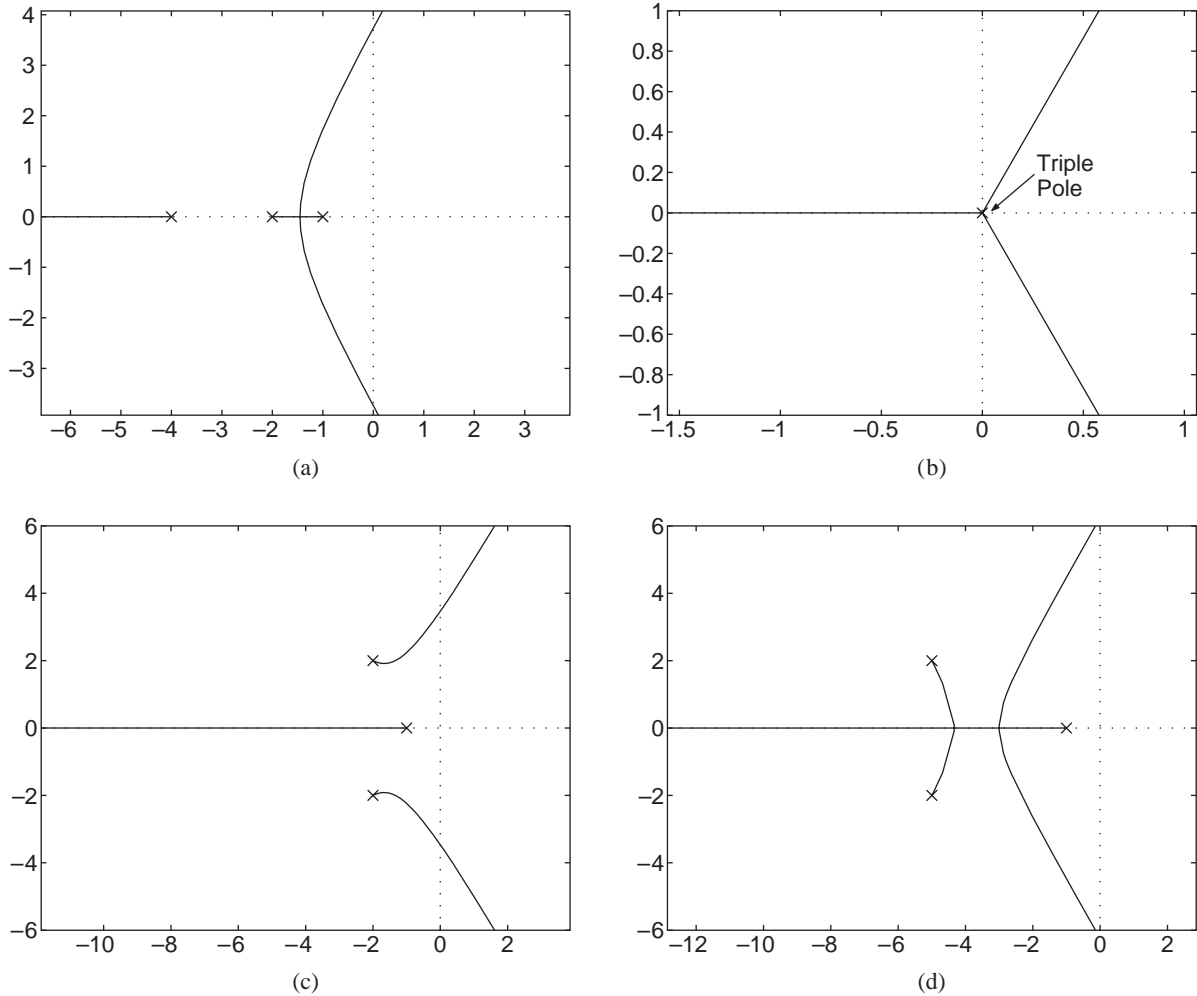
THE COMPLEMENTARY ROOT LOCUS

The development of the plotting guides was based on the angle and magnitude criteria, under the assumption that the parameter K is positive or zero. The corresponding root locus is called the *primary* root locus. There are applications, however, where we cannot formulate the problem such that $K \geq 0$. This occurs with some plant models that have negative coefficients. For example, proportional control with the plant

$$G_p(s) = \frac{-s + 5}{s(s + 8)} \tag{11.2.19}$$

results in the following root locus equation.

$$1 + K \frac{s - 5}{s(s + 8)} = 0 \tag{11.2.20}$$

where $K = -K_P$. Thus to investigate the root locus for positive values of K_P, we must let K vary through negative values.

Figure 11.2.10 Root locus plots for some third-order equations having real poles and one zero.

(a)

(b)

(c)

(d)
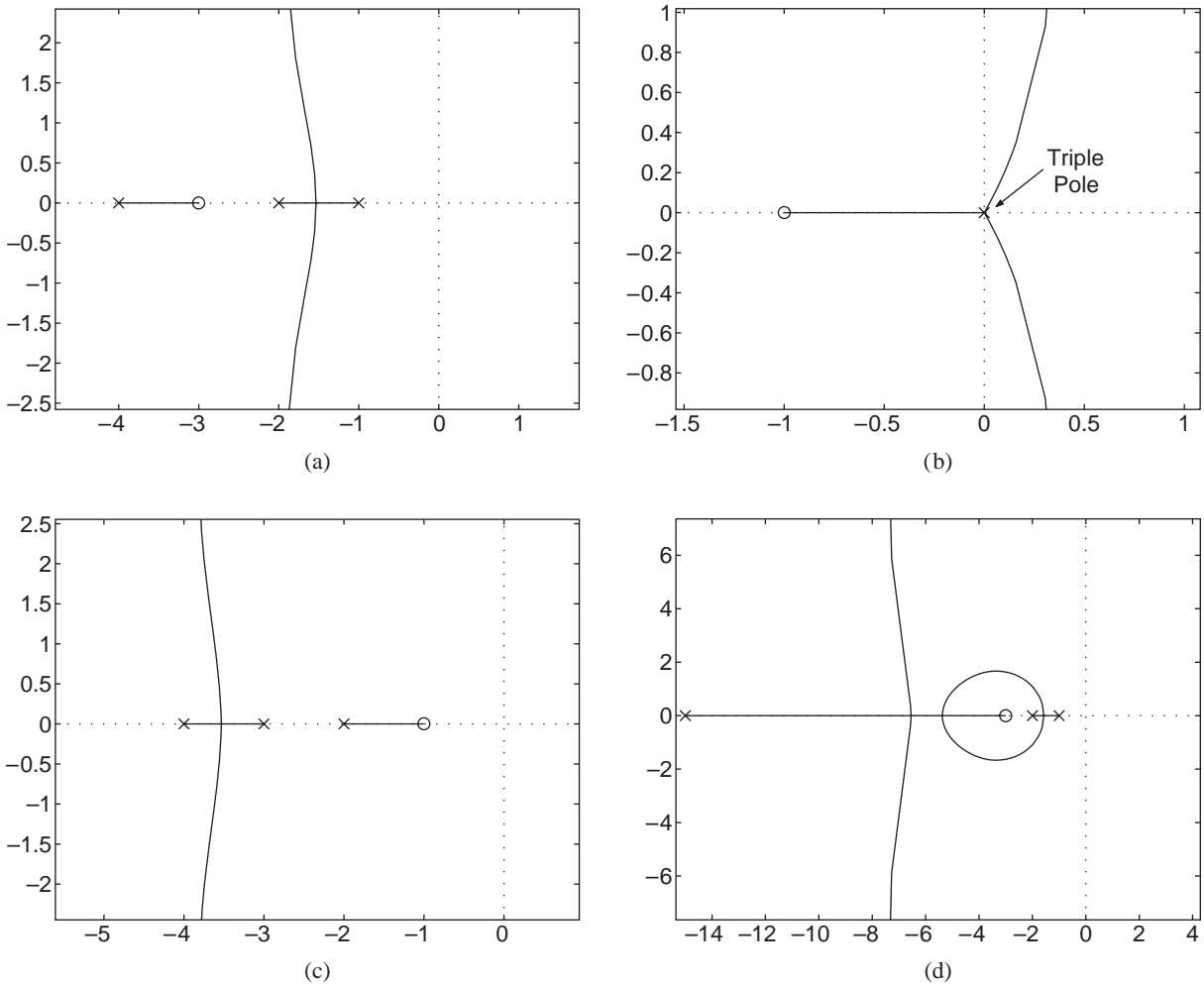
The guides for the complementary root locus of the equation

$$1 + K\frac{N(s)}{D(s)} = 0 \qquad K \leq 0 \tag{11.2.21}$$

are found from the same criteria, repeated here.

$$\left| K\frac{N(s)}{D(s)} \right| = 1 \tag{11.2.22}$$

$$\angle K + \angle N(s) - \angle D(s) = \angle(-1) = (2n + 1)180° \tag{11.2.23}$$

If $K \leq 0$, (11.1.22) becomes $-K|P(s)| = 1$ or

$$K = -\frac{1}{|P(s)|} \tag{11.2.24}$$

Thus, the same scaling procedure is used, with a sign reversal included.

Figure 11.2.11 Root locus plots for some third-order equations having complex poles and one zero.

(a)

(b)
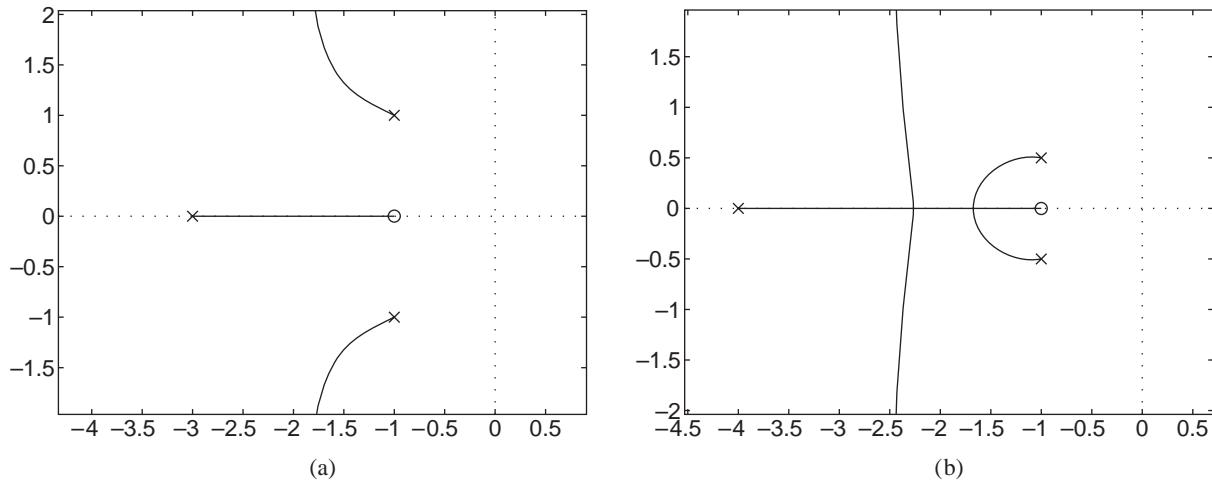
If $K \leq 0$, then $\angle K = (2k + 1)180°$, and (11.2.23) becomes

$$(2k + 1)180° + \angle N(s) - \angle D(s) = (2n + 1)180°$$

which reduces to

$$\angle N(s) - \angle D(s) = (2n + 1)180° - (2k + 1)180° = m360° \qquad m = 0, 1, 2, 3, \ldots$$

(11.2.25)

The guides for the complementary locus are identical to those for the primary locus except for guides 4 and 6, which we renumber as 4a and 6a.

Guide 4a The locus exists in a section on the real axis only if the number of real poles and/or zeros to the *right* of the section is *even;* furthermore, it must exist there. The number zero is taken to be even. In other words, the complementary locus exists on the real axis wherever the primary locus does not exist.

Guide 6a The paths that do not terminate at a zero leave the plot and approach asymptotes. The asymptotic angles relative to the positive real axis are found from

$$\theta = \frac{m360°}{z - p} \qquad m = 0, 1, 2, 3 \ldots$$

(11.2.26)

where z and p are the number of zeros and poles and m is increased until enough angles have been found.

Note that the extraneous roots found when looking for breakaway and breakin points on the primary locus are the breakaway or breakin points on the complementary locus. Note also that the intersection point of the asymptotes is the same for the primary and the complementary loci. When determining the angles of arrival and departure for the complementary locus, be sure to use (11.2.25).

These guides can be used to show that the root locus of (11.2.20) is a circle of radius 8.06 centered on the zero at $s = 5$.

Figure 11.2.12 Root locus plots for some third-order equations having two zeros.

(a)

(b)

(c)
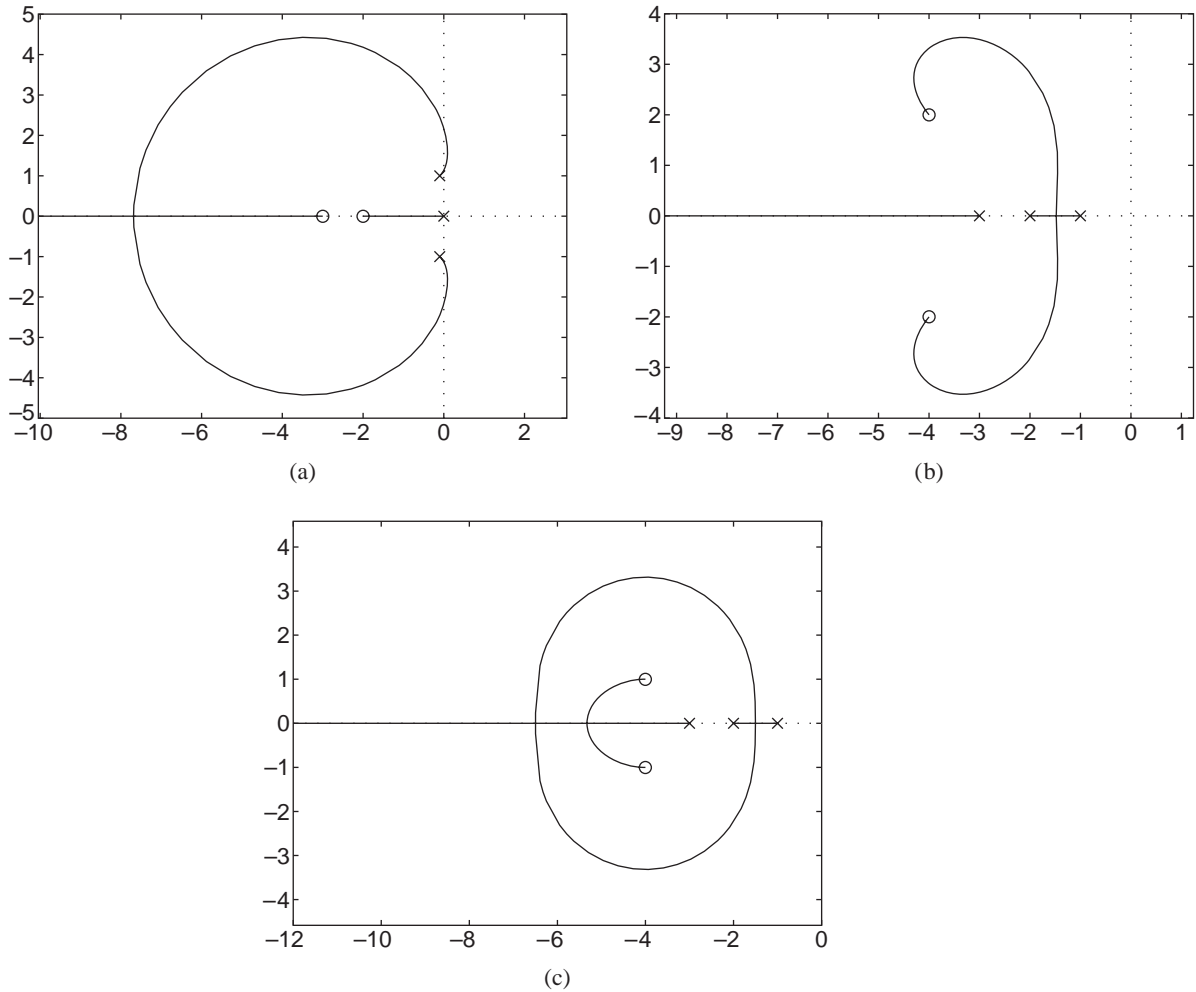
ROOT LOCUS EXAMPLES

EXAMPLE 11.2.2   Two Real Poles, One Zero

■ Problem
Sketch the root locus of the equation

$$s^2 + (K - 4)s + 6K = 0$$

■ Solution
In the standard form the equation is

$$1 + K\frac{s + 6}{s(s - 4)} = 0 \tag{1}$$

The poles are $s = 0$ and $s = 4$. The zero is $s = -6$. Thus, there are two paths. One starts at $s = 0$ and the other at $s = 4$, with $K = 0$. One path terminates at $s = -6$, while the other must leave the plot as $K \to \infty$.

Guide 4 shows that the locus exists on the real axis between $s = 0$ and $s = 4$ and to the left of $s = -6$. Therefore, the two paths must break away from the real axis between $s = 0$ and $s = 4$. From the location of the termination point at $s = -6$, we know that the locus must return to the real axis somewhere to the left of $s = -6$.

The breakaway and breakin points are found as follows. Solve equation (1) for K, compute dK/ds, and solve $dK/ds = 0$.

$$K = -\frac{s(s-4)}{s+6}$$

$$\frac{dK}{ds} = -\frac{(s+6)(2s-4) - s(s-4)}{(s+6)^2} = 0$$

This is satisfied for finite s if the numerator is zero. Thus $s^2 + 12s - 24 = 0$. The candidates are $s = 1.75$ and $s = -13.7$.

There is no need to check for a minimum or a maximum of K, because we know from guide 4 that the breakaway point must be $s = 1.75$ and the breakin point must be $s = -13.7$.

The crossover point is found by substituting $s = j\omega$ into equation (1), and collecting real and imaginary parts. This gives

$$6K - \omega^2 + (K-4)\omega j = 0$$

Both the real and imaginary parts must be zero, so we have $6K - \omega^2 = 0$ and $(K-4)\omega = 0$. The latter equation gives either $\omega = 0$, which corresponds to the pole at $s = 0$, or $K = 4$. Substituting this value into the equation for the real part gives $6(4) - \omega^2 = 0$, or $\omega = \pm\sqrt{24} = \pm4.9$. This gives the crossover points, which are $s = \pm4.9j$ when $K = 4$. The system is unstable for $0 \le K < 4$, and neutrally stable if $K = 4$.

This leaves only the shape of the locus off the real axis to be determined. For equations of order higher than two the locus does not have a simple shape, but for second-order equations it is often a circle. We note here that the breakaway and breakin points are symmetrically placed with a distance of 7.75 from the zero at $s = -6$. The simplest way for this to occur is with a circle of radius 7.75 centered at the zero. Another confirmation of this is the fact that the crossover points are a distance $\sqrt{6^2 + 4.9^2} = 7.75$ from the zero. The root locus is shown in Figure 11.2.13a.

Figure 11.2.13 (a) Root locus for Example 11.2.2. (b) Plot of the gain K versus real values of s.

(a)

(b)
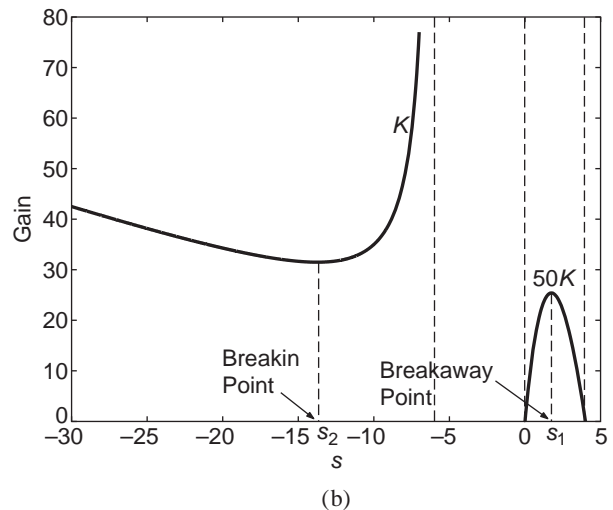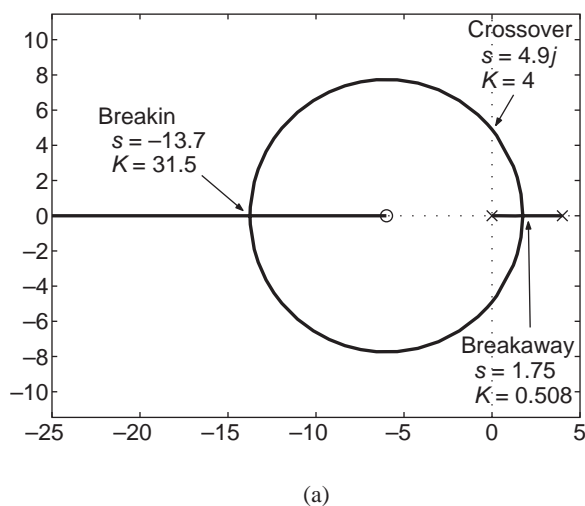
Part (b) of the figure shows the variation in K as s varies through real values. Note that K reaches a relative minimum at a breakin point and a relative maximum at a breakaway point. The plot for $0 \leq s \leq 4$ has been expanded to show the variation more clearly.

Elementary geometry can be used to show that the root locus of the equation

$$(s + b)(s + c) + K(s + a) = 0 \qquad (11.2.27)$$

is a circle centered on the zero at $s = -a$ if c is positive and $a > b > c$. The radius of the circle can be determined once the breakaway and breakin points are found.

EXAMPLE 11.2.3

PD Control

■ Problem

The equation of motion of an object rotating under an applied torque T is $I\ddot{\theta} = T$. Use PD action to control this system. Assume that $I = 1$ and $T_D = 2$. Determine the gain values (a) to achieve the smallest possible dominant time constant and (b) to achieve a damping ratio of 0.707. (c) Compare the step responses of the two designs.

■ Solution

The plant is

$$G_p(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{s^2}$$

The controller transfer function for PD action is

$$G_c(s) = K_D s + K_P = K_P(T_D s + 1) = K_P T_D \left(s + \frac{1}{T_D} \right)$$

The root locus equation is $1 + G_c(s)G_p(s) = 0$, or

$$1 + K_P T_D \frac{s + 1/T_D}{s^2} = 0$$

Thus PD action places a zero on the root locus plot at $s = -1/T_D$. There are two poles at $s = 0$.

a.   The root locus is a circle centered at the zero (Figure 11.2.14). Its radius is the distance from the zero to the poles and is $1/T_D$. Thus the breakin point is at $s = -2/T_D$. This point represents the smallest possible dominant time constant, which is $\tau = T_D/2 = 2/2 = 1$. The characteristic equation is

$$s^2 + K_P T_D s + K_P = 0$$

Figure 11.2.14 Root locus for Example 11.2.3.
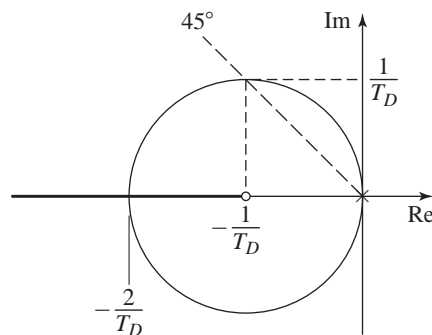
At the breakin point, $\zeta = 1$, where

$$\zeta = \frac{K_P T_D}{2\sqrt{K_P}} = 1$$

This gives $K_P = 4/T_D^2 = 4/4 = 1$ at the breakin point. This is the gain value that will give the smallest time constant for the given value of T_D.

b. From the root locus plot we see that $\zeta = 0.707$ corresponds to the top of the circle where $s = (-1 + j)/T_D$. At this point,

$$\zeta = \frac{K_P T_D}{2\sqrt{K_P}} = 0.707$$

This gives $K_P = 2/T_D^2 = 2/4 = 0.5$ for $\zeta = 0.707$.

c. For $T_D = 2$ and $K_P = 1$, the command transfer function is

$$\frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + 2s + 1}$$

With $T_D = 2$ and $K_P = 0.5$, the command transfer function is

$$\frac{C(s)}{R(s)} = \frac{s + 0.5}{s^2 + s + 0.5}$$

The step response for $K_P = 1$ has an overshoot of 13.5% and a settling time of 5.39. The step response for $K_P = 0.5$ has an overshoot of 20.8% and a settling time of 6.82. So $K_P = 1$ gives better response.

Three Poles, No Zero | **EXAMPLE 11.2.4**

■ **Problem**

Sketch the root locus plots of the equations

$$1 + K \frac{1}{s[(s + 2)^2 + 4]} = 0 \tag{1}$$

$$1 + K \frac{1}{s[(s + 2)^2 + 1]} = 0 \tag{2}$$

■ **Solution**

For equation (1) the poles are $s = 0$ and $s = -2 \pm 2j$. Thus, there are three paths. One starts at $s = 0$ and the other two at $s = -2 \pm 2j$, with $K = 0$. Because there is no zero, all three paths must leave the plot, approaching asymptotes as $K \to \infty$. Because there are three more poles than zeros, the asymptotes make angles of $\pm 60°$ and $180°$ with the positive real axis. The asymptotes intersect the real axis at

$$\sigma = \frac{\sum s_p - \sum s_z}{p - z} = \frac{0 - 2 + 2j - 2 - 2j}{3 - 0} = -\frac{4}{3}$$

The crossover points are found from the characteristic equation

$$s\left[(s + 2)^2 + 4\right] + K = s^3 + 4s^2 + 8s + K = 0$$

with $s = j\omega$. This gives

$$-j\omega^3 - 4\omega^2 + 8j\omega + K = 0$$

or

$$K - 4\omega^2 = 0$$

$$\omega(8 - \omega^2) = 0$$

Thus the crossover points are given by $\omega = \pm\sqrt{8} = \pm 2.83$, where $K = 4(\omega^2) = 32$.

Figure 11.2.15 Root locus
for the open-loop transfer
function $1/s\left[(s+2)^2+4\right]$.
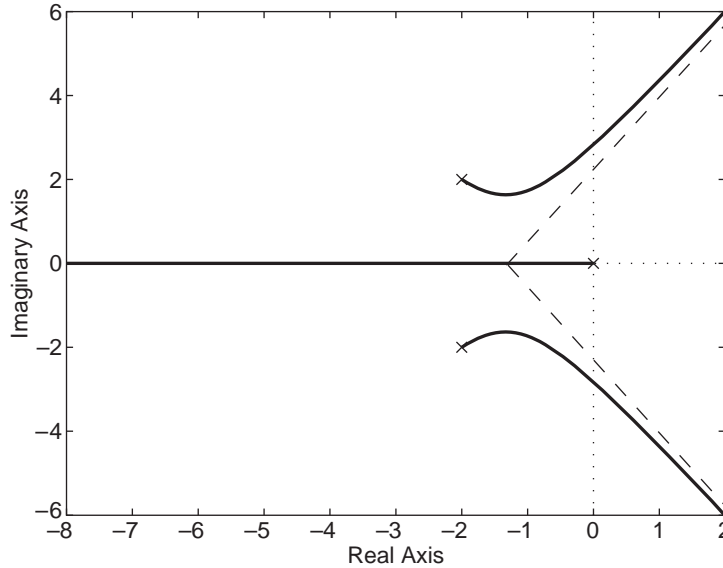
To examine for breakaway and breakin points, solve equation (1) for K, obtain dK/ds, and
solve $dK/ds = 0$ for s.

$$K = -\left(s^3 + 4s^2 + 8s\right)$$

$$\frac{dK}{ds} = -\left(3s^2 + 8s + 8\right) = 0$$

The solutions are $s = -1.33 \pm 0.942j$. Since these are complex, there are no breakaway or
breakin points. The root locus is shown in Figure 11.2.15 and can be generated with the MATLAB
`rlocus` function. Note that MATLAB does not display the asymptotes, so Figure 11.2.15 was
edited to show the asymptotes.

For equation (2) the poles are $s = 0$ and $s = -2 \pm j$. The asymptotes are $\pm 60°$ and $180°$.
The asymptotes intersect the real axis at

$$\sigma = \frac{\sum s_p - \sum s_z}{p - z} = \frac{0 - 2 + j - 2 - j}{3 - 0} = -\frac{4}{3}$$

The crossover points are found from the characteristic equation

$$s\left[(s+2)^2 + 1\right] + K = s^3 + 4s^2 + 5s + K = 0$$

with $s = j\omega$. This gives $\omega = \pm\sqrt{5} = \pm2.24$ where $K = 4(5) = 20$.

Because equations (1) and (2) are so similar, after sketching the locus for equation (1),
however, we may be lulled into not looking for breakaway or breakin points. Equation (2) gives

$$K = -(s^3 + 4s^2 + 5s) \tag{3}$$

$$\frac{dK}{ds} = -(3s^2 + 8s + 5) = 0$$

The solutions are $s = -1.67$ and $s = -1$. So there are breakaway or breakin points for this
equation! The corresponding values of K are found from equation (3). For $s = -1.67$, $K = 1.85$,
and for $s = -1$, $K = 2$. Computing d^2K/ds^2 for each solution identifies the nature of each point.

$$\frac{d^2K}{ds^2} = -(6s + 8) = \begin{cases} 2.02 > 0 & \text{for } s = -1.67 \\ -2 < 0 & \text{for } s = -1 \end{cases}$$
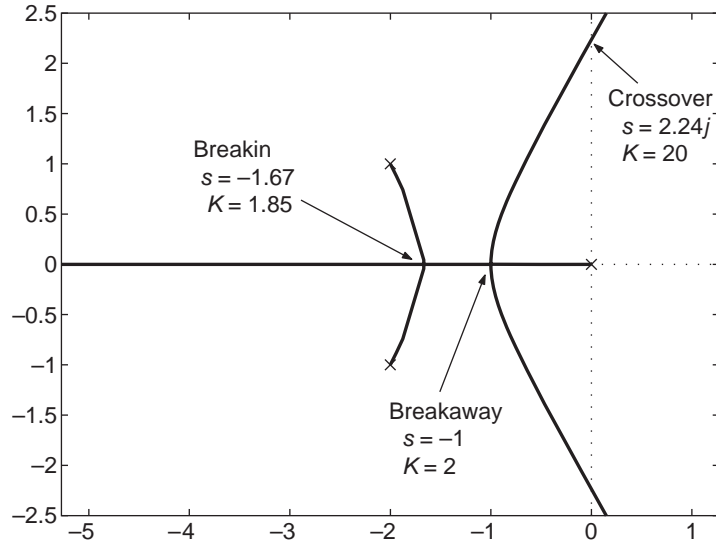
Figure 11.2.16 Root locus for the open-loop transfer function $1/s[(s+2)^2 + 1]$.

Thus $s = -1.67$ is a breakin point and $s = -1$ is a breakaway point. The root locus is shown in Figure 11.2.16.

Two Complex Poles, Two Complex Zeros

EXAMPLE 11.2.5

■ **Problem**

Sketch the root locus plots of the equation

$$1 + K\frac{(s+4)^2 + 9}{[(s+1)^2 + 1])} = 0 \tag{1}$$

■ **Solution**

The poles are $s = -1 \pm j$, so there are two paths. The zeros are $s = -4 \pm 3j$. Both paths must end at the zeros, so we need not look for asymptotes. Checking in the usual way for breakaway, breakin, and crossover points shows there are none.

The angles of arrival and departure are found as follows. Figure 11.2.17 shows the test points placed near the pole or zero in question, in order to calculate the vector angles. We use the general root locus equation

$$\angle N(s) - \angle D(s) = (2n+1)180° \qquad n = 0, 1, 2, 3, \ldots$$

The angle of departure ϕ is found from part (a) of the figure.

$$\phi + 90° - \phi_1 - \phi_2 = (2n+1)180°$$

where $\phi_1 = 360 - \tan^{-1}(2/3) = 326°$ and $\phi_2 = \tan^{-1}(4/3) = 53°$. Thus

$$\phi = \phi_1 + \phi_2 - 90° + (2n+1)180° = 289° + (2n+1)180°$$

Choosing $n = 0$ gives $\phi = 469°$, which is equivalent to $469 - 360 = 109°$.

Figure 11.2.17
(a) Calculation of the
departure angle.
(b) Calculation of the arrival
angle.

(a)

(b)
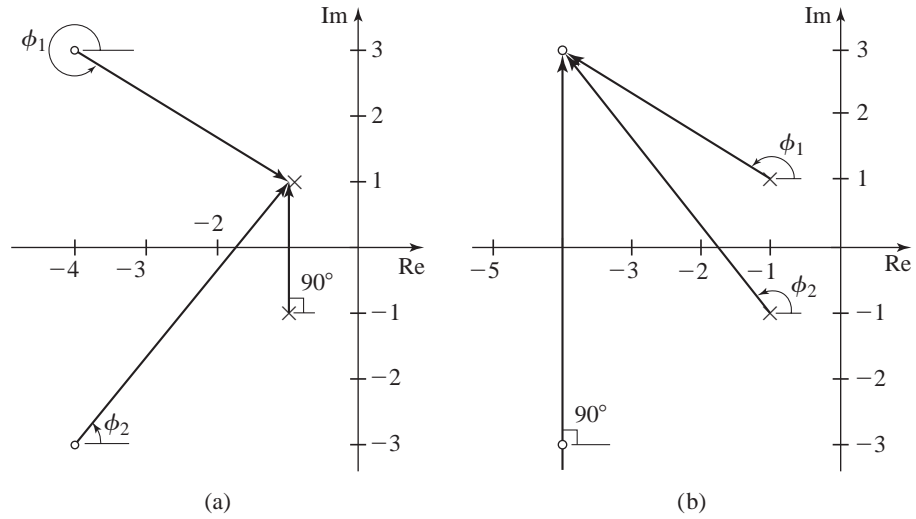
Figure 11.2.18 Root locus
for Example 11.2.5.
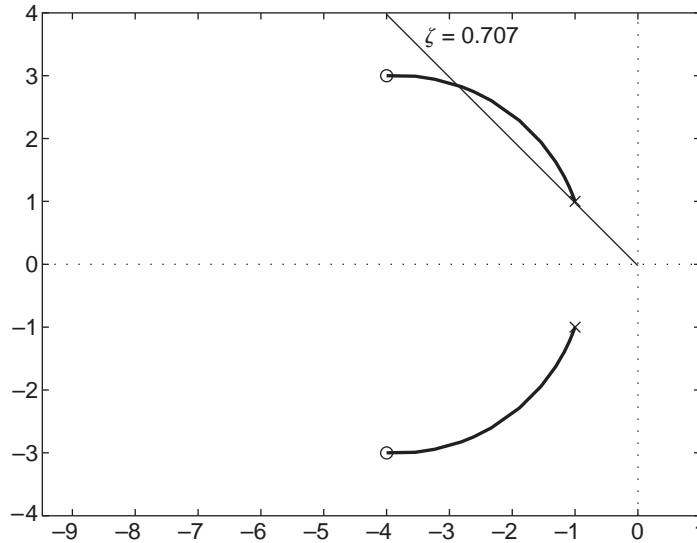
The angle of arrival ϕ is found from part (b) of the figure.

$$\phi + 90° - \phi_2 - \phi_1 = (2n + 1)180°$$

where $\phi_1 = 180 - \tan^{-1}(2/3) = 146°$ and $\phi_2 = 180° - \tan^{-1}(4/3) = 127°$. Thus

$$\phi = \phi_1 + \phi_2 - 90° + (2n + 1)180° = 183° + (2n + 1)180°$$

Choosing $n = 0$ gives $\phi = 363°$, which is equivalent to $3°$. Choosing $n = 1$ gives $\phi = 723°$, which is equivalent to $3°$.

These angles of departure and arrival indicate that the root locus will pass above the $45°$ line, and thus it is possible to obtain a value for K to achieve $\zeta \leq 0.707$. The root locus is shown in Figure 11.2.18.

Control of the Plant $1/(s+3)(s+5)$ **EXAMPLE 11.2.6**

■ Problem

Design a control algorithm for the plant

$$G_p(s) = \frac{1}{(s+3)(s+5)}$$

The dominant time constant must be no greater than 0.5 and the steady-state error for a step command must be zero.

■ Solution

a. Proportional Action: With P action the command transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P}{s^2 + 8s + 15 + K_P}$$

The root locus equation is

$$1 + K_P \frac{1}{(s+3)(s+5)} = 0$$

Figure 11.2.19 shows the root locus plot, from which we can tell that the smallest possible time constant is $\tau = 1/4$. Therefore the time constant specification can be satisfied.

   The steady-state response for a unit step command, however, is $c_{ss} = K_P/(15 + K_P)$. Thus the error is $e_{ss} = 1 - c_{ss} = 15/(15 + K_P)$, which cannot be made zero.

b. I Action: I action tends to eliminate steady-state error. The command transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_I}{s^3 + 8s^2 + 15s + K_I}$$

The steady-state response for a unit-step command is $c_{ss} = K_I/K_I = 1$, and so the error is zero. The root locus equation is

$$1 + K_I \frac{1}{s(s+3)(s+5)} = 0$$
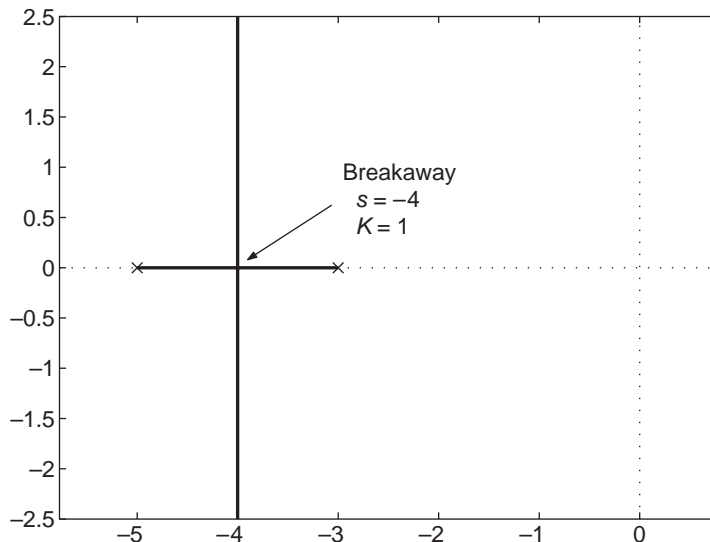
Breakaway
$s = -4$
$K = 1$

Figure 11.2.19 Root locus for the open-loop transfer function $1/(s+3)(s+5)$.

Figure 11.2.20 Root locus
for the open-loop transfer
function $1/s(s+3)(s+5)$.
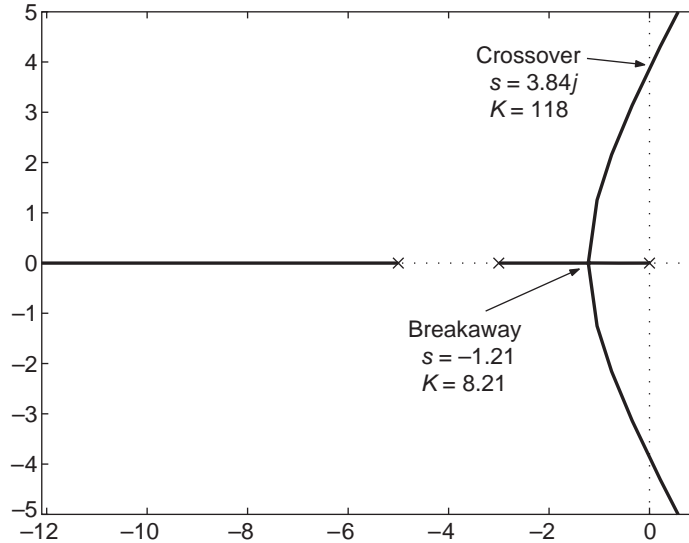
The breakaway point is found from

$$\frac{dK_I}{ds} = -\left(3s^2 + 16s + 15\right) = 0$$

or $s = -1.21$ and $s = -4.12$. The root locus does not exist at $s = -4.12$ for $K_I \geq 0$, so the breakaway point must be at $s = -1.21$.

Figure 11.2.20 shows the root locus plot, from which we can tell that the smallest possible time constant occurs at the breakaway point and is $\tau = 1/1.21 = 0.836$. Therefore the time constant specification cannot be satisfied. Note also that the system can now be unstable, whereas with P action it is always stable.

c.   PI Action: With PI action the command transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P s + K_I}{s^3 + 8s^2 + (15 + K_P)s + K_I}$$

The steady-state response for a unit step command is $c_{ss} = K_I/K_I = 1$, and so the error is zero. Expressing the integral gain as $K_I = K_P/T_I$, the root locus equation may be expressed as

$$1 + K_P \frac{s + 1/T_I}{s(s+3)(s+5)} = 0$$

The root locus parameter K is seen to be K_P. We now have two parameters to select: K_P and T_I. The usual procedure is to use a trial value for T_I, obtain the root locus, and use it to set the gain K_P. Note that the value of $1/T_I$ determines the location of the zero.

If we place the zero between the poles of the plant, we obtain a root locus plot like that shown in Figure 11.2.21. The particular plot shown corresponds to the choice $1/T_I = 3.5$. The dominant time constant at the breakaway point is $1/1.8 = 0.556$, which is larger than required, so we must use a gain value somewhat greater than 4.07. The larger the gain, the smaller will be the damping ratio. Because the $\pm 90°$ asymptotes intersect the real axis at $\sigma = -2.25$, the smallest possible dominant time constant is $1/2.25 = 0.444$, but this requires an infinite gain value. So a time constant of 0.5 can be achieved. By using MATLAB the necessary gain was found to be $K = 8.11$. This puts the dominant roots at $s = -2 \pm 1.76j$, for which $\zeta = 0.75$. The third root is $s = -3.99$.
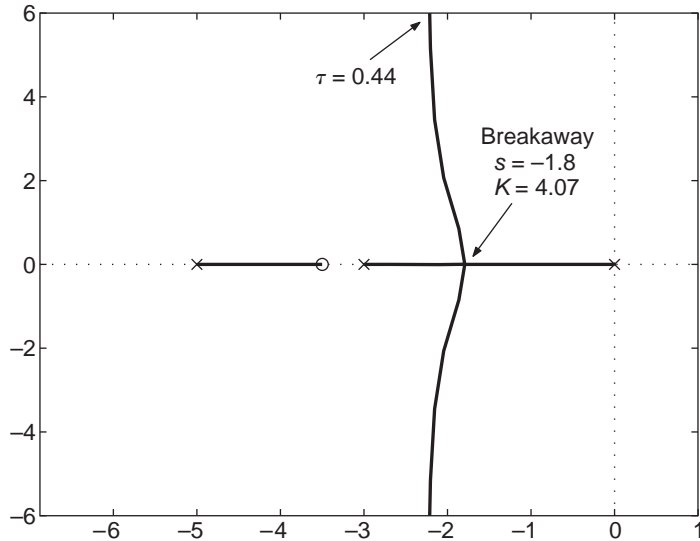
Figure 11.2.21 Root locus for the open-loop transfer function $(s + 3.5)/s(s + 3)(s + 5)$.
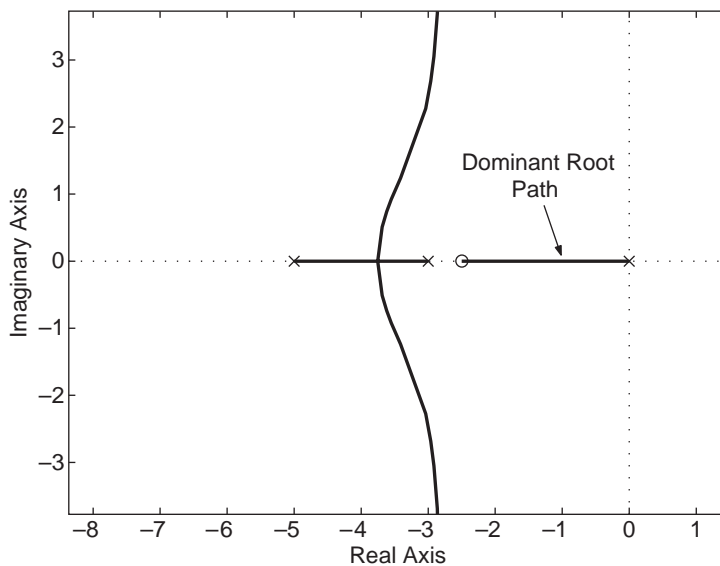
Figure 11.2.22 Root locus for the open-loop transfer function $(s + 2.5)/s(s + 3)(s + 5)$.

The placement of the zero at -3.5 was arbitrary. Normally with this design process, the designer experiments with the placement of the zero to see how it affects other aspects of the performance, such as overshoot and maximum actuator output, for example. The MATLAB interface `rltool` enables you to move the poles and zeros and see the effect on the root locus. It also enables you to compute the resulting step response.

If we place the zero to the right of the pole at $s = -3$, we obtain a root locus plot like that shown in Figure 11.2.22. The particular plot shown corresponds to the choice $1/T_I = 2.5$. The dominant root path is now always real. The dominant root is -2 when $K = 12$. The other roots are $s = -3 \pm 2.45j$. These are somewhat close to the dominant root, so they might influence the response significantly.

Simulation of the two PI designs for a step command shows that their responses are very similar. The design with the zero at -3.5 has a 3.3% overshoot with a settling time of

2.16. The design with the zero at -2.5 has no perceptible oscillation, no overshoot, and a settling time of 1.29. Thus its response is better, but it requires a larger gain, and other aspects of its performance should be checked before accepting it as the final design.

EXAMPLE 11.2.7

Tracking the Dominant Roots

■ Problem

A significant application of the root locus plot is to keep track of the dominant roots. Consider the design of a control algorithm for the plant

$$G_p(s) = \frac{1}{s^2 + 6s + 13}$$

It is required to have zero steady-state error for a step command. Discuss the design of the control algorithm with respect to (a) achieving a desired damping ratio of $\zeta = 0.707$ and (b) minimizing the time constant.

■ Solution

We rule out P action because of the error requirement. Using PI action gives the command transfer function

$$\frac{C(s)}{R(s)} = \frac{K_P s + K_I}{s^3 + 6s^2 + (13 + K_P)s + K_I}$$

The steady-state response for a unit step command is $c_{ss} = K_I/K_I = 1$, and so the error is zero. The root locus equation is

$$1 + K_P \frac{s + 1/T_I}{s^3 + 6s^2 + 13s} = 0$$

The plot is shown in Figure 11.2.23 for the trial value of $1/T_I = 4$. The solution for $\zeta = 0.707$ is shown in the figure and requires a gain of $K = K_P = 3.81$. This solution, however, corresponds to the secondary root. The dominant root for $K = 3.81$ is at $s = -1.53$.

Figure 11.2.24 shows what happens if we forego the damping ratio requirement and instead try to minimize the time constant. For small values of K, the dominant root is real. As we increase K, the real root moves to the left and the complex roots move to the right and eventually

Figure 11.2.23 Root locus for the open-loop transfer function $(s + 4)/(s^3 + 6s^2 + 13s)$ with the design solution $\zeta = 0.707$.

become dominant as K is increased. The smallest possible dominant time constant occurs when the real parts of all the roots are equal. To determine when this occurs, we can use guide 10. With $1/T_I = 4$, the characteristic equation is $s^3 + 6s^2 + (13 + K_P)s + 4K_P = 0$. If all three roots have the same real part $-a$, and the two complex roots have the imaginary part b, then the three roots are $s = -a$ and $s = -a \pm bj$. From guide 10, the sum of the roots must be -6, so we have

$$-6 = -a - a + bj - a - bj = -3a$$

or $a = 2$. Substituting $s = -2$ into the characteristic equation gives $K_P = 5$. Thus, when $K_P < 5$, the dominant root is real. When $K_P > 5$, the dominant roots are complex. The smallest possible dominant time constant is $1/2$.

11.3 STATE-VARIABLE FEEDBACK

In some examples in Chapter 10 we used an internal feedback loop to improve the response of a system. One example is the use of P action with rate feedback in addition to position feedback to replace PD control. This eliminated the numerator dynamics and the resulting overshoot. State-variable feedback is a generalization of that technique and uses some or all of the system's state variables to modify the control signal. With state-variable feedback, we have a better chance of placing the characteristic roots of the closed-loop system in locations that will give desirable performance. In our examples, we will assume that all of the state variables are measurable or at least derivable from other information. For example, a position signal can theoretically be passed through a differentiator to obtain the velocity if a tachometer cannot be used.

| Motor Control Using State-Variable Feedback | **EXAMPLE 11.3.1** |

■ Problem
Consider PD action (Figure 11.3.1a) and state-variable feedback (Figure 11.3.1b) used to control the angular displacement of a dc motor. For the state-variable feedback, the state variables are θ, ω, and the motor current i, and the controller uses P action. Compare the performance of these two systems.

(a)

(b)
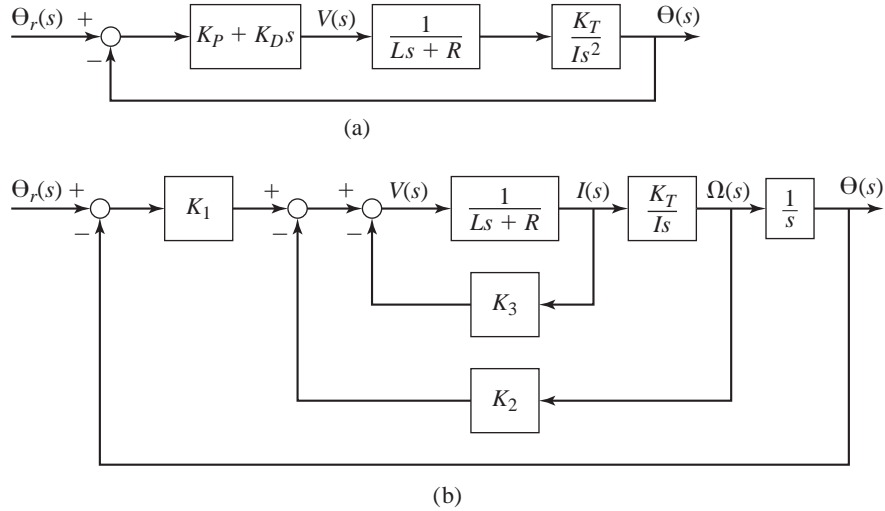
■ Solution

Consider PD control first (Figure 11.3.1a). The closed-loop transfer function is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_T(K_P + K_D s)}{L I s^3 + I R s^2 + K_T K_D s + K_T K_P}$$

The system is stable if $I R K_T K_D > I L K_T K_P$; that is, if $K_P / K_D < R/L$. Note that the coefficient of s^2 is unaffected by the control gains, so it might not be possible to select these gains to achieve the desired transient response. That is, since the highest coefficient LI is fixed and we cannot set the values of the last three coefficients, we cannot specify the locations of the three roots, in general.

Now, consider P action with state-variable feedback (Figure 11.3.1b). The transfer function can be obtained by successively reducing the inner loops of the diagram. The result is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_1 K_T}{L I s^3 + I(R + K_3)s^2 + K_T K_2 s + K_T K_1}$$

There are no numerator dynamics and the system is stable if $(R + K_3)K_2 > LK_1$. The coefficient of s^2 is now affected by the gains, as are the coefficient of s and the constant term. We therefore have more influence over the transient behavior by feeding back all the state variables.

If the current is not fed back, $K_3 = 0$, and the s^2 coefficient is again beyond our influence. The system is unstable if no velocity feedback is used (if $K_2 = 0$).

If the system is stable, the steady-state error is 0 for a step command and is K_2/K_1 for a unit ramp.

Suppose we require the roots to be $s = -1 \pm j$ and -5. These correspond to the polynomial $s^3 + 7s^2 + 12s + 10$. Divide the denominator of the transfer function by LI to obtain

$$s^3 + \frac{R + K_3}{L}s^2 + \frac{K_T K_2}{LI}s + \frac{K_T K_1}{LI}$$

Comparing the coefficients of these two polynomials, we obtain $K_1 = 10LI/K_T$, $K_2 = 12LI/K_T$, and $K_3 = 7L - R$. Given values for K_T, L, R, and I, the next step in the design process would be to determine whether the required motor torque $K_T i$ exceeds the maximum available torque, whose value depends on the specific motor and can be obtained from the motor manufacturer. Because the system is third order, determination of the maximum required torque is best done by simulation. Simulink can be used for this purpose, as demonstrated in Section 11.7.

Integral Control with State-Variable Feedback | **EXAMPLE 11.3.2**

■ Problem

A *type-1 system* requires no additional integrations to obtain a zero steady-state error for a step *command*. P action with state-variable feedback, however, will not give a zero error for a step *disturbance,* even though the plant in the previous example is a type 1 system. Therefore, we replace the P action with the integral controller shown in Figure 11.3.2. Evaluate its performance when enhanced with state-variable feedback.

Figure 11.3.2 Integral control using state-variable feedback.
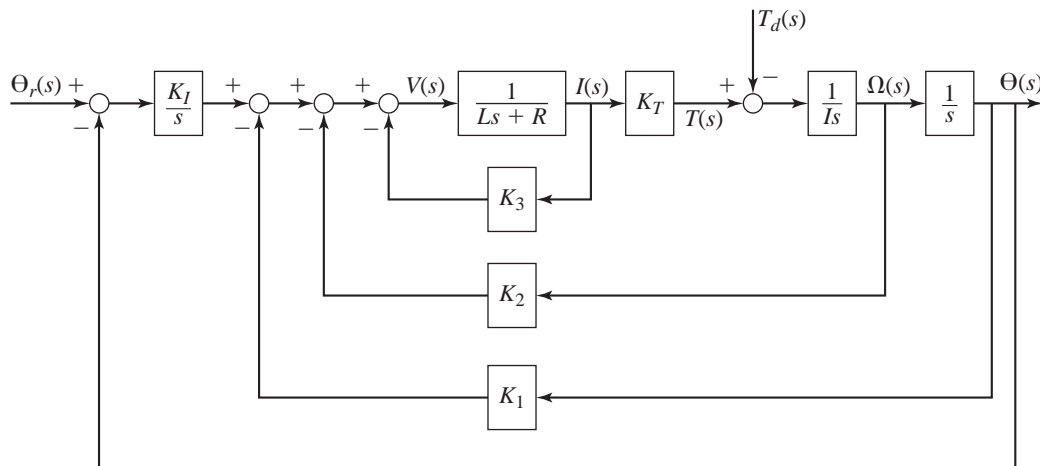
■ Solution

The output equation obtained from the diagram is

$$\Theta(s) = \frac{K_T K_I}{LIs^4 + I(R + K_3)s^3 + K_T K_2 s^2 + K_T K_1 s + K_T K_I}\Theta_r(s)$$

$$- \frac{s(Ls + R + K_3)}{LIs^4 + I(R + K_3)s^3 + K_T K_2 s^2 + K_T K_1 s + K_T K_I}T_d(s)$$

From this we see that if the system is stable, the steady-state error for a step disturbance is zero.

Using the Routh-Hurwitz results of Table 10.5.3, and assuming the gains and parameters are positive, we see that the system is stable if and only if

$$K_T K_1[K_T K_2 I(R + K_3) - K_T K_1 LI] - K_T K_I[I(R + K_3)]^2 > 0$$

which reduces to

$$K_1 K_2 K_T(R + K_3) - IK_I(R + K_3)^2 - LK_T K_1^2 > 0$$

This shows that if $K_1 = 0$ or if $K_2 = 0$, the system is unstable. Thus, the position θ must be fed back in the inner loop as well as in the outer loop. Now, however, with integral control as opposed to proportional control, the gain K_3 is not required for stability.

In general, the form of state-variable feedback using P action is

$$f(t) = K_1[r(t) - x_1] - K_2 x_2 - K_3 x_3 - \cdots \tag{11.3.1}$$

where $r(t)$ is the command input and $f(t)$ is the controller output. If I action is used instead, the form is

$$f(t) = K_I \int [r(t) - x_1]\, dt - K_1 x_1 - K_2 x_2 - K_3 x_3 - \cdots \qquad (11.3.2)$$

CHOICE OF FEEDBACK VARIABLES

With the motor example, the choice of state variables and measured quantities is perhaps obvious. This might not always be the case. Consider the plant

$$\frac{C(s)}{F(s)} = \frac{1}{s(s^2 + 8s + 15)} \qquad (11.3.3)$$

where $f(t)$ is the controller output (the actuating variable) and $c(t)$ is the output to be controlled. The differential equation is

$$\frac{d^3 c}{dt^3} + 8\frac{d^2 c}{dt^2} + 15\frac{dc}{dt} = f(t)$$

A logical choice for state variables is $x_1 = c$, $x_2 = \dot{c}$, and $x_3 = \ddot{c}$. The state equations are

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = f(t) - 15x_2 - 8x_3$$

With proportional control and state-variable feedback, these equations lead to the diagram shown in Figure 11.3.3. This assumes that we can measure or compute $c(t)$, $\dot{c}(t)$, and $\ddot{c}(t)$.

Since this is a type 1 plant, I action is not needed to eliminate the steady-state error for a step command. This can be confirmed from the transfer function.

$$\frac{C(s)}{R(s)} = \frac{K_1}{s^3 + (8 + K_3)s^2 + (15 + K_2)s + K_1}$$

Note that the last three coefficients of the denominator can be independently set with the three gains. This means that we can place the roots anywhere.

Some choices of root locations can result in very large gain values, which might cause very large actuator outputs. Such designs might also be very sensitive to slight changes or uncertainties in the plant parameters or control gains. Simulation is recommended before accepting the design.

The diagram in Figure 11.3.3 implies that we can measure the state variables. The diagram in Figure 11.3.4 shows the controller for the case where we compute x_2

Figure 11.3.3 State-variable feedback for the plant $1/(s^3 + 8s^2 + 15s)$.
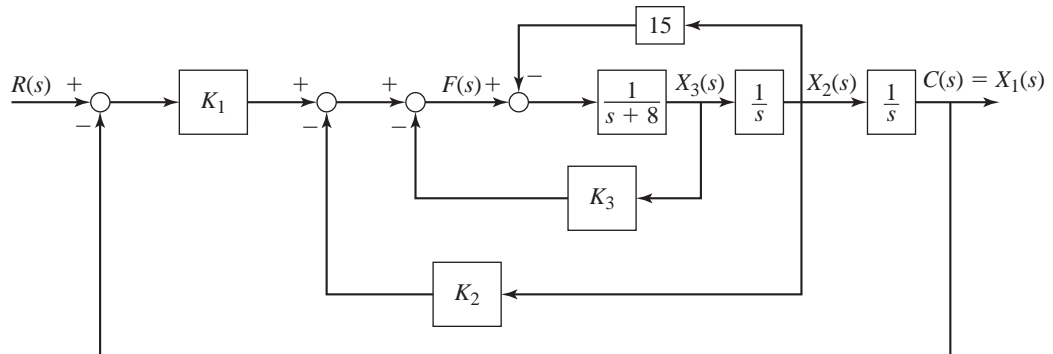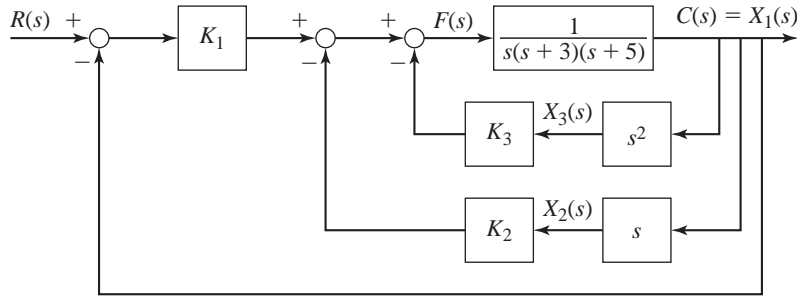
Figure 11.3.4 An equivalent representation of Figure 11.3.3.

Figure 11.3.5 Integral control with state-variable feedback for the plant $1/(s^3 + 8s^2 + 15s)$.
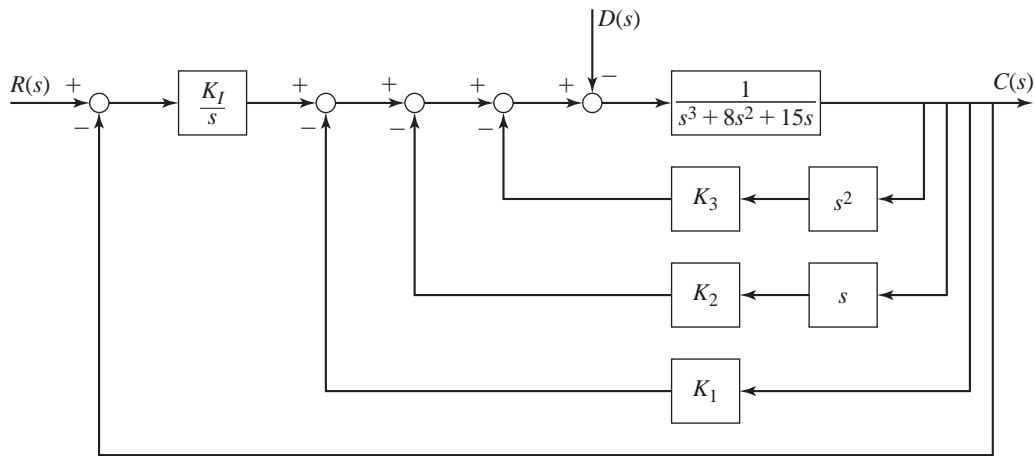
Figure 11.3.6 An equivalent representation of Figure 11.3.5.
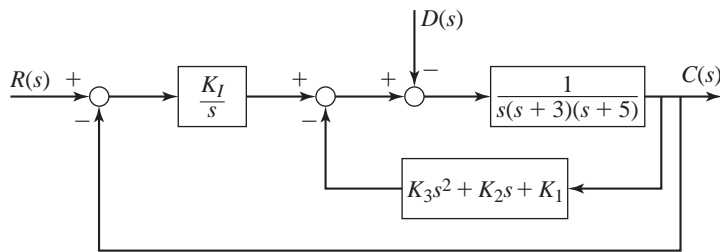
and x_3 from $\dot{c}$ and $\ddot{c}$, respectively. Even if we can measure the state variables, this representation has the advantage of making it easier to obtain the transfer functions, especially if there is a disturbance (Figure 11.3.5). With it we can easily combine the loops as in Figure 11.3.6 and obtain the following output equation.

$$C(s) = \frac{K_I}{s^4 + (8 + K_3)s^3 + (15 + K_2)s^2 + K_1 s + K_I} R(s)$$
$$- \frac{s}{s^4 + (8 + K_3)s^3 + (15 + K_2)s^2 + K_1 s + K_I} D(s)$$
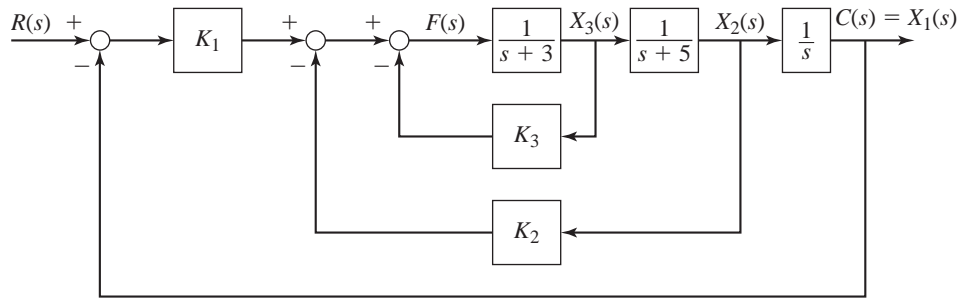
To place the roots at $s = -6 \pm 6j$, $s = -7$, and $s = -8$, the characteristic polynomial must be $s^4 + 27s^3 + 308s^2 + 1752s + 4032$. This requires the gains to be $K_1 = 1752$, $K_2 = 293$, $K_3 = 19$, and $K_I = 4032$.

There are other choices for state variables for (11.3.3). For example, because we can factor the plant model as

$$\frac{C(s)}{F(s)} = \frac{1}{s}\frac{1}{s+5}\frac{1}{s+3}$$

we can choose

$$X_1(s) = C(s) \qquad X_2(s) = \frac{1}{s+5}X_3(s) \qquad X_3(s) = \frac{1}{s+3}F(s)$$

The corresponding diagram using P action is given in Figure 11.3.7. This choice of state variables, however, might or might not correspond to measurable variables.

ROOT LOCUS ANALYSIS

The root locus plot and Bode plot are useful for analyzing the choice of root locations and the resulting gains. For example, suppose we want to analyze the effect of the integral gain K_I for the system shown on Figure 11.3.5. The root locus equation is

$$1 + K_I\frac{1}{s[s^3 + (8 + K_3)s^2 + (15 + K_2)s + K_1]} = 0$$

With the gain values given previously, this becomes

$$1 + K_I\frac{1}{s(s^3 + 27s^2 + 308s + 1752)} = 0$$

There are no zeros and the poles are $s = -13.91$, $s = -6.54 \pm 9.12j$, and $s = 0$. The root locus is shown in Figure 11.3.8. Increasing K_I moves the two secondary roots even farther to the left, but the dominant roots move to the right. This increases the dominant time constant above $1/6$ and increases ζ above 0.707. The dominant time constant cannot be reduced much smaller than $1/6$ by decreasing K_I, because one of the real roots moves to the right and becomes the dominant root. This occurs when $K_I \approx 3950$ and the real root is $s = -5.93$. Thus we cannot make the dominant time constant less than $1/5.93 = 0.169$ by adjusting K_I.

MATRIX METHODS

For higher-order models the algebra required to obtain the characteristic equation can be time-consuming. Just as we use computer methods to obtain the response of such
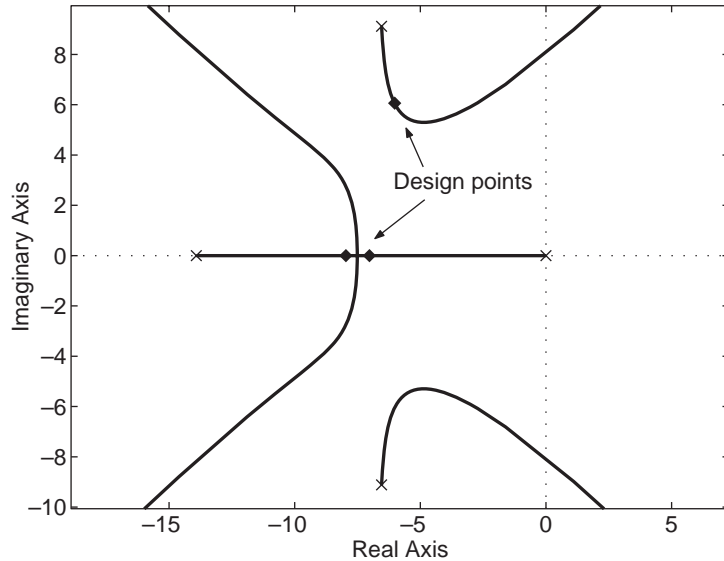
Figure 11.3.8 Root locus plot for the system shown in Figure 11.3.5 for variable K_1.

models, rather than spending time to obtain closed-form solutions, we also may use computer methods to determine the gain values required to place the roots at the desired locations. MATLAB provides the `acker` function for this purpose. It is named for J. Ackermann, who developed the algorithm. The function requires the matrices of the state-variable model.

Consider the model

$$\dot{x}_1 = -x_1 + x_2 + f \quad \dot{x}_2 = x_1 - 2x_2 \tag{11.3.4}$$

With P action, the control algorithm is

$$f(t) = K_1[r(t) - x_1] - K_2 x_2$$

The `acker` function requires the **A** and **B** matrices of the model (see Chapter 5 for more discussion of these matrices). For this model,

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The syntax of the `acker` function is $\mathtt{K = acker(A,B,p)}$, where $\mathtt{p}$ is the array containing the desired closed-loop roots and $\mathtt{K}$ is the array of gains $K_1, K_2, \ldots$ returned by the function. Suppose we want to place the roots at $s = -5$ and $s = -20$. The script file is

```
A = [-1, 1; 1, -2]; B = [1; 0];
K = acker(A,B,[-5, -20])
```

The answer given is $\mathtt{K = [22, 55]}$. Thus the control law is $f(t) = 22[r(t) - x_1] - 55x_2$.

If I action is to be used, the form is

$$f(t) = K_I \int [r(t) - x_1] \, dt - K_1 x_1 - K_2 x_2 - K_3 x_3 - \cdots$$

and we need to augment the state variable model with an additional state variable. This state, denoted w, is defined to be the output of the integrator. That is,

$$w = \int [r(t) - x_1] \, dt \tag{11.3.5}$$

This implies that

$$\dot{w} = r(t) - x_1 \tag{11.3.6}$$

which gives an additional state equation that is appended to the first n equations.

The control algorithm is then expressed as

$$f(t) = -K_1 x_1 - K_2 x_2 - K_3 x_3 - \cdots - (-K_I w)$$

Thus we see that the *negative* of the integral gain corresponds to the gain for the *last* state variable. So the array K returned by the acker function must be interpreted as $K = [K_1, K_2, K_3, \ldots, -K_I]$.

Using I action with the model (11.3.4), we obtain the state equations

$$\dot{x}_1 = -x_1 + x_2 + f \quad \dot{x}_2 = x_1 - 2x_2 \quad \dot{w} = r(t) - x_1$$

So the new state variable is $x_3 = w$, and the new matrices are

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

To place the roots at $s = -1$, $s = -2$, and $s = -2$, the script file is

```
A = [-1, 1, 0; 1, -2, 0; -1, 0, 0]; B = [1; 0; 0];
K = acker(A,B,[-1, -2, -2])
```

The answer given is K = [2, -1, -2], which corresponds to the gains $K_1 = 2$, $K_2 = -1$, and $K_I = 2$. Thus the control law is

$$f(t) = 2 \int [r(t) - x_1]\, dt - 2x_1 + x_2$$

CONTROLLABILITY WITH LINEAR STATE FEEDBACK

We have noted that an advantage of state-variable feedback is that it enables us to place the roots where we want. This is not always possible, however. Consider the following model.

$$\dot{x}_1 = -2x_1 + f \quad \dot{x}_2 = x_1 - x_2 - f$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using P action, the control law is $f(t) = K_1[r(t) - x_1] - K_2 x_2$. Substituting this into the state equations and rearranging gives

$$\dot{x}_1 = (-2 - K_1)x_1 - K_2 x_2 + K_1 r(t) \quad \dot{x}_2 = (1 + K_1)x_1 + (K_2 - 1)x_2 - K_1 r(t)$$

Using the Laplace transform with zero initial conditions gives

$$(s + 2 + K_1)X_1(s) + K_2 X_2(s) = K_1 R(s)$$

$$-(1 + K_1)X_1(s) + (s + 1 - K_2)X_2(s) = -K_1 R(s)$$

The determinant of these equations gives the characteristic equation

$$s^2 + (3 + K_1 - K_2)s + 2 + K_1 - K_2 = 0$$

Letting $b = K_1 - K_2$ gives $s^2 + (3 + b)s + 2 + b = 0$, and we thus see that the equation has only one parameter! Therefore, we cannot specify both roots in general.

Such systems where we cannot specify all the roots by using linear state-variable feedback are said to be "uncontrollable" with linear state feedback. This means that we cannot control the response of every state variable. The `acker` function will not work for such systems, and gives a warning to that effect.

There are advanced methods for determining whether or not a system is controllable, but for our purposes, the warning issued by the `acker` function is sufficient.

	Design of an Active Suspension	**EXAMPLE 11.3.3**

■ Problem

Active suspensions are now used in many vehicles to improve the performance under a variety of conditions. The control gains can be changed according to conditions to provide improved response, whereas with traditional spring-damper systems, the system response depends on the spring and damping constants, which are fixed. Figure 11.3.9 illustrates an active suspension for the quarter car model. An electrohydraulic actuator between the chassis and the wheel assembly provides a force that acts on both and is under feedback control. The system model is

Figure 11.3.9 An active suspension.

$$m_1 \ddot{x}_1 = k_1(y - x_1) - k_2(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) - f$$

$$m_2 \ddot{x}_2 = k_2(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) + f$$

The given parameter values are $m_1 = 36 \, \text{kg}, m_2 = 240 \, \text{kg}, k_1 = 1.6 \times 10^5 \, \text{N/m}, k_2 = 8000 \, \text{N/m}$, and $c = 50 \, \text{N} \cdot \text{s/m}$.

a. Put the model into state variable form.
b. Assume that we can measure all four state variables, and use P action with state-variable feedback. In the original passive suspension system, $k_2 = 1.6 \times 10^4 \, \text{N/m}$ and $c = 98 \, \text{N} \cdot \text{m/s}$, which resulted in characteristic roots at $s = -1.397 \pm 69.94j$, $s = -0.168 \pm 7.779j$. Compute the values of the feedback gains so that the closed-loop roots will be near those of the passive system.

■ Solution

a. Let the state variables be

$$z_1 = x_1 \qquad z_2 = \dot{x}_1 \qquad z_3 = x_2 \qquad z_4 = \dot{x}_2$$

The input is f. The state equations are

$$\dot{z}_1 = z_2 \qquad \dot{z}_2 = \frac{1}{m_1}[k_1 y - (k_1 + k_2)z_1 - cz_2 + k_2 z_3 + cz_4 - f]$$

$$\dot{z}_3 = z_4 \qquad \dot{z}_4 = \frac{1}{m_2}[k_2 z_1 + cz_2 - k_2 z_3 - cz_4 + f]$$

where the disturbance input is the road displacement y. The matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c}{m_1} & \frac{k_2}{m_1} & \frac{c}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c}{m_2} & -\frac{k_2}{m_2} & -\frac{c}{m_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4667 & -1.389 & 222.2 & 1.389 \\ 0 & 0 & 0 & 1 \\ 33.33 & 0.2083 & -33.33 & -0.2083 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{1}{m_1} \\ 0 \\ \frac{1}{m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0278 \\ 0 \\ 0.0042 \end{bmatrix}$$

b. The MATLAB script file is

```
A = [0, 1, 0, 0; -4667, -1.389, 222.2, 1.389;...
     0, 0, 0, 1; 33.33, 0.2083, -33.23, -0.2083];
B = [0; -0.0278; 0; 0.0042];
p = [-1.397+69.94j, -1.397-69.94j, -0.168+7.779j,
     -0.168-7.779j];
K = acker(A,B,p)
```

The result is $K = [-7963, -48, 7934, 47.5]$. For this application, $r(t) = 0$, and the control algorithm is $f(t) = K_1(0 - x_1) - K_2 x_2 - K_3 x_3 - K_4 x_4 = 7963 x_1 + 48 x_2 - 7934 x_3 - 47.5 x_4$.

11.4 TUNING CONTROLLERS

Thus far we have seen several methods for computing appropriate gain values for the controller. In practice, however, the gains should be varied or adjusted from their computed values to determine how sensitive is the performance to the specific gain values. This process is part of "tuning the controller," and it is often possible to improve the performance in this way. Tuning also means computing a preliminary set of gain values based on experiments with the plant.

Before computers became available for simulation, controller tuning meant collecting plant response data and adjusting the gains of an analog controller "in the field" with the controller installed and connected to the plant. Tuning is still done this way with both analog and digital controllers, but it is not always possible to do so for safety or cost reasons. For example, one would not tune the autopilot of a spacecraft during launch!

Now, with computer simulation, it is easier to investigate how a change in gain values or other parameter values will affect the system performance.

THE ZIEGLER-NICHOLS METHODS

Some processes, particularly those involving thermal processes, fluid flow, or chemical reactions, are very difficult to model, especially with linear differential equations. In such applications, if we can do experiments with the plant, there are guidelines available for obtaining preliminary values of the controller gains. These values can then be adjusted by tuning.

The most commonly used methods are those developed by Ziegler and Nichols. They have proved so useful that they are still in use 60 years after their development in the early 1940s. Other methods have been developed; see for example [Seborg, 1989]. Ziegler and Nichols developed two methods. The first method requires the open-loop step response of the plant, while the second method uses the results of experiments performed with the controller already installed. While primarily intended for use with systems for which no analytical model is available, the methods are also helpful sometimes even when a model can be developed, as we will demonstrate.

From a number of experiments and analysis, Ziegler and Nichols found that controllers adjusted according to the following methods usually had a step response that was oscillatory but with enough damping so that the second overshoot was less than 25% of the first (peak) overshoot. This is the *quarter-decay criterion,* and it is sometimes used as a specification.

PROCESS REACTION METHOD

The first method, the *process reaction method,* relies on the fact that many processes have an open-loop unit-step response like that shown in Figure 11.4.1. This process signature is characterized by two parameters, R and L. The constant R is the slope of a line tangent to the steepest part of the response curve, and the constant L is the time at which this line intersects the time axis. Note that first- and second-order linear systems without dead time do not yield positive values for L, and so the method cannot be applied to such systems. Linear systems of third order and higher, with sufficient damping, do yield such a response, however. If so, the Ziegler-Nichols process reaction method recommends the controller gains given in Table 11.4.1.

Note that the table expresses the gains in terms of the *reset time T_I* and the *derivative time T_D*. The relation between the two forms of the PID control law is

$$G(s) = K_P + \frac{K_I}{s} + K_D s = K_P \left(1 + \frac{1}{T_I s} + T_D s \right) \tag{11.4.1}$$

Thus, $K_I = K_P/T_I$ and $K_D = K_P T_D$. Many industrial controllers require you to enter values for K_P, T_I, and T_D, rather than values for K_I and K_D. Some controllers require the value of $1/T_I$, which is called the *reset rate*.

Note that the process reaction method assumes that the response data is from a unit-step input and that the output starts from zero. In practice, this is sometimes difficult to
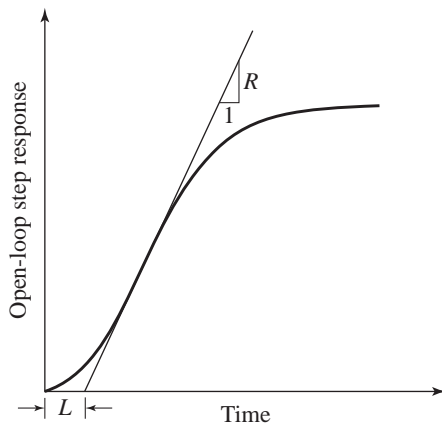
Figure 11.4.1 Open-loop step response of a process.

Table 11.4.1 The Ziegler-Nichols settings.

Controller transfer function

$$G(s) = K_P + \frac{K_I}{s} + K_D s \quad \text{or} \quad G(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s \right)$$

$$K_I = \frac{K_P}{T_I} \qquad\qquad K_D = T_D K_P$$

Control Mode	Process Reaction Method	Ultimate Cycle Method
P control	$K_P = 1/RL$	$K_P = 0.5 K_{P_u}$
PI control	$K_P = 0.9/RL$	$K_P = 0.45 K_{P_u}$
	$T_I = 3.3L$	$T_I = 0.83 P_u$
PID control	$K_P = 1.2/RL$	$K_P = 0.6 K_{P_u}$
	$T_I = 2L$	$T_I = 0.5 P_u$
	$T_D = 0.5L$	$T_D = 0.125 P_u$

do. If the response data $c_i(t)$ starts at a nonzero value c_0, subtract it from the other data to obtain $x_i(t) = c_i(t) - c_0$. Then, if the input has a magnitude M, divide the shifted data by M to obtain $y_i(t) = x_i(t)/M$. The modified data $y_i(t)$ is then used with the Ziegler-Nichols process reaction method. This technique assumes that the process is linear and thus scalable, but it might not be in practice.

It might be difficult to identify the maximum slope R by looking at a plot of the data, and therefore also difficult to estimate L. A polynomial curve fit of the modified data $y_i(t)$ can be used to obtain R and L as follows. After obtaining the curve fit $p_n(t)$ of order n, differentiate it with respect to t to obtain $dp_n(t)/dt = p_{n-1}(t)$, which is a polynomial one order lower than $p_n(t)$. Determine the maximum value of $p_{n-1}(t)$ and the time $t_{\max}$ at which the maximum occurs, either by plotting $p_{n-1}(t)$, by calculus, or with the MATLAB `max` function. The maximum gives the value of R, which can be used along with $t_{\max}$ to construct a straight line of slope R passing through the point $[p_n(t_{\max}), t_{\max}]$. The intersection of this line with the t axis gives L. See Section 11.6.

ULTIMATE-CYCLE METHOD

The *ultimate-cycle method* of Ziegler and Nichols uses experiments with the controller in place. All control modes except proportional are turned off, and the process is started with the proportional gain K_P set at a low value. The gain is slowly increased until the process output starts to show sustained oscillations. The period of these oscillations is called the *ultimate period,* and is denoted by P_u. Let the proportional gain setting at this condition be denoted by K_{P_u}, which is called the *ultimate gain*. The Ziegler-Nichols recommendations are given in Table 11.4.1 in terms of these parameters.

The proportional gain is lower for PI control than for P control and higher for PID control, because I action increases the order of the system, which tends to destabilize it; thus, a lower gain is needed. On the other hand, D action tends to stabilize the system; therefore, the proportional gain can be increased without degrading the stability characteristics. Because the settings were developed for typical cases out of many types of processes, final tuning of the gains by simulation or in the field is usually necessary.

Sometimes the Ziegler-Nichols methods can be applied to an analytical model. Example 11.4.1 shows how this is done with the ultimate-cycle method.

EXAMPLE 11.4.1

Ultimate-Cycle Method with a Third-Order Plant

■ Problem
Use the ultimate-cycle method to obtain the gains for (a) P action, (b) PI action, and (c) PID action. Analyze the unit-step response of each design. The plant is

$$G_p(s) = \frac{10}{2s^3 + 12s^2 + 22s + 12}$$

■ Solution
The ultimate-cycle method starts by using proportional action only. The transfer function for P action with this plant is

$$\frac{C(s)}{R(s)} = \frac{10K_P}{2s^3 + 12s^2 + 22s + 12 + 10K_P}$$

The characteristic equation is

$$2s^3 + 12s^2 + 22s + 12 + 10K_P = 0 \tag{1}$$

To apply the ultimate-cycle method, we must first find the ultimate period P_u and the associated gain K_{P_u}. To do this analytically, we note that sustained oscillations occur when a pair of roots is purely imaginary and the rest of the roots have negative real parts. To find when this occurs, set $s = j\omega_u$, where ω_u is the as-yet-unknown frequency of oscillation. Then $s^2 = -\omega_u^2$ and $s^3 = -j\omega_u^3$. Substituting these into the characteristic equation we obtain

$$-2j\omega_u^3 - 12\omega_u^2 + 22j\omega_u + 12 + 10K_{P_u} = 0$$

Collect the real and imaginary parts.

$$\left(12 + 10K_{P_u} - 12\omega_u^2\right) + j\left(-2\omega_u^3 + 22\omega_u\right) = 0$$

To satisfy this equation, both the real and imaginary terms must be zero. So we obtain two equations to solve for the two unknowns ω_u and K_{P_u}. These are

$$12 + 10K_{P_u} - 12\omega_u^2 = 0 \qquad (2)$$

$$-2\omega_u^3 + 22\omega_u = 0$$

The latter equation yields the nonoscillatory solution $\omega_u = 0$ as well as the one of interest—namely,

$$\omega_u = \pm\sqrt{11} = \pm 3.317$$

Substituting this into equation (2), we see that the ultimate gain is $K_{P_u} = 12$. The ultimate period is

$$P_u = \frac{2\pi}{\omega_u} = \frac{2\pi}{\sqrt{11}} = 1.89$$

For sustained oscillations to occur, as required by the ultimate-cycle method, the third root must be negative. The first two roots are $s_1 = j\omega_u$ and $s_2 = -j\omega_u$. The third root can be determined from guide 10 of the root locus guides. Dividing equation (1) by 2 gives

$$s^3 + 6s^2 + 11s + 6 + 5K_P = 0$$

Guide 10 says that the sum of the three roots must be -6. Thus,

$$+j\omega_u - j\omega_u + r_3 = -6$$

which gives $r_3 = -6$. Since $r_3 < 0$, the assumption of neutral stability is justified. Thus the solutions for ω_u and K_{P_u} are valid and can be used with Table 11.4.1 to compute the PID gains.

We obtain the following results. For P action, $K_P = 0.5K_{P_u} = 6$. For PI action,

$$K_P = 0.45K_{P_u} = 5.4 \qquad T_I = 0.83P_u = 1.57 \qquad K_I = \frac{K_P}{T_I} = 3.44$$

For PID action,

$$K_P = 0.6K_{P_u} = 7.2 \qquad T_I = 0.5P_u = 0.947 \qquad K_I = \frac{K_P}{T_I} = 7.62$$

$$T_D = 0.125P_u = 0.236 \qquad K_D = K_P T_D = 1.7$$

The closed-loop transfer function for P action is

$$\frac{C(s)}{R(s)} = \frac{60}{2s^3 + 12s^2 + 22s + 72}$$

For PI action,

$$\frac{C(s)}{R(s)} = \frac{54s + 34.4}{2s^4 + 12s^3 + 22s^2 + 66s + 34.4}$$

For PID action,

$$\frac{C(s)}{R(s)} = \frac{17s^2 + 72s + 76.2}{2s^4 + 12s^3 + 39s^2 + 84s + 76.2}$$

The unit-step responses are shown in Figure 11.4.2. Note the improvement as the control action is changed from P to PI to PID. First, the steady-state error is eliminated, and then the oscillations and settling time are reduced.

Figure 11.4.2 Unit-step responses of the controllers obtained in Example 11.4.1. (a) P action. (b) PI action. (c) PID action.
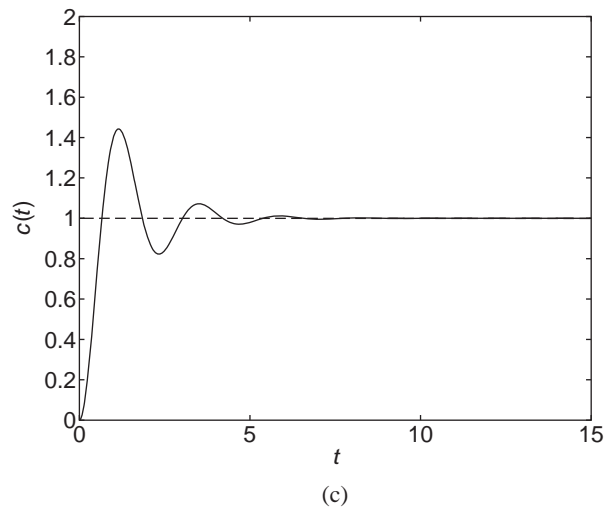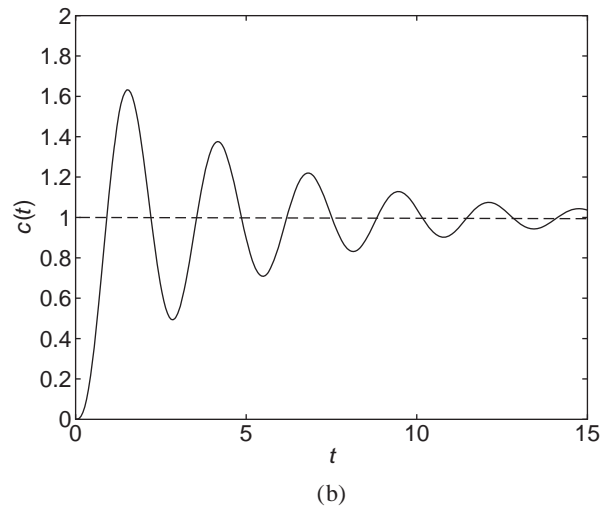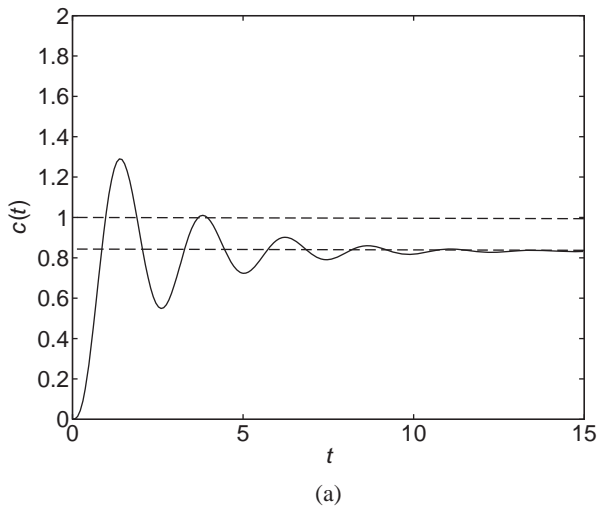
Note that this analysis uses only proportional control, but the gains for other control laws can be computed with these results. The power of the method is shown by the fact that we did not have to solve for the roots of a cubic or quartic equation. Note also that I action applied to this system would result in a fourth-order system to analyze—a difficult task—were it not for the Ziegler-Nichols method.

The Ziegler-Nichols methods are used to obtain gain values that are "in the ball-park." The engineer can use these values as a starting point for tuning the gains to obtain improved performance.

Tuning a PID Controller | **EXAMPLE 11.4.2**

■ Problem
Consider the PID controller designed in Example 11.4.1. Use the root locus plot to adjust the gains to reduce the overshoot.

■ Solution
The closed-loop transfer function for PID control of this plant is

$$\frac{C(s)}{R(s)} = \frac{10K_D s^2 + 10K_P s + 10K_I}{2s^4 + 12s^3 + (22 + 10K_D)s^2 + (12 + 10K_P)s + 10K_I}$$

When the controller transfer function is expressed in the form

$$G(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s \right)$$

we can see that changing the proportional gain K_P changes the proportional, integral, and derivative gains by the same factor. For PID action, the Ziegler-Nichols settings give the following values of K_D and K_I in terms of K_P.

$$K_D = T_D K_P = 0.236 K_P \qquad K_I = \frac{K_P}{T_I} = \frac{K_P}{0.947} = 1.056 K_P$$

Thus, we can express the characteristic equation as

$$2s^4 + 12s^3 + (22 + 2.36K_P)s^2 + (12 + 10K_P)s + 10.56K_P = 0$$

and the root locus equation in terms of the parameter K_P is

$$1 + 1.18K_P \frac{s^2 + 4.24s + 4.47}{s(s^3 + 6s^2 + 11s + 6)} = 0$$

Thus the root locus parameter is $K = 1.18K_P$.

The root locus plot is shown in Figure 11.4.3. When K_P has the value recommended by the Ziegler-Nichols method, $K_P = 7.2$, the roots are $s = -0.77 \pm 2.67j$, $s = -1.98$, and $s = -2.48$. The dominant root is marked on the root locus, and has a time constant of $\tau = 1/0.77 = 1.3$ and a small damping ratio of $\zeta = 0.28$. The root locus shows that decreasing K_P from 7.2 increases the time constant slightly to $\tau = 1/0.523 = 1.9$ but increases the damping ratio quite a bit. Thus the system will respond somewhat more slowly, but the overshoot should be greatly reduced to perhaps none at all.

The root locus shows that we can improve the response by decreasing K_P until the dominant root has a damping ratio of 1. This occurs at the breakaway point ($s = -0.523$) and requires that $K_P = 0.306$, $K_D = T_D K_P = 0.236(0.306) = 0.070$, $K_I = K_P/T_I = K_P/0.947 = 0.323$. The improved step response, marked "Tuned," is shown in Figure 11.4.4, along with the response obtained with the Ziegler-Nichols settings, marked "Original."

Figure 11.4.3 Root locus plot for Example 11.4.2.
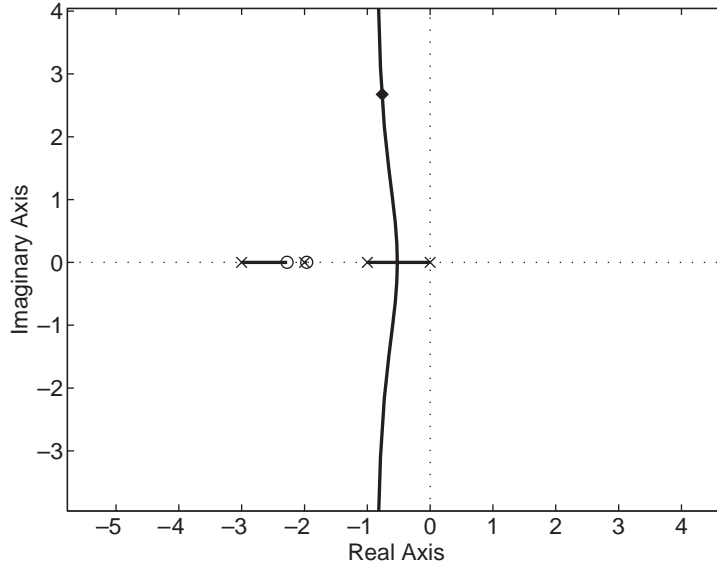
Figure 11.4.4 Unit-step responses for Example 11.4.2.
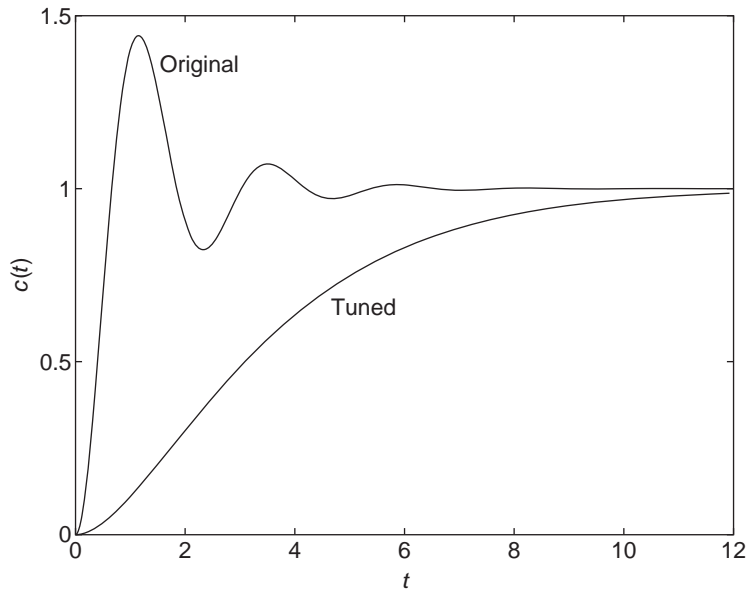
11.5 SATURATION AND RESET WINDUP

We have seen that a motor-amplifier combination can produce a torque proportional to the input voltage over only a limited range. No amplifier can supply an infinite current, and there is a maximum current that the motor can withstand without overheating or demagnetizing. Thus, there is a maximum torque the motor can produce in either direction (clockwise or counterclockwise). Whenever an actuator reaches its limit, it is said to be *saturated*. When the controller commands the actuator to produce an output greater than its limit, the actuator is said to be *overdriven*.
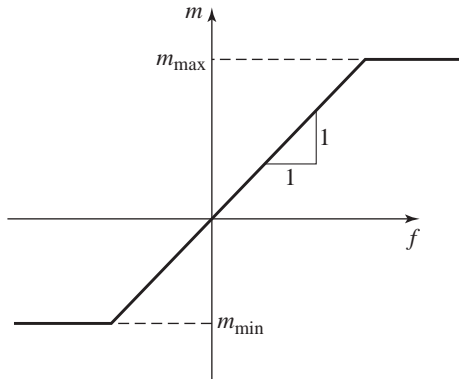
Figure 11.5.1 Saturation nonlinearity.

A graph of the saturation nonlinearity is shown in Figure 11.5.1, where $m(t)$ is the actuator response and $f(t)$ is the commanded value of $m(t)$ generated by the controller.

For the motor, the graph of the saturation nonlinearity is symmetric, so $m_{\min} = -m_{\max}$, but this is not always the case. An example is a flow control valve. Once it is fully open, the flow rate $m(t)$ cannot be made greater without increasing the supply pressure. Of course, no flow occurs when the valve is fully closed, so $m_{\min} = 0$. Note, however, that even if the nonlinearity is nonsymmetric, its slope is always 1 because $m(t) = f(t)$ when there is no saturation.

The possibility of actuator saturation must be considered when designing control systems, otherwise the mathematical model can predict unrealistic results. For example, we can use the closed-loop transfer function to select gain values to achieve a desired settling time, but the real system will respond slower than predicted if the actuator saturates.

RESET WINDUP

Any controller with integral action can exhibit the phenomenon called *reset windup* or *integrator buildup* when overdriven, if it is not properly designed.

Consider the PI-control law.

$$m(t) = K_p e(t) + K_I \int_0^t e(t)\, dt \tag{11.5.1}$$

For a step command input, the proportional term responds instantly and causes saturation immediately if the command input is large enough. The integral term, however, does not respond as quickly. It integrates the error signal and can cause saturation some time later if the error remains large for a long enough time. As the error decreases, the proportional term no longer causes saturation. The integral term, however, continues to increase as long as the error has not changed sign, and thus the manipulated variable remains saturated. Even though the output is near its desired value, the manipulated variable remains saturated until sometime after the error has reversed sign. The result can be a large overshoot in the response of the controlled variable.

Consider the PI control system shown in Figure 11.5.2. Suppose the plant is $G_p(s) = 1/4s$. To obtain a time constant of $\tau = 1.5$ and a damping ratio of $\zeta = 1$, we compute the gains to be $K_P = 16/3$ and $K_I = 16/9$. Ignoring any limits on actuator output, we predict the unit-step response to be that of the curve labeled "Unlimited" shown in Figure 11.5.3. Now suppose that the actuator limits are $m_{\max} = 1$ and $m_{\min} = -1$. Simulation, using Simulink for example, gives the curve labeled "Actuator Limited." Clearly the overshoot is much greater and the response time is longer.

Figure 11.5.2 PI control with saturation nonlinearity.
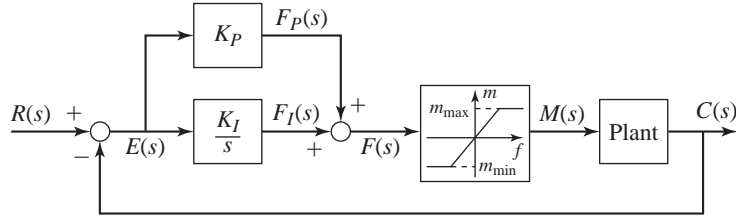
Figure 11.5.3 Effects of actuator saturation on step response.
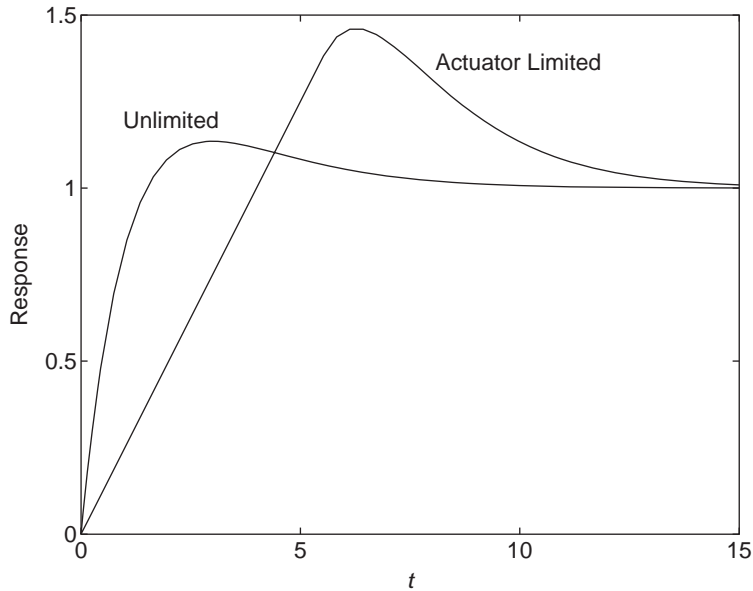
To see what is happening, consider Figure 11.5.4. The actuator output is plotted along with the output of the I action and the error, $e(t) = 1 - c(t)$. For zero initial conditions, the error is initially $e(0) = 1 - 0 = 1$, and the actuator immediately saturates because the proportional term is $K_P e = 16/3 > 1$. In the saturated mode, $m(t) = 1$, $\dot{c}(t) = m(t)/4 = 1/4$, $c(t) = t/4$, and $e(t) = 1 - t/4$, until $f < 1$. Note that the error eventually changes sign, which means that the output c is now larger than the desired value, but the I action term keeps the system saturated until some time later. This is what causes the overshoot often observed due to reset windup.

SELECTING GAINS TO AVOID SATURATION

Consider again the PI control system shown in Figure 11.5.2 for an arbitrary plant. If the maximum step command is r_{max} and if the output c is zero at startup at $t = 0$, then the output f_I of the integral term is zero and the output of the proportional term is $f_P = K_P[r_{max} - c(0)] = K_P r_{max}$. Then the maximum value K_P can have without overdriving the actuator is

$$K_P = \frac{m_{max}}{r_{max}} \tag{11.5.2}$$

This analysis does not account for the possibility that the integral term might cause saturation at some later time, but (11.5.2) provides a starting point for the design.
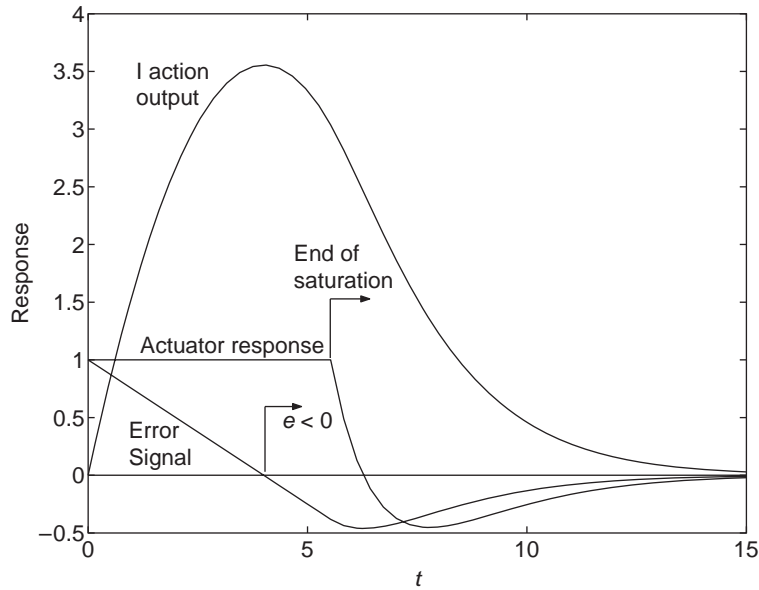
Figure 11.5.4 Effects of reset windup.

Computing K_P in this way removes some flexibility from the design. Suppose the plant is $G_p(s) = 1/(Is + c)$. In the linear (unsaturated) region, the transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P s + K_I}{Is^2 + (c + K_P)s + K_I}$$

With K_P fixed from (11.5.2), the only remaining design variable is K_I, for which the root locus equation is

$$1 + \frac{K_I}{I} \frac{1}{s[s + (c + K_P)/I]} = 0$$

The poles are $s = 0$ and $s = -(c + K_P)/I$. The $\pm 90°$ asymptotes intersect the real axis at the breakaway point at $s = -(c + K_P)/2I$, and thus the smallest possible dominant time constant is $\tau = 2I/(c + K_P)$.

If we again consider the plant $G_p(s) = 1/4s$ with $r_{max} = m_{max} = 1$, then $I = 4$, $c = 0$, and $K_P = 1$ from (11.5.2). The smallest possible dominant time constant is $\tau = 8$. So we see that our earlier design with $\tau = 1.5$ was unrealistic and will cause saturation. With $K_P = 1$, any choice of $\zeta \leq 1$ will give a time constant of 8. Choosing $\zeta = 1$ requires that $K_I = 1/16$. Figures 11.5.5 and 11.5.6 compare the two designs. With this choice of gains, no reset windup occurs, the actuator is not overdriven, and the overshoot is much less than before. However, the response now takes longer to settle down to its steady-state value.

To decide which response is optimal, the designer must look more closely at the requirements for the particular application. In many systems, the occurrence of saturation and response overshoot indicates that the system is using more energy than is necessary to accomplish the task.

Reset windup can be prevented by selecting the gains so that saturation will never occur. This requires knowledge of the maximum input magnitude that the system will encounter. In a PI controller, any saturation initially caused by the proportional term is easy to predict, because it responds instantly to the error. The integral term might cause saturation also, but at a later time, because its output requires time to accumulate.

Figure 11.5.5 Response of two PI control systems.
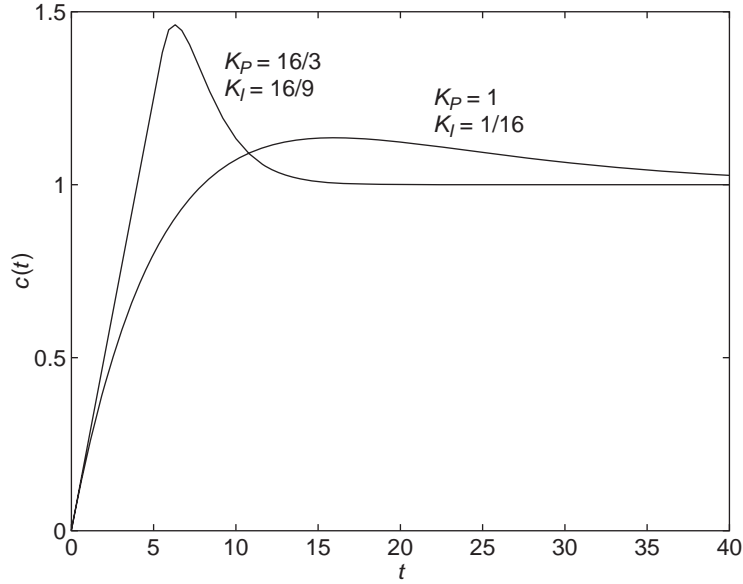
Figure 11.5.6 Actuator response of two PI control systems.
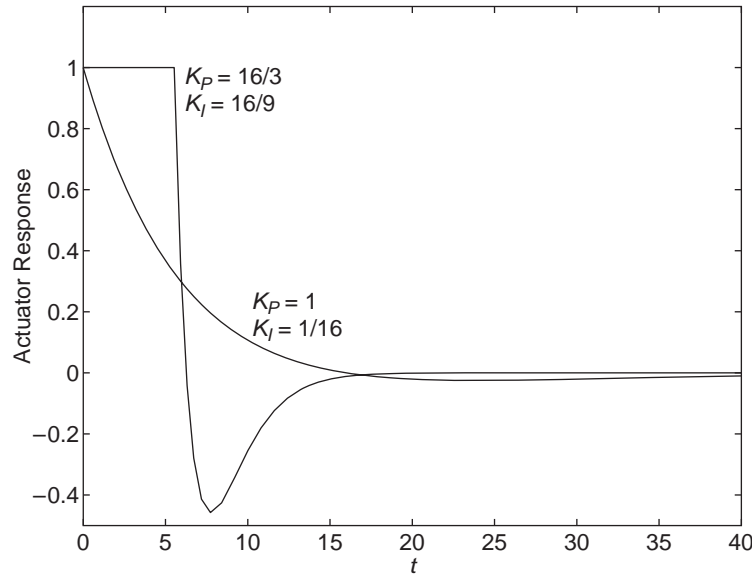
OTHER CONTROL STRUCTURES

Control structures other than the classic PID structure sometimes can be used to reduce the problems associated with actuator saturation and reset windup. One of these is the use of an internal feedback loop in conjunction with the main controller, which uses I action (Figure 11.5.7). In Chapter 10 we saw that such an arrangement requires a much smaller actuator output at the cost of a slightly longer response time.

For the plant $G_p(s) = 1/Is$ with a step command and $\zeta = 1$, we can obtain the analytical solution for the actuator output $m(t)$. From this we can determine that $m(t)$ reaches a maximum of $r_{max} K_2/5.44$ at $t = 2I/K_2$. Thus to avoid overdriving the

actuator, the gain K_2 should be chosen to be no larger than

$$K_2 = 5.44 \frac{m_{max}}{r_{max}} \tag{11.5.3}$$

To obtain $\zeta = 1$ requires that the integral gain be

$$K_I = \frac{K_2^2}{4I} = \frac{29.6}{4I} \left(\frac{m_{max}}{r_{max}} \right)^2 \tag{11.5.4}$$

Suppose the plant is $G_p(s) = 1/4s$ with $r_{max} = m_{max} = 1$. Then $K_2 = 5.44$ and $K_I = 29.6/16 = 1.85$. Figures 11.5.8 and 11.5.9 show the output and the actuator response for this design and the PI design using $K_P = 1$ and $K_I = 1/16$. It is apparent that the system with internal feedback gives a better response without demanding more from the actuator. Further analysis, however, should be done to compare the two designs' responses to a disturbance and to other inputs such as ramps. This comparison is given in [Palm, 2000] and [Phelan, 1977].

AN ANTI-WINDUP SYSTEM

It is sometimes impossible to compute the gains to avoid reset windup and saturation because either the input magnitudes are not known well enough in advance or because the required mathematics is too cumbersome. There are some designs, however, that act to prevent reset windup. One of these anti-windup designs is shown in Figure 11.5.10 [Franklin, 1994]. It consists of a feedback loop with a gain K_A around the integrator.

Figure 11.5.9 Actuator
responses for PI control and I
action with internal feedback.
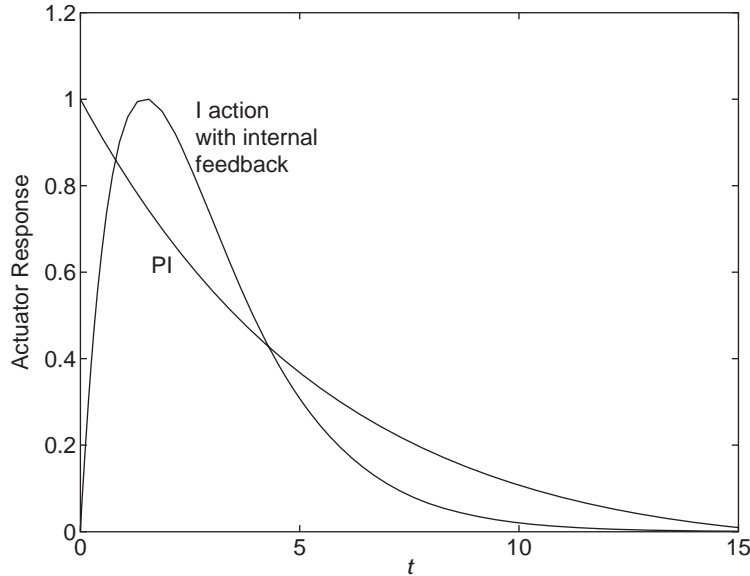
Figure 11.5.10 An
anti-windup system.
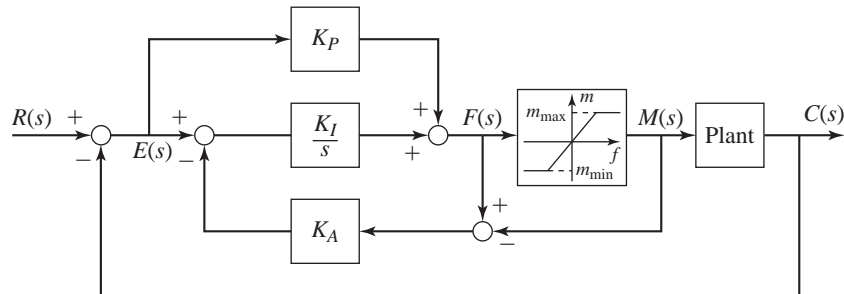
The actuator output is $M(s)$. Its output commanded by the controller is $F(s)$. When the actuator is not saturated, $m = f$ and the comparator in the feedback loop produces a zero output. Thus the loop has no effect. When the actuator is saturated, however, so that $m = m_{max}$ but $f > m_{max}$, a positive signal is fed by the comparator to the gain K_A, which decreases the signal to the integrator. Thus, the integrator output begins to decrease, causing the actuator to become unsaturated.

Note that the P action bypasses the inner feedback loop, so the proportional signal is unaffected by the anti-windup loop. The gain K_A should be chosen large enough to allow the loop to follow changes in the error signal. Figure 11.5.11 compares the response of this system using $K_P = K_I = 16$ and $K_A = 10$ for the plant $G_p(s) = 1/4s$ with $m_{max} = r_{max} = 1$ to the response of the classic PI controller using $K_P = K_I = 16$. Note that the anti-windup system does not prevent the actuator from being overdriven, but it does eliminate reset windup that causes overshoot.

Limits on the actuator output obviously can prevent us from achieving the desired response time. We can, however, reduce the effects of saturation by choosing gains that prevent it and reset windup, by choosing alternative control structures that place less demand on the actuator, or by using a control structure that prevents reset windup. For plants other than first order, the mathematics required to analyze such designs is formidable, so the design approach is to assume the system is operating in the linear
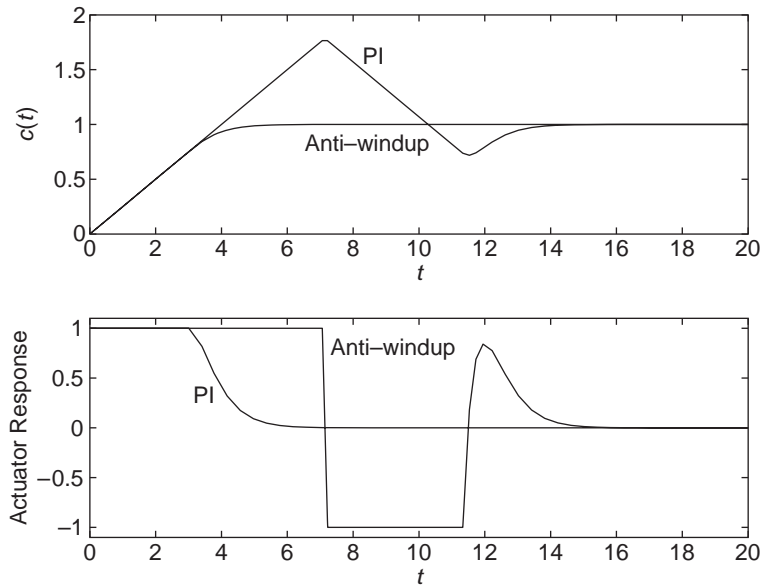
Figure 11.5.11 Responses for PI control and an anti-windup system.

region, and compute the response for the most severe inputs expected. Then simulation is used to check for saturation, and the gains are adjusted to prevent it. Because of the saturation nonlinearity, Simulink is well suited for such simulations.

11.6 MATLAB APPLICATIONS

We have already seen many of the MATLAB features that are particularly helpful for doing the design tasks described in this chapter. These include the transfer function and state-variable functions, such as `tf` and `ss`; the transient response functions `step` and `lsim`; and the frequency response function `bode`. In Section 11.3 we introduced the `acker` function, which is used to compute the state-variable feedback gains.

The polynomial regression tools, including `polyfit`, which were introduced in Chapter 1, are useful for applying the Ziegler-Nichols process reaction method discussed in Section 11.4.

This section introduces the MATLAB functions relevant to this chapter.

ROOT LOCUS ANALYSIS WITH MATLAB

The MATLAB Control System toolbox provides several useful commands for producing root locus plots and for extracting information from them. The basic function is `rlocus(sys)`, which displays the root locus. Additional commands are available for enhancing the plot and for obtaining root locations and gain values from the plot.

The function `rlocus(sys)` computes and displays the root locus plot for the equation

$$D(s) + KN(s) = 0 \tag{11.6.1}$$

where `sys` corresponds to the transfer function $N(s)/D(s)$ and the parameter K varies from 0 through a large positive value determined by MATLAB. The order of $N(s)$ must be no greater than the order of $D(s)$. It is recommended that the highest coefficients in $N(s)$ and $D(s)$ both be *unity* so that the values of K displayed by MATLAB will be interpreted correctly.

Figure 11.6.1 Root locus
plot of $2s^2 + cs + 8 = 0$.
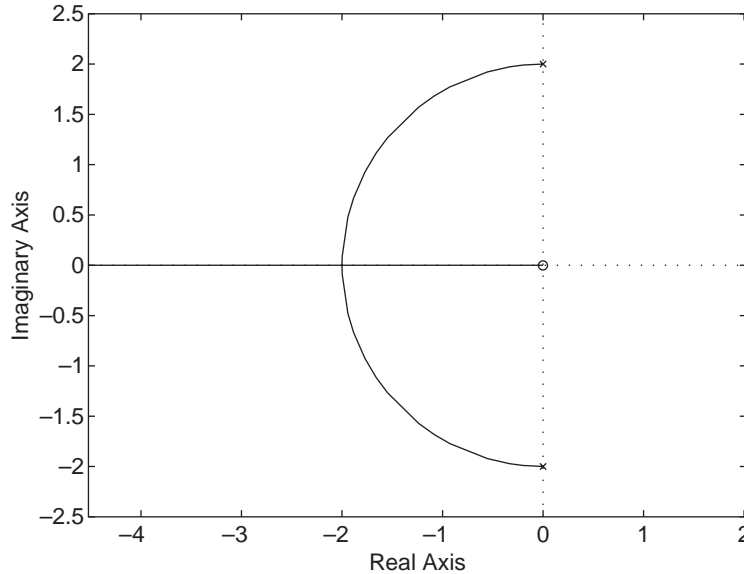
MATLAB refers to the parameter K as the "gain." The starting points of the roots corresponding to $K = 0$ are called the "poles" and are marked with a cross ($\times$). These are the roots of $D(s) = 0$. The finite termination points that the roots approach as $K \to \infty$ are called the "zeros" and are marked with a circle ($\bigcirc$). These are the roots of $N(s) = 0$.

You should use equal scaling on the real and imaginary axes so that circular root loci will display as circles and so that the angle θ associated with the damping ratio, $\zeta = \cos\theta$, can be properly interpreted. This can be done by including the command `axis equal` after the `rlocus` function.

The plot shown in Figure 11.1.3b corresponds to the characteristic equation $2s^2 + cs + 8 = 0$. This may be put into the standard form (11.6.1) by rewriting it as

$$s^2 + 4 + \frac{c}{2}s = 0$$

where $K = c/2$, $D(s) = s^2 + 4$, and $N(s) = s$. The poles are $s = \pm j2$. The zero is $s = 0$. To obtain this root locus plot, the session is

```
≫sys1 = tf([1,0],[1,0,4]);
≫rlocus(sys1),axis equal
```

Note that the numerator must be expressed as `[1, 0]` because the numerator polynomial is actually $s + 0$. The result is shown in Figure 11.6.1.

PLACING A (ζ, ω_n) GRID

The `sgrid` command superimposes an s plane grid of constant ζ and constant ω_n lines on the root locus plot. This grid is useful for locating roots that satisfy performance specifications stated in terms of ζ and ω_n. The alternate syntax `sgrid(zeta,omega)` superimposes an s plane grid of constant ζ and constant ω_n lines on the root locus plot, using the values contained in the vectors `zeta` and `omega`.

Suppose we want to see if any dominant roots of the following equation have damping ratios in the range $0.5 \leq \zeta \leq 0.707$, and undamped natural frequencies in the range $0.5 \leq \omega_n \leq 0.75$.

$$s^3 + 3s^2 + 2s + K = 0$$

The required session is

```
≫sys2 = tf(1,[1,3,2,0]);
≫rlocus(sys2),sgrid([0.5,0.707],[0.5,0.75]),axis equal
```

You may use empty brackets if you want to omit either lines of constant ζ or lines of constant ω_n. For example, to see just the damping ratio lines corresponding to $\zeta = 0.5$ and $\zeta = 0.707$, you would use the form `sgrid([0.5,0.707],[])`.

OBTAINING INFORMATION FROM THE PLOT

Once the plot is displayed you can pick a point from this plot to find the gain K required to achieve ζ and ω_n values in the desired range. To do this you can use the `rlocfind` function discussed later in this section, or you can place the cursor on the plot at the desired point. Left click to display the root value (although labeled "pole," this is not the same as the starting points marked by ×). The gain value, the damping ratio (labeled "damping"), the percent overshoot, and the undamped natural frequency ω_n (labeled "frequency") are also displayed. You can move the cursor along the plot and view the updated values. Right-clicking anywhere within the plot area brings up a menu that enables the plot to be edited.

Note that the percent overshoot value displayed on the screen with this method is computed from the formula for $M_\%$ given in Table 9.3.2, and thus it does *not* include the effects of numerator dynamics on the overshoot.

You can use this method to see how sensitive the root location is to a change in the value of the gain K. Sometimes it is inadvisable to design a critically damped system, which corresponds to two or more equal, real roots, because such designs can be very sensitive to parameter variation and parameter uncertainty. Near such roots, a slight change in the gain K can cause a large change in the root location.

An alternate syntax to obtain the root values for specified values of K is `r = rlocus(sys,K)`. This returns the root locations in the vector `r` corresponding to the user-specified vector of gain values `K`. The matrix `r` has `length(K)` columns and its jth column contains the roots for the gain value `K(j)`. No plot is displayed.

MATLAB provides the `rlocfind` function to obtain information about roots and gain values from the plot on the screen. The syntax `[K,r] = rlocfind(sys)` enables you to use a cursor to obtain the value of the gain `K` corresponding to a specified point on a root locus plot. The vector `r` contains the roots corresponding to this gain value. The advantage of using `rlocfind` is that it returns *all* the roots for a given value of K, whereas left-clicking on the curve shows only some of the roots. The `rlocfind` function, which must follow the `rlocus` command, generates a cursor on the screen and waits for the user to press the mouse button after positioning the cursor over the desired point on the locus. Once the button is pressed, you will see on the screen the coordinates of the selected point, the gain value at that point, the roots closest to that point, and the other roots that correspond to the gain value.

For example, the following session displays the root locus with a grid line corresponding to $\zeta = 0.707$. This enables us to use the cursor to find the dominant roots that have a damping ratio of $\zeta = 0.707$ and the corresponding gain value K.

```
>>sys3 = tf(1,[1,3,2,0]);
>>rlocus(sys3),axis equal,sgrid(0.707,[]),...
    [K,r] = rlocfind(sys3)
```

The dominant roots selected this way will not have a damping ratio of $\zeta = 0.707$ exactly, because you cannot position the cursor exactly at the intersection of the locus and the $\zeta = 0.707$ line. To reduce this inaccuracy, you can enlarge the plot by enabling the Edit Plot button and clicking the magnifying glass icon on the menu bar of the figure window.

Another syntax is `[K,r] = rlocfind(sys,p)`, which computes a root locus gain K for each desired root location specified in the vector `p` (or a gain for which one of the closed-loop roots is near the desired location). The jth entry in the vector `K` is the computed gain for the root location `r(j)`. The jth column of the matrix `r` contains the resulting closed loop roots.

The real power of the root locus method only becomes apparent when it is applied to a model higher than second-order, as in the following example.

The `rlocus` function is especially useful for applications in this chapter where the model is third order or higher, because simple expressions for the time constant, damping ratio, overshoot, etc. are not available to assist in setting the gains. The root locus plot is useful for analyzing the effects on the roots of changing a control system gain. It is also helpful for understanding how the uncertainty in a parameter value will affect the performance of the control system. We now give two examples of these applications. More applications of the root locus for control system design are given in Chapter 12.

EXAMPLE 11.6.1   Varying the Integral Gain

■ Problem

The characteristic equation of the liquid-level control system of Example 10.8.5 is

$$8s^3 + (6 + 8K_1)s^2 + (1 + 4K_1 + K_2)s + K_I = 0$$

In that example, the gain values were selected to give the characteristic roots $s = -0.5 \pm 0.5j$ and $s = -5$. The gain values are $K_1 = 21/4$, $K_2 = 22$, and $K_I = 20$. Investigate the effects on the roots and on the steady-state ramp-command error of changing the value of K_I.

■ Solution

The steady-state error for a unit-ramp command is $e_{ss} = 44/K_I$, and thus its nominal value for $K_I = 20$ is $e_{ss} = 44/20 = 2.2$.

Substituting $K_1 = 21/4$ and $K_2 = 22$ into the characteristic equation gives

$$8s^3 + 48s^2 + 44s + K_I = 0$$

With the Routh-Hurwitz condition we immediately see that the system is unstable if $48(44) < 8K_I$; that is, if $K_I > 264$.

Rearranging the characteristic equation into the standard root-locus form, we have

$$1 + \frac{K_I}{8} \frac{1}{s^3 + 6s^2 + 5.5s} = 0$$

With the highest coefficients normalized to unity, the root locus gain is $K = K_I/8$. The poles are the roots of the denominator, which are $s = 0, -1.129, -4.87$. There are no zeros because the numerator is a constant.

The MATLAB session is

```
>>sys = tf(1, [1, 6, 5.5, 0]);
>>rlocus(sys)
```
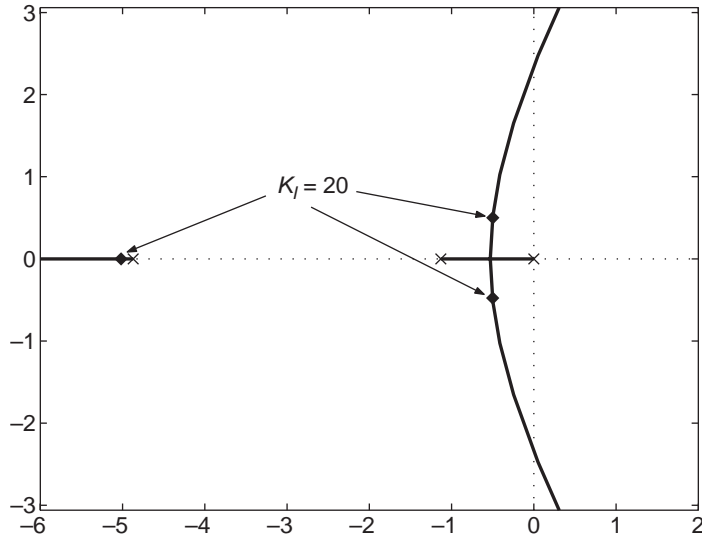
Figure 11.6.2 Root locus plot for Example 11.6.1.

The root locus plot is shown in Figure 11.6.2. The Plot Editor has been used to annotate the plot. The locations of the roots when $K_I = 20$ have been marked. We see that as K_I is increased the system becomes unstable when the dominant roots cross the imaginary axis and move into the right-half plane, but we knew this from the Routh-Hurwitz condition.

However, the plot also tells us that as K_I is increased above 20, the dominant roots move up and to the right and the third root moves to the left. Thus the damping ratio decreases (increasing the overshoot), the oscillation frequency increases, and the dominant time constant increases so the response becomes slower—all undesirable effects—but the ramp error becomes smaller. The plot also shows that the third root is never the dominant root.

As K_I is decreased, the dominant root eventually becomes real. The damping ratio of the dominant root is 1 when the two roots are real and equal. Use the cursor to move along the curve to the point where the two roots meet and read the corresponding gain displayed in the box that appears. At this point $s = -0.502$ and the gain is $K = 1.38$, which corresponds to $K_I = 8K = 11.04$. (Root locations are very sensitive to the gain near points where two roots are equal, so these values are approximate.) Thus for $K_I < 11.04$ the dominant root is real and moves to the right as K_I is further decreased. We see, therefore, that the *smallest* possible dominant time constant for the given values of K_1 and K_2 occurs when $K_I = 11.04$ and is $\tau = 1/0.502 = 1.99$. For this value, the ramp error is $44/11.04 = 3.99$.

In conclusion, the root locus plot shows that the designer must choose between a faster system by using $K_I = 11.04$ or minimizing the ramp error $44/K_I$ but at the cost of a very oscillatory system that is approaching instability. This insight could not have been obtained without the root locus plot.

Uncertainty in the Damping Constant | **EXAMPLE 11.6.2**

■ **Problem**

The characteristic equation for PI control of the plant $1/(10s^2 + cs)$ is

$$10s^3 + cs^2 + K_P s + K_I = 0$$

Some discussion was devoted to the effect of uncertainty in value of the damping c on the effectiveness of the control system. If $c = 80$ and if the gains are $K_P = 240$ and $K_I = 320$, the

Figure 11.6.3 Root locus
plot for Example 11.6.2.
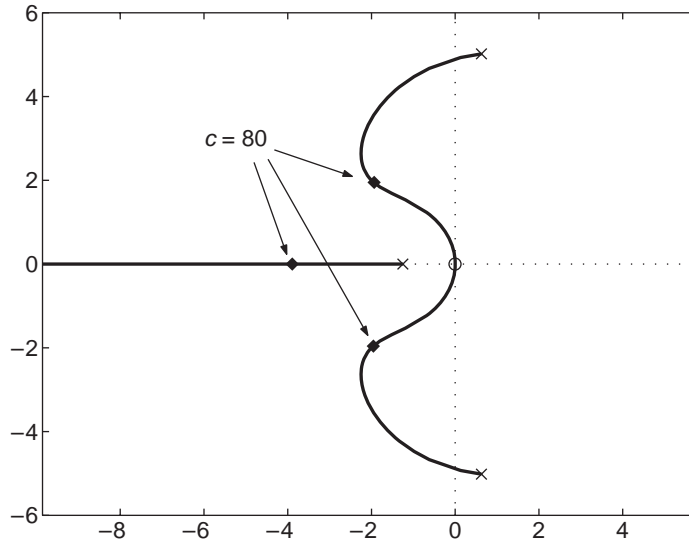
roots are $s = -2 \pm 2j$ and $s = -4$. Suppose that $c = 80$ is just a rough estimate. Investigate the effects of having a different value of c.

■ **Solution**
Substituting the gains $K_P = 240$ and $K_I = 320$ into the characteristic equation gives

$$10s^3 + cs^2 + 240s + 320 = 0$$

With the Routh-Hurwitz condition we immediately see that the system is unstable if $240c < 10(320)$; that is, if $c < 13.33$.

Rearranging this equation into the standard root-locus form, we have

$$1 + \frac{c}{10} \frac{s^2}{s^3 + 24s + 32} = 0$$

With the highest coefficients normalized to unity, the root locus gain is $K = c/10$. The poles are the roots of the denominator, which are $s = 0.626 \pm 5.02j$ and $s = -1.25$. There are two zeros, which are the roots of the numerator. They are both at $s = 0$.

The MATLAB session is

```
>>sys = tf([1, 0, 0],[1, 0, 24, 32]);
>>rlocus(sys)
```

The root locus plot is shown in Figure 11.6.3. The Plot Editor has been used to annotate the plot. The locations of the roots when $c = 80$ have been marked. We see that as c is decreased, the system becomes unstable when the dominant roots cross the imaginary axis and move into the right-half plane, but we knew this from the Routh-Hurwitz condition.

Remember we have no influence over the value of c. If $c < 80$, the dominant time constant is smaller, and the smallest possible dominant time constant occurs when the dominant root path is the farthest to the left. This occurs at $s = -2.25 \pm 2.67j$ when the root locus gain is near $K = 7.1$, which was determined by moving the cursor along the plot. The smallest possible dominant time constant is $\tau = 1/2.25 = 0.44$. This gain value corresponds to $c = 10K = 71$. Thus, regardless of the value of c, the time constant will be no less than 0.44.

THE `rltool`, `ltitool`, AND `ltiview` FUNCTIONS

The MATLAB Control System Toolbox also contains the Root Locus Tool, which is a graphical interface for interacting with the root locus plot. You activate it by typing `rltool`. It enables you to interactively design a control system using the root locus technique. The `ltitool` and `ltiview` functions activate other interfaces for interacting with the frequency response and step response plots, for example. Detailed information on their use is available with the MATLAB online help feature.

FITTING DATA FOR THE PROCESS REACTION METHOD

When applying the Ziegler-Nichols process reaction method, it is usually difficult to determine the slope R and intercept L from a plot of the data. As described in Section 11.4, a procedure using polynomial regression can be applied to compute R and L. The following script file illustrates the procedure for the unit-step open-loop response of a hypothetical thermal process.

```
% Times at which measurements were made (minutes).
td = (0:1:12);
% Measured Temperatures (deg C).
yd = [0, 3, 10, 20, 37, 53, 63, 77, 83, 87, 90, 93, 93];
% Fit a 5th order polynomial to the data.
coef = polyfit(td,yd,5);
% Evaluate the polynomial at a large number of points.
dt = 0.01; t = (0:dt:12); y = polyval(coef,t)';
% Find the coefficients of the polynomial's derivative.
coef2 = [5*coef(1), 4*coef(2), 3*coef(3), 2*coef(4), coef(5)];
% der is the derivative polynomial.
der = polyval(coef2,t)';
% Find the maximum slope R, and ...
   the time tmax at which it occurs.
[R,i] = max(der); tmax = i*dt;
% Compute the temperature where the slope is the maximum.
ymax = y(i);
% Compute the intercept L on the time axis.
L = (R*tmax-ymax)/R;
disp(R),disp(L)
% Compute Gains For PID Control
KP = 1.2/(R*L)
KI = KP/(2*L)
KD = 0.5*L*KP
```

The results are $R = 14.8433$, $L = 1.5417$, $K_P = 0.0524$, $K_I = 0.0170$, and $K_D = 0.0404$. The controller transfer function is

$$G_c(s) = \frac{0.0404s^2 + 0.0524s + 0.017}{s} \qquad (11.6.1)$$

11.7 SIMULINK APPLICATIONS

This section contains a number of the chapter's applications that can benefit from the capabilities of Simulink.

AN ANTI-WINDUP SYSTEM

The anti-windup design shown in Figure 11.5.10 is a good example of a system that is easy to simulate in Simulink. The Simulink model shown in Figure 11.7.1 can be developed directly from the block diagram in Figure 11.5.10, and it was obtained with the parameter values given in Section 11.5. Figure 11.5.11 was obtained with this model, and the value of $K_A = 10$ was obtained from simulations.

THE RATE-LIMITER ELEMENT

In addition to being limited by saturation, some actuators have limits on how fast they can react. This limitation is independent of the time constant of the actuator, and might be due to deliberate restrictions placed on the unit by its manufacturer. An example is a flow control valve whose rate of opening and closing is controlled by a "rate limiter." Simulink has such a block, and it can be used in series with the Saturation block to model the valve behavior.

Consider the model of the height h of liquid in a tank, whose input is a flow rate q_i. For specific parameter values, such a model has the form

$$\frac{H(s)}{Q_i(s)} = \frac{2}{5s + 1}$$

A Simulink model is shown in Figure 11.7.2 for a specific PI controller whose gains are $K_P = 4$ and $K_I = 5/4$. Suppose that the minimum and maximum flow rates available

Figure 11.7.1 Simulink model of an anti-windup system.
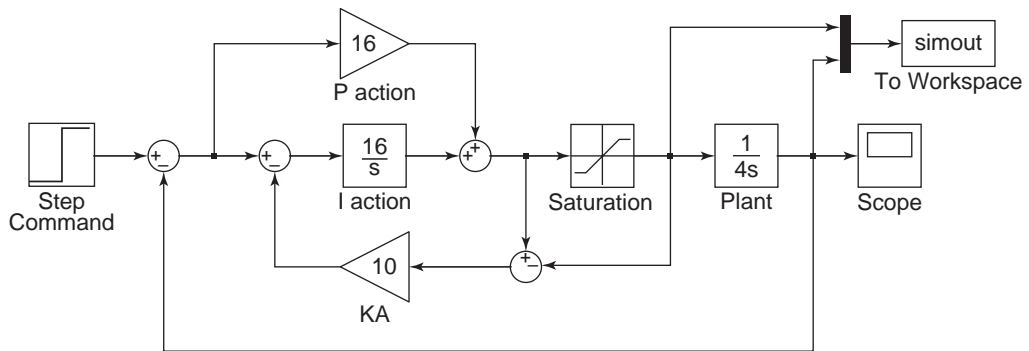
Figure 11.7.2 Application of the Rate Limiter block.
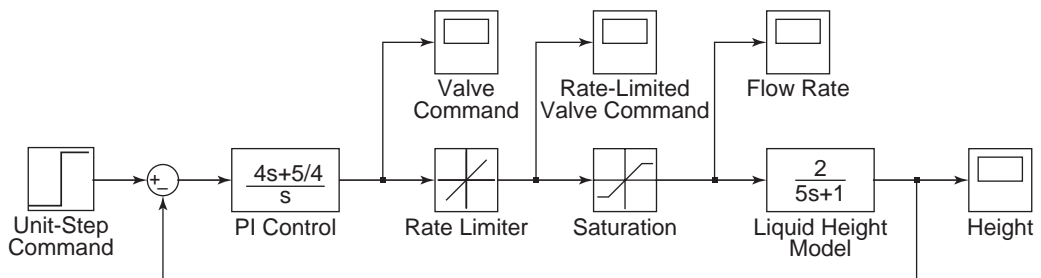
from the valve are 0 and 2. These are the limits for the Saturation block. The model enables us to experiment with the lower and upper limits of the Rate Limiter block to see its effect on the system performance.

11.8 CHAPTER REVIEW

This chapter shows how to use the root locus plot to design control systems. This type of plot enables a systematic design approach, by showing the effects of pole and zero placement on the transient response and stability.

Section 11.1 introduced the root locus plot. Section 11.2 illustrates the use of the root locus plot to design PID controllers. When PID control action fails to yield the desired performance, it must be either modified or replaced by an entirely different control scheme.

State-variable feedback, treated in Section 11.3, is a generalization of rate feedback that uses some or all of the system's state variables to modify the control signal. This eliminates numerator dynamics and the resulting overshoot. With state-variable feedback, we have a better chance of placing the characteristic roots of the closed-loop system in locations that will give desirable performance.

In many applications, especially in process control involving thermodynamic, fluid, or chemical processes, a transfer function model of the plant is not available, and the gains must be computed from experimentally determined open-loop response data or from computer simulations of the plant. In addition, the computed gain values sometimes do not yield the desired performance, and their values must be adjusted either in simulation or with the actual controller hardware. This process is called tuning and is discussed in Section 11.4.

Using high gain values tends to drive the control elements to such an extent that they overload or "saturate" and thus exhibit nonlinear behavior. Controllers having I action can exhibit reset windup, which can cause overshoot and saturation. Designs that avoid these unwanted effects are covered in Section 11.5.

Root locus plots for simple systems can be sketched by hand with the aid of a calculator. If these methods are to be used for higher-order systems, however, the use of computer methods is highly recommended. The MATLAB `rlocus`, `rltool`, and `sisotool` functions make it easy to view the root locus and transient response plots to see the effect of changing parameter values. These methods are reviewed in Section 11.6.

When dealing with systems having nonlinear, discontinuous elements such as the saturation nonlinearity, or dead-time elements, Simulink provides a quick and easy way of simulating such systems. The Simulink features relevant to the topics of this chapter are summarized in Section 11.7.

Now that you have finished this chapter, you should be able to

1. Sketch a root locus plot.
2. Determine the major features of a root locus plot, and use it to assess the results of pole and zero placement.
3. Design a controller incorporating state-variable feedback.
4. Use the Ziegler-Nichols methods to design controllers.
5. Design a control system to avoid actuator saturation.
6. Apply MATLAB and Simulink to analyze and design control systems using the concepts presented in this chapter.

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PROBLEMS

Section 11.1 Root Locus Plots

11.1   Sketch the root locus plot of $3s^2 + 12s + k = 0$ for $k \geq 0$. What is the smallest
possible dominant time constant, and what value of k gives this time constant?

11.2   Sketch the root locus plot of $3s^2 + cs + 12 = 0$ for $c \geq 0$. What is the smallest
possible dominant time constant, and what value of c gives this time constant?
What is the value of ω_n if $\zeta < 1$?

11.3   Sketch the root locus of the armature-controlled dc motor model in terms of
the damping constant c, and evaluate the effect on the motor time constant. The
characteristic equation is

$$L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T = 0$$

Use the following parameter values:

$$K_b = K_T = 0.1 \text{ N} \cdot \text{m/A} \qquad I = 12 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

$$R_a = 2 \ \Omega \qquad\qquad L_a = 3 \times 10^{-3} \ \text{H}$$

11.4   Sketch the root locus plot of $ms^2 + 12s + 10 = 0$ for $m \geq 2$. What is the
smallest possible dominant time constant, and what value of m gives this time
constant?

Section 11.2 Design Using the Root Locus Plot

11.5   In the following equations, identify the root locus plotting parameter K and its
range in terms of the parameter p, where $p \geq 0$.
   a. $6s^2 + 8s + 3p = 0$
   b. $3s^2 + (6 + p)s + 5 + 2p = 0$
   c. $4s^3 + 4ps^2 + 2s + p = 0$

11.6 Consider a unity feedback system with the plant $G_p(s)$ and the controller $G_c(s)$. PID control action is applied to the plant

$$G_p(s) = \frac{s + 10}{(s + 1)(s + 2)}$$

The PID controller has the transfer function

$$G_c(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s \right)$$

Use the values $T_I = 0.2$ and $T_D = 0.5$.
a. Identify the open-loop poles and zeros.
b. Identify the root locus parameter K in terms of K_P.
c. Identify the closed-loop poles and zeros for the case $K_P = 10$.

11.7 In parts (a) through (f), sketch the root locus plot for the given characteristic equation for $K \geq 0$.
a. $s(s + 5) + K = 0$
b. $s(s + 7)(s + 9) + K = 0$
c. $s^2 + 3s + 5 + K(s + 3) = 0$
d. $s(s + 4) + K(s + 5) = 0$
e. $s(s^2 + 3s + 5) + K = 0$
f. $s(s + 3)(s + 7) + K(s + 4) = 0$

11.8 PID control action is applied to the plant

$$G_p(s) = \frac{s + 10}{(s + 2)(s + 5)}$$

The PID controller has the transfer function

$$G_c(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s \right)$$

Use the values $T_I = 0.2$ and $T_D = 0.5$. Plot the root locus with the proportional gain K_P as the parameter.

11.9 Consider the following equation where the parameter p is nonnegative.

$$4s^3 + (25 + 5p)s^2 + (16 + 30p)s + 40p = 0$$

a. Put the equation in standard root locus form and define a suitable root locus parameter K in terms of the parameter p.
b. Obtain the poles and zeros, and sketch the root locus plot.

11.10 In the following equation, $K \geq 0$.

$$s^2(s + 9) + K(s + 1) = 0$$

a. Obtain the root locus plot.
b. Obtain the value of K at the breakaway point, and obtain the third root for this value of K.
c. What is the smallest possible dominant time constant for this equation?

11.11 Consider the following equation where the parameter K is nonnegative.

$$(2s + 5)(2s^2 + 14s + 49) + Ks(2s + 1)(2s + 3) = 0$$

a. Determine the poles and zeros, and sketch the root locus plot.

    b.   Use the plot to set the value of K required to give a dominant time constant of $\tau = 0.5$. Obtain the three roots corresponding to this value of K.

11.12 In the following equations, identify the root locus plotting parameter K and its range in terms of the parameter p, where $p \geq 0$.

    a. $9s^3 + 6s^2 - 5ps + 2 = 0$

    b. $4s^3 - ps^2 + 2s + 7 = 0$

    c. $s^2 + (3 - p)s + 4 + 4p = 0$

11.13 In parts (a) through (f), Obtain the root locus plot for $K \leq 0$ for the given characteristic equation.

    a. $s(s + 5) + K = 0$

    b. $s^2 + 3s + 3 + K(s + 3) = 0$

    c. $s(s^2 + 3s + 3) + K = 0$

    d. $s(s + 5)(s + 7) + K = 0$

    e. $s(s + 3) + K(s + 4) = 0$

    f. $s(s + 6) + K(s - 4) = 0$

11.14 The plant transfer function for a particular process is

$$G_p(s) = \frac{8 - s}{s^2 + 2s + 3}$$

We wish to investigate the use of proportional control action with this plant.

    a.   Obtain the root locus and determine the range of values of the proportional gain K_P for which the system is stable.

    b.   Determine the value of K_P required to give a time constant of $\tau = 2/3$.

    c.   Plot the unit step response of the plant for $K_P = 1$. A process containing a negative sign in the numerator of its transfer function is called a "reverse reaction" process. What is the effect of the negative sign in the numerator of $G_p(s)$?

11.15 The plant transfer function for a particular process is

$$G_p(s) = \frac{26 + s - 2s^2}{s(s + 2)(s + 3)}$$

We wish to investigate the use of proportional control action with this plant.

    a.   Obtain the root locus and determine the range of values of the proportional gain K_P for which the system is stable.

    b.   Determine the value of K_P required to give $\zeta = 1$.

    c.   Plot the unit step response of the plant. What is the effect of the negative sign in the numerator of $G_p(s)$?

11.16 Control of the attitude θ of a missile by controlling the fin angle ϕ, as shown in Figure P11.16, involves controlling an inherently unstable plant. Consider the specific plant transfer function

$$G_p(s) = \frac{\Theta(s)}{\Phi(s)} = \frac{1}{s^2 - 5}$$

    a.   Determine the PD control gains so that the steady-state error for a step command is zero, the closed-loop damping ratio is 0.707, and the dominant closed-loop time constant is 0.1.
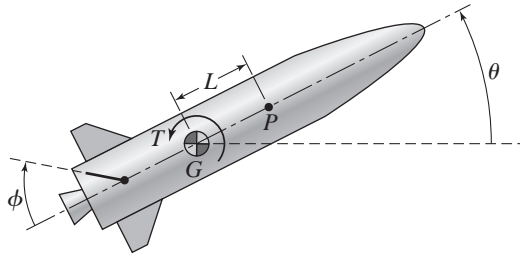
b. Use the root locus to evaluate the performance of the resulting controller in light of the specifications $\zeta = 0.707$ and $\tau = 0.1$ if the plant transfer function $G_p(s)$ has an uncertainty Δ due to fuel consumption, where

$$G_p(s) = \frac{1}{s^2 - 5 - \Delta}, \quad 0 \le \Delta \le 1$$

11.17 The use of a motor to control the rotational displacement of an inertia I is shown in Figure P11.17. The open-loop transfer function of the plant for a specific application is

$$G_p(s) = \frac{6}{s(2s + 2)(3s + 24)}$$

a. Use the root locus plot to show that it is not possible with proportional control to achieve a dominant time constant of less than 2.07 sec for this plant.

b. Use PD control to improve the response, so that the dominant time constant is 0.5 sec or less and the damping ratio is 0.707 or greater. To do this, first select a suitable value for T_D, then plot the locus with K_P as the variable.

Figure P11.17

(a)   (b)
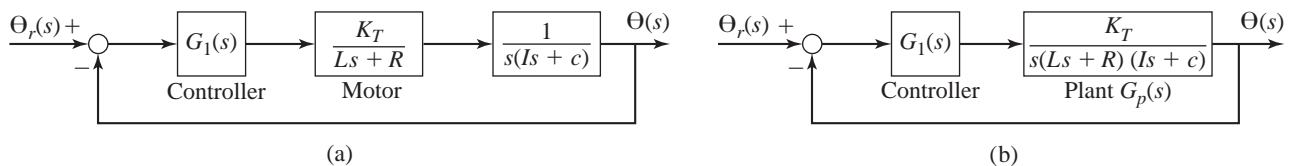
11.18 Proportional control action applied to the heat flow rate q_i can be used to control the temperature of the oven shown in Figure P11.18. Consider the specific plant

$$G_p(s) = \frac{T_1(s)}{Q_i(s)} = \frac{s + 10}{s^2 + 5s + 6}$$
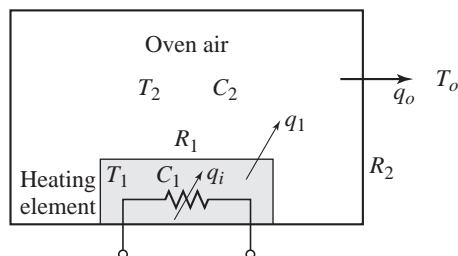
Use the root locus plot to obtain the smallest damping ratio this system can have. Obtain the value of the proportional gain K_P required to minimize the dominant time constant with $\zeta = 0.707$, and determine this time constant.

11.19 Proportional control action applied to the flow rate q_{mi} can be used to control the liquid height, as shown in Figure P11.19. Consider the specific plant

$$G_p(s) = \frac{H_2(s)}{Q_{mi}(s)} = \frac{5(s+4)}{(s+3)(s+p)}$$

The proportional gain is $K_P = 2$. The likely value of p is $p = 7$, but it is known that p might be greater than 7. Use the root locus plot to investigate the roots of the system for $p \geq 7$.

Figure P11.19
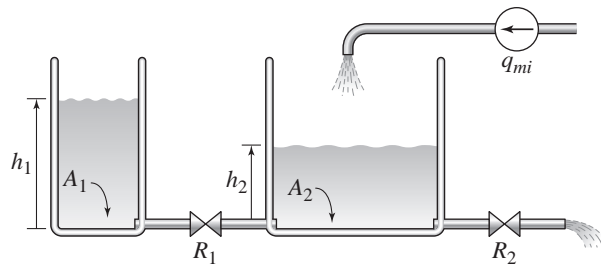
11.20 Proportional control action applied to the flow rate q_{mi} can be used to control the liquid height of the system shown in Figure P11.20. Consider the specific plant

$$G_p(s) = \frac{H_2(s)}{Q_{mi}(s)} = \frac{1}{s^2 + 3s + 2}$$

Use the root locus plot to design a PI controller for this system to minimize the dominant time constant, with a damping ratio of $\zeta = 0.707$.

11.21 Design a PID controller applied to the motor torque T to control the robot arm angle θ shown in Figure P11.21. Consider the specific plant

$$G_p(s) = \frac{\Theta(s)}{T(s)} = \frac{4}{3s^2 + 3}$$

The dominant closed-loop roots must have $\zeta = 0.5$ and a time constant of 1.

11.22 a) The equations of motion of the inverted pendulum model were derived in Example 2.4.4 in Chapter 2. Linearize these equations about $\phi = 0$, assuming that ϕ is very small. b) Obtain the linearized equations for the following values: $M = 10$ kg, $m = 50$ kg, $L = 1$ m, $I = 0$, and $g = 9.81$ m/s^2. c) Use the linearized model developed in part (b) to design a PID controller to stabilize the pendulum angle near $\phi = 0$. It is required that the 2% settling time be no greater that 4 s and that the response be nonoscillatory. This means that the dominant root should be real and no greater than -1. No restriction is placed on the motion of the base. Assume that only ϕ and $\dot{\phi}$ can be measured.

Figure P11.20
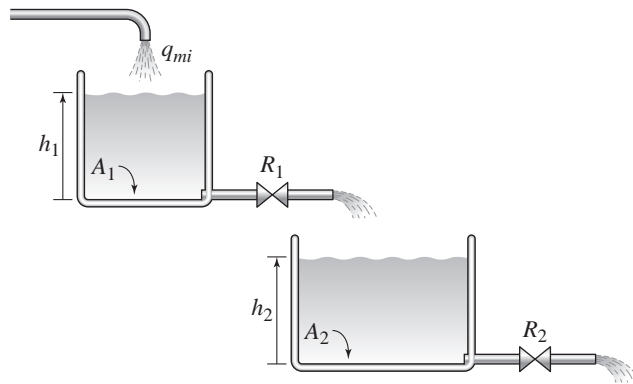
Figure P11.21
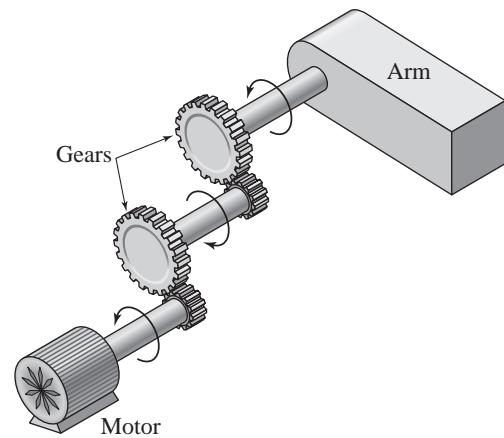
Section 11.3 State-Variable Feedback

11.23 A certain field-controlled dc motor with load has the following parameter
values.

$$L = 2 \times 10^{-3} \text{ H} \qquad R = 0.6 \, \Omega$$
$$K_T = 0.04 \text{ N} \cdot \text{m/A} \qquad c = 0$$
$$I = 6 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

Compute the gains for a state variable feedback controller using P action to
control the motor's angular position. The desired dominant time constant is
0.5 s. The secondary roots should have a time constant of 0.05 s and a damping
ratio of $\zeta = 0.707$.

11.24 In Figure P11.24 the input u is an acceleration provided by the control system
and applied in the horizontal direction to the lower end of the rod. The
horizontal displacement of the lower end is y. The linearized form of Newton's
law for small angles gives

$$mL\ddot{\theta} = mg\theta - mu$$

a. Put this model into state variable form by letting $x_1 = \theta$ and $x_2 = \dot{\theta}$.
b. Construct a state variable feedback controller by letting $u = k_1 x_1 + k_2 x_2$.
Over what ranges of values of k_1 and k_2 will the controller stabilize the
system? What does this formulation imply about the displacement y?

11.25 Figure 11.3.9 illustrates an active suspension for the quarter car model. An
electrohydraulic actuator between the chassis and the wheel assembly
provides a force that acts on both and is under feedback control. The system
model is

$$m_1\ddot{x}_1 = k_1(y - x_1) - k_2(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) - f$$
$$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) + f$$

The given parameter values are $m_1 = 50$ kg, $m_2 = 250$ kg, $k_1 = 1.5 \times 10^5$ N/m, $k_2 = 1.2 \times 10^4$ N/m, and $c = 100$ N $\cdot$ s/m.

Figure P11.24
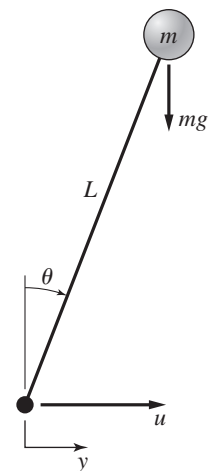
a. Put the model into state variable form.
b. Assume that we can measure all four state variables, and use P action with state-variable feedback. In the original passive suspension system, $k_2 = 1.6 \times 10^4$ N/m and $c = 98$ N · m/s, which resulted in characteristic roots at $s = -1.397 \pm 69.94j$, $s = -0.168 \pm 7.779j$. Compute the values of the feedback gains so that the closed-loop roots will be near those of the passive system.

11.26 Figure P11.26a is the circuit diagram of a speed-control system in which the dc motor voltage v_a is supplied by a generator driven by an engine. This system has been used on locomotives whose diesel engine operates most efficiently at one speed. The efficiency of the electric motor is not so sensitive to speed and thus can be used to drive the locomotive at various speeds. The motor voltage v_a is varied by changing the generator input voltage v_f. The voltage v_a is related to the generator field current i_f by $v_a = K_f i_f$.

Figure P11.26b is a diagram of a feedback system for controlling the speed by measuring it with a tachometer and varying the voltage v_f. Use the following values in SI units.

$$L_f = 0.2 \quad R_f = 2 \quad K_t = 1$$
$$L_a = 0.2 \quad R_a = 1 \quad K_b = K_T = 0.5$$
$$K_f = 50 \quad I = 10 \quad c = 20$$

Figure P11.26

(a)

(b)
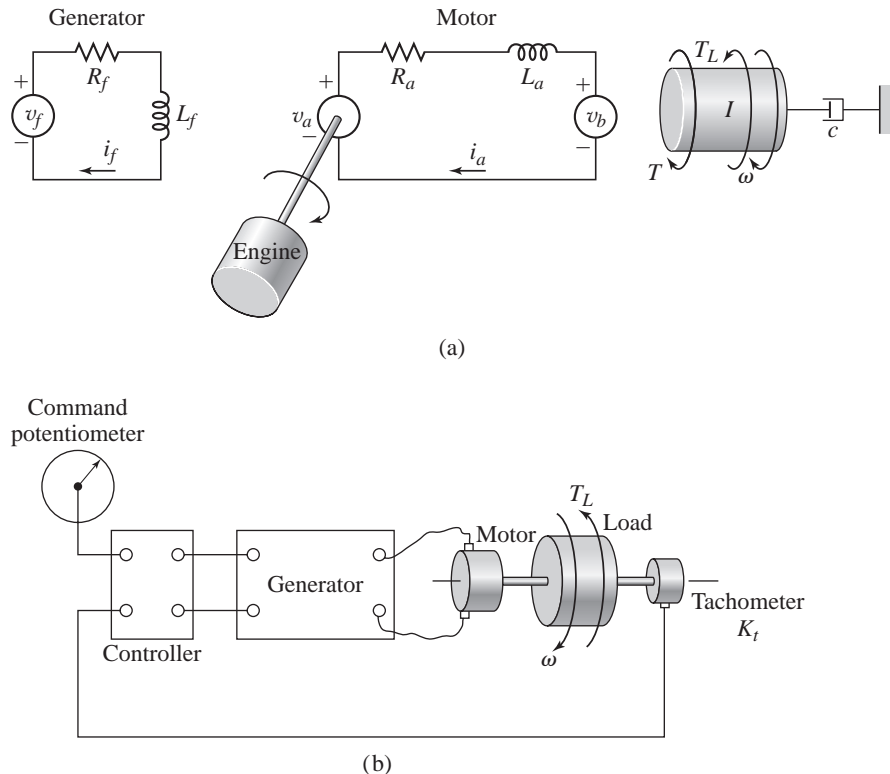
   a. Develop a state variable model of the plant that includes the generator, the
      motor, and the load. Include the load torque T_L as a disturbance.
   b. Develop a proportional controller assuming all the state variables can be
      measured. Analyze its steady-state error for a step command input and for
      a step disturbance.

11.27 The following equations are the model of the roll dynamics of a missile
([Bryson, 1975]). See Figure P11.27.

$$\dot{\delta} = u$$

$$\dot{\omega} = -\frac{1}{\tau}\omega + \frac{b}{\tau}\delta$$

$$\dot{\phi} = \omega$$

where δ = aileron deflection
        b = aileron effectiveness constant
        u = command signal to the aileron actuator
        ϕ = roll angle, ω = roll rate
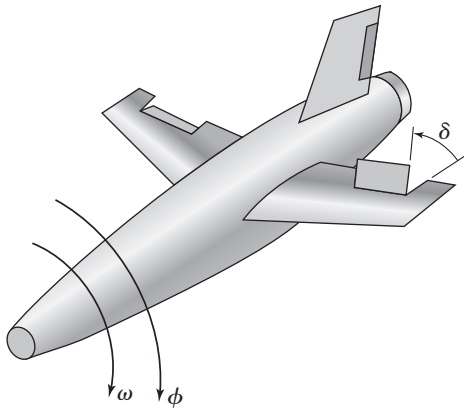
Figure P11.27

Using the specific values $b = 10$ s^{-1} and $\tau = 1$ s, and assuming that the state
variables δ, ω, and ϕ can be measured, develop a linear state-feedback
controller to keep ϕ near 0. The dominant roots should be $s = -10 \pm 10j$, and
the third root should be $s = -20$.

11.28 Many winding applications in the paper, wire, and plastic industries require a
control system to maintain proper tension. Figure P11.28 shows such a system
winding paper onto a roll. The paper tension must be held constant to prevent
internal stresses that will damage the paper. The pinch rollers are driven at a
speed required to produce a paper speed v_p at the rollers. The paper speed as it
approaches the roll is v_r. The paper tension changes as the radius of the roll
changes or as the speed of the pinch rollers change. The paper has an elastic
constant k so that the rate of change of tension is

$$\frac{dT}{dt} = k(v_r - v_p)$$
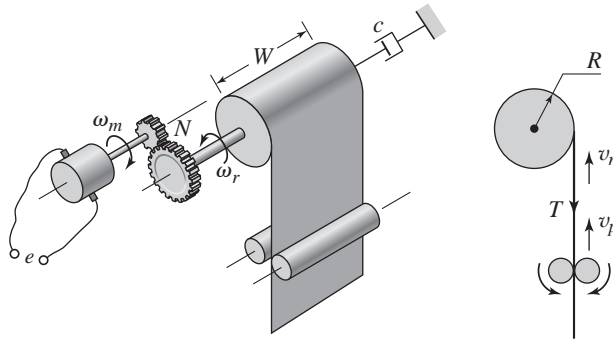
For a paper thickness d, the rate of change of the roll radius is

$$\frac{dR}{dt} = \frac{d}{2}W$$

The inertia of the windup roll is $I = \rho \pi W R^4 / 2$, where ρ is the paper mass density and W is the width of the roll. So R and I are functions of time.

The viscous damping constant for the roll is c. For the armature-controlled motor driving the windup roll, neglect its viscous damping and armature inertia.

a.   Assuming that the paper thickness is small enough so that $\dot{R} \approx 0$ for a short time, develop a state-variable model with the motor voltage e and the paper speed v_p as the inputs.

b.   Modify the model developed in part (a) to account for R and I being functions of time.

11.29 An electro-hydraulic positioning system is shown in Figure P11.29. Use the following values.

$$K_a = 10 \text{ V/A} \quad K_1 = 10^{-2} \text{ in./V}$$
$$K_2 = 3 \times 10^5 \text{ sec}^{-3} \quad K_3 = 20 \text{ V/in.}$$
$$\zeta = 0.8 \quad \omega_n = 100 \text{ rad/sec} \quad \tau = 0.01 \text{ sec}$$

a.   Develop a state-variable model of the plant with the controller current i_c as the input and the displacement y as the output.

b.   Assuming that proportional control is used so that $G_c(s) = K_P$, develop a state model of the system with y_r as the input and y as the output. Draw the root locus and use it to determine whether or not the system can be made stable with an appropriate choice for the value of K_P.

11.30 a) The equations of motion of the inverted pendulum model were derived in Example 2.4.4 in Chapter 2. Linearize these equations about $\phi = 0$, assuming that $\dot{\phi}$ is very small. b) Obtain the linearized equations for the following values: $M = 10$ kg, $m = 50$ kg, $L = 1$ m, $I = 0$, and $g = 9.81$ m/s^2. c) Use the linearized model developed in part (b) to design a state variable feedback controller to stabilize the pendulum angle near $\phi = 0$. It is required that the 2% settling time be no greater that 4 s and that the response be nonoscillatory. This means that the dominant root should be real and no greater than -1. No restriction is placed on the motion of the base. Assume that ϕ, $\dot{\phi}$, x, and $\dot{x}$ can be measured.
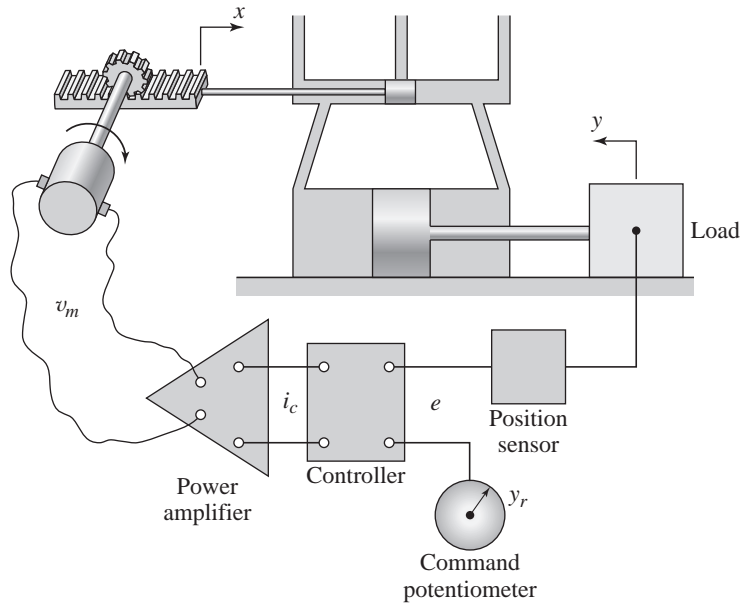
(a)
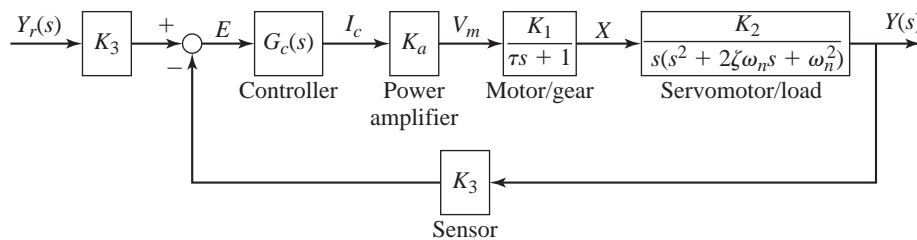
(b)

Section 11.4 Tuning Controllers

Note: See the problems for Section 11.6 for problems dealing with the process reaction method.

11.31 Use of a motor to control the position of a certain load having inertia, damping, and elasticity gives the following plant transfer function. See Figure P11.31.

$$G_p(s) = \frac{\Theta(s)}{V(s)} = \frac{0.5}{(s^2 + s + 1)(s + 0.5)}$$

a. Use the ultimate cycle method to compute the controller gains for P, PI, and PID control.

b. Plot and compare the unit-step responses for the three designs obtained in (a). If the PID response is unsatisfactory, tune the gains to improve the performance.

11.32 Figure P11.32 shows an electrohydraulic position control system whose plant transfer function for a specific system is

$$G_p(s) = \frac{Y(s)}{F(s)} = \frac{5}{2s^3 + 10s^2 + 2s + 4}$$

Figure P11.31
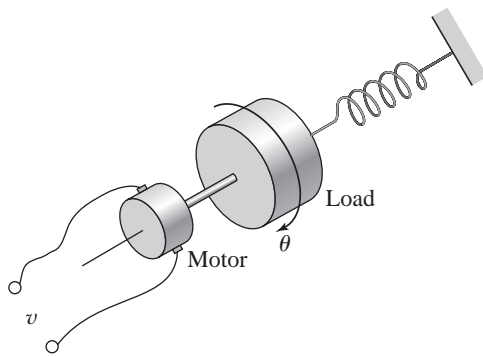
Figure P11.32
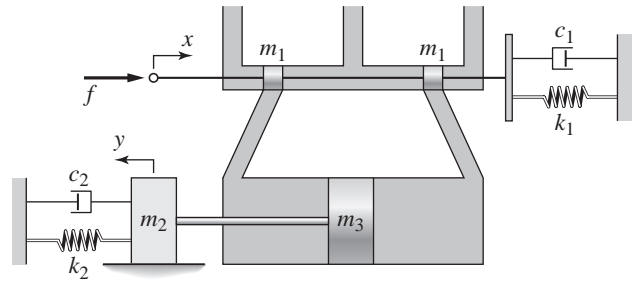
a. Use the ultimate cycle method to design P, PI, and PID controllers.

b. Plot and compare the unit-step responses for the three designs obtained in (a). If the PID response is unsatisfactory, tune the gains to improve the performance.

11.33 A certain plant has the transfer function

$$G_p(s) = \frac{4p}{(s^2 + 4\zeta s + 4)(s + p)}$$

where the nominal values of ζ and p are $\zeta = 0.5$ and $p = 1$.

a. Use Ziegler-Nichols tuning to compute the PID gains. Obtain the resulting closed-loop characteristic roots.

b. Use the root locus to determine the effect of a variation in the parameter p over the range $0.4 \leq \zeta \leq 0.6$.

c. Use the root locus to determine the effect of a variation in the parameter p over the range $0.5 \leq p \leq 1.5$.

11.34 The plant transfer function of the system in Figure P11.34 for a specific case is

$$G_p(s) = \frac{8}{(2s + 2)(s + 2)(4s + 12)}$$

Figure P11.34
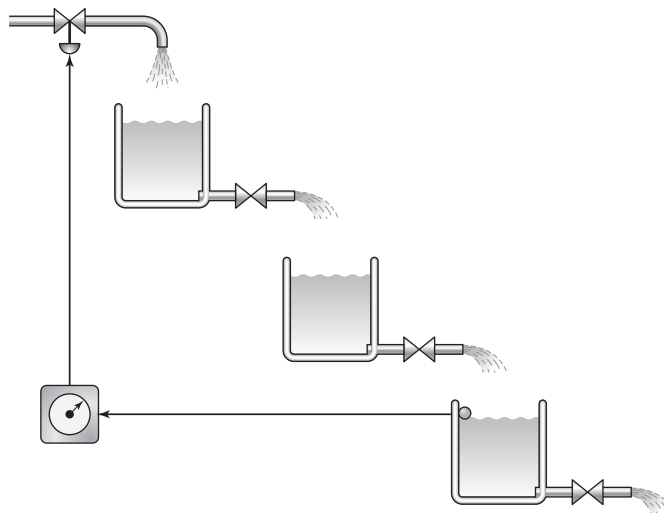
a. Use the ultimate cycle method to compute the PID gains.

b. Plot the unit-step response. If the response is unsatisfactory, use the root locus plot to explain the result, and try to improve the response by tuning the gains.

Section 11.5 Saturation and Reset Windup

11.35 Consider the PI-control system shown in Figure P11.35 where $I = 5$ and $c = 0$. It is desired to obtain a closed-loop system having $\zeta = 1$ and $\tau = 0.1$. Let $m_{max} = 20$ and $r_{max} = 2$. Obtain K_P and K_I.
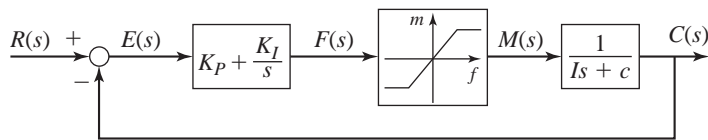
11.36 Consider the PI-control system shown in Figure P11.35 where $I = 10$ and $c = 20$. It is desired to obtain a closed-loop system having $\zeta = 1$ and $\tau = 0.1$.

  a. Obtain the required values of K_P and K_I, neglecting any saturation of the control elements.

  b. Let $m_{max} = r_{max} = 1$. Obtain K_P and K_I.

  c. Compare the unit-step response of the two designs.

11.37 Consider the PI-control system shown in Figure P11.35 where $I = 7$ and $c = 5$. It is desired to obtain a closed-loop system having $\zeta = 1$ and $\tau = 0.2$. Let $m_{max} = 20$ and $r_{max} = 5$. Obtain K_P and K_I.

11.38 a. Design a PI and an I controller with internal feedback for the plant $G_p(s) = 1/4s$. See Figure P11.38. We are given that $m_{max} = 6$ and $r_{max} = 3$. Set $\zeta = 1$.

  b. Evaluate the unit-step response of each design.

  c. Evaluate the unit-ramp response of each design.

(a)

(b)
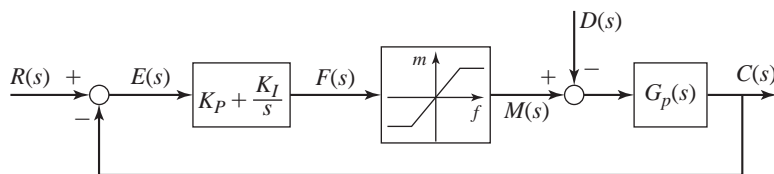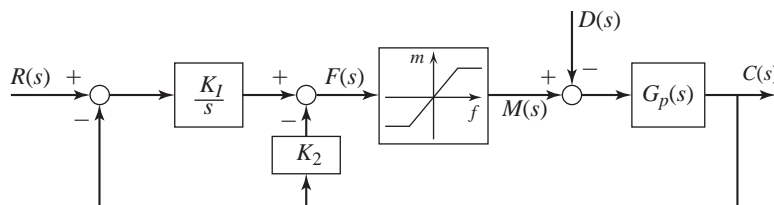
11.39 Compare the performance of the critically damped controllers shown in
Figure P11.38 with the plant $G_p(s) = 1/Is$ having the following inputs:
   a. A unit-ramp disturbance
   b. A sinusoidal disturbance
   c. A sinusoidal command input

Section 11.6 MATLAB Applications

11.40 The following table gives the measured open-loop response of a system to a
unit-step input. Use the process reaction method to find the controller gains for
P, PI, and PID control.

Time (min)	Response
0	0
0.5	4
1.0	20
1.5	32
2.0	56
2.5	84
3.0	116
3.5	140
4.0	160
4.5	172
5.0	184
5.5	190
6.0	194
7.0	196

11.41 A liquid in an industrial process must be heated with a heat exchanger through
which steam passes. The exit temperature of the liquid is controlled by
adjusting the rate of flow of steam through the heat exchanger with the control
valve. An open-loop test was performed in which the steam pressure was
suddenly changed from 15 to 18 psi above atmospheric pressure. The exit
temperature data are shown in the following table. Use the Ziegler-Nichols
process reaction method to compute the PID gains.

Time (min)	Temperature ($°F$)
0	156
1	157
2	159
3	162
4	167
5	172
6	175
7	179
8	181
9	182
10	183
11	184
12	184

11.42 Use MATLAB to obtain the root locus plot of $5s^2 + cs + 45 = 0$ for $c \geq 0$.

11.43 Use MATLAB to obtain the root locus plot of the system shown in Figure P11.43 in terms of the variable $k \geq 0$. Use the values $m = 4$ and $c = 8$. What is the smallest possible dominant time constant and the associated value of k?

11.44 Use MATLAB to obtain the root locus plot of the system shown in Figure P11.43 in terms of the variable $c \geq 0$. Use the values $m = 4$ and $k = 64$. What is the smallest possible dominant time constant and the associated value of c?

11.45 Use MATLAB to obtain the root locus plot of the system shown in Figure P11.45 in terms of the variable $k_2 \geq 0$. Use the values $m = 2$, $c = 8$, and $k_1 = 26$. What is the value of k_2 required to give $\zeta = 0.707$?
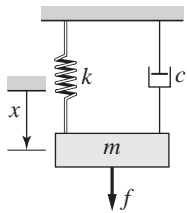
Figure P11.43   **Figure P11.45**   **Figure P11.46**
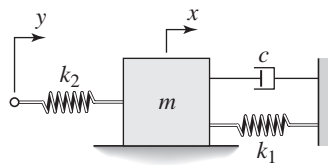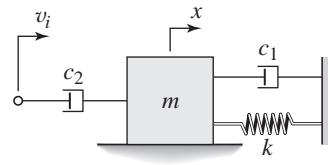
11.46 Use MATLAB to obtain the root locus plot of the system shown in Figure P11.46 in terms of the variable $c_2 \geq 0$. Use the values $m = 2$, $c_1 = 8$, and $k = 26$. What is the smallest possible dominant time constant and the associated value of c_2?

11.47 Use MATLAB to obtain the root locus plot of $s^3 + 13s^2 + 52s + 60 + K = 0$ for $K \geq 0$. Is it possible for any dominant roots of this equation to have a damping ratio in the range $0.5 \leq \zeta \leq 0.707$ and an undamped natural frequency in the range $3 \leq \omega_n \leq 5$?

11.48 (a) Use MATLAB to obtain the root locus plot of $2s^3 + 12s^2 + 16s + K = 0$ for $K \geq 0$. (b) Obtain the value of K required to give a dominant root pair having $\zeta = 0.707$. (c) For this value of K, obtain the unit-step response and the maximum overshoot, and evaluate the effects of the secondary root. The closed-loop transfer function is $K/(2s^3 + 12s^2 + 16s + K)$.

11.49 Use MATLAB to obtain the root locus of the armature-controlled dc motor model in terms of the damping constant c, and evaluate the effect on the motor time constant. The characteristic equation is

$$L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T = 0$$

Use the following parameter values:

$$K_b = K_T = 0.1 \text{ N} \cdot \text{m/A} \qquad I = 4 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$
$$R_a = 2 \, \Omega \qquad\qquad L_a = 3 \times 10^{-3} \text{ H}$$

11.50 Consider the two-mass model shown in Figure P11.50. Use the following numerical values: $m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 4$, and $c_2 = 8$.

a. Use MATLAB to obtain the root locus plot in terms of the parameter c_1.

b. Use the root locus plot to determine the value of c_1 required to give a dominant root pair having a damping ratio of $\zeta = 0.707$.

c. Use the root locus plot to determine the value of c_1 required to give a dominant root that is real and has a time constant equal to 4.
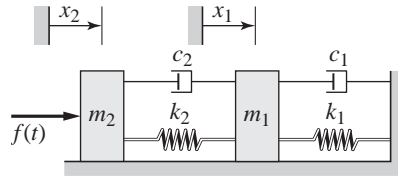
    d. Using the value of c_1 found in part (c), obtain a plot of the unit-step response.

11.51 In parts (a) through (f), use MATLAB to obtain the root locus plot for the given characteristic equation for $K \geq 0$.

    a. $s(s + 5) + K = 0$

    b. $s(s + 7)(s + 9) + K = 0$

    c. $s^2 + 3s + 5 + K(s + 3) = 0$

    d. $s(s + 4) + K(s + 5) = 0$

    e. $s(s^2 + 3s + 5) + K = 0$

    f. $s(s + 3)(s + 7) + K(s + 4) = 0$

11.52 In parts (a) through (f), use MATLAB to obtain the root locus plot for $K \leq 0$ for the given characteristic equation.

    a. $s(s + 5) + K = 0$

    b. $s^2 + 3s + 3 + K(s + 3) = 0$

    c. $s(s^2 + 3s + 3) + K = 0$

    d. $s(s + 5)(s + 7) + K = 0$

    e. $s(s + 3) + K(s + 4) = 0$

    f. $s(s + 6) + K(s - 4) = 0$

11.53 Consider the equation

$$s^3 + 10s^2 + 24s + K = 0$$

    a. Use MATLAB to obtain the value of K required to give dominant roots with $\zeta = 0.707$. Obtain the three roots corresponding to this value of K.

    b. Use MATLAB to obtain the value of K required to give a dominant time constant of $\tau = 2/3$. Obtain the three roots corresponding to this value of K.

11.54 Consider the equation

$$s^3 + 9s^2 + (8 + K)s + 2K = 0$$

    a. Use MATLAB to obtain the value of K required to put the dominant root at the breakaway point. Obtain the three roots corresponding to this value of K.

    b. Investigate the sensitivity of the dominant root when K varies by $\pm 10\%$ about the value found in part (a).

11.55 Consider the equation

$$s^3 + 9s^2 + (8 + K)s + 2K = 0$$

Use the sgrid function to determine if it is possible to obtain a dominant root having a damping ratio in the range $0.5 \leq \zeta \leq 0.707$. If so, use MATLAB to

obtain the value of K required to give the largest possible value of ζ in the range $0.5 \leq \zeta \leq 0.707$.

11.56 Consider the equation

$$s^3 + 10s^2 + 24s + K = 0$$

Use the `sgrid` function to determine if it is possible to obtain a dominant root having a damping ratio in the range $0.5 \leq \zeta \leq 0.707$, and an undamped natural frequency in the range $2 \leq \omega_n \leq 3$. If so, use MATLAB to obtain the value of K required to give the largest possible value of $\zeta \omega_n$ in the ranges stated.

11.57 In Example 10.7.4 the steady-state error for a unit-ramp disturbance is $1/K_I$. For the gains computed in that example, this error is $1/25$. We want to see if we can make this error smaller by increasing K_I. Using the values given for K_P and K_D in that example, obtain a root locus plot with K_I as the variable. Discuss what happens to the damping ratio and time constant of the dominant root as K_I is increased.

11.58 In Example 10.8.3 the steady-state error for a unit-ramp command is $-4/K_I$. For the gains computed in that example, this error is $1/1000$. We want to see if we can make this error smaller by increasing K_I. Using the values given for K_P and K_D in that example, obtain a root locus plot with K_I as the variable. Discuss what happens to the damping ratio and time constant of the dominant root as K_I is increased.

Section 11.7 Simulink Applications

11.59 With the PI gains set to $K_P = 6$ and $K_I = 50$ for the plant

$$G_p(s) = \frac{1}{s + 4}$$

the time constant is $\tau = 0.2$ and the damping ratio is $\zeta = 0.707$.

a. Suppose the actuator saturation limits are ± 5. Construct a Simulink model to simulate this system with a unit-step command. Use it to plot the output response, the error signal, the actuator output, and the outputs of the proportional term and the integral term versus time.

b. Construct a Simulink model of an anti-windup system for this application. Use it to select an appropriate value for K_A and to plot the output response and the actuator output versus time.

11.60 With the PI gains set to $K_P = 6$ and $K_I = 50$ for the plant

$$G_p(s) = \frac{1}{s + 4}$$

the time constant is $\tau = 0.2$ and the damping ratio is $\zeta = 0.707$. Suppose there is a rate limiter of ± 0.1 between the controller and the plant. Construct a Simulink model of the system and use it to determine the effect of the limiter on the speed of response of the system. Use a unit-step command.

11.61 A certain dc motor has the following parameter values:

$$L = 2 \times 10^{-3}\,\text{H} \qquad R = 0.6\,\Omega$$
$$K_T = 0.04\,\text{N} \cdot \text{m/A} \qquad c = 0$$
$$I = 6 \times 10^{-5}\,\text{kg} \cdot \text{m}^2$$
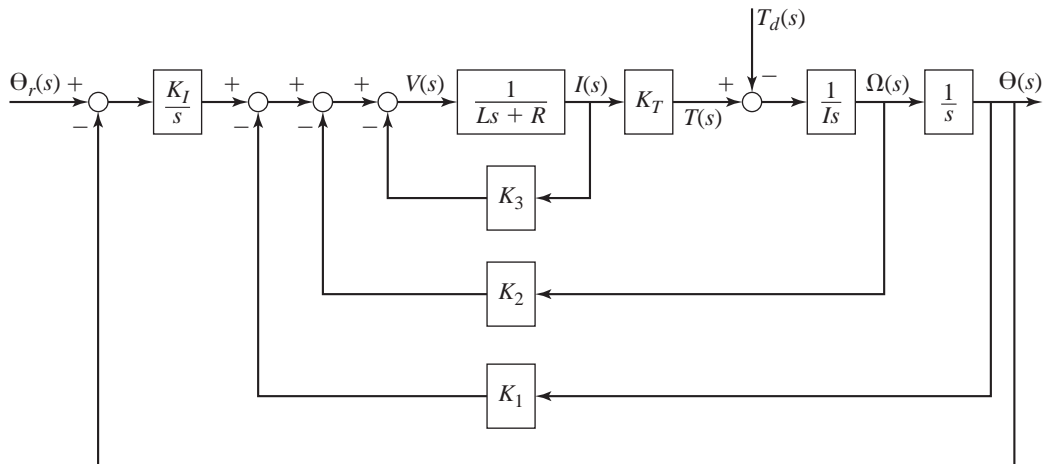
Figure P11.61

Figure P11.61 shows an integral controller using state-variable feedback to control the motor's angular position.

a. Compute the gains to give a dominant time constant of 0.5 s. The secondary roots should have a time constant of 0.05 s and a damping ratio of $\zeta = 0.707$. The fourth root should be $s = -20$.

b. Construct a Simulink model of the system and use it to plot the response of the system to a step disturbance of magnitude 0.1.

c. Suppose the motor current is limited to ± 2 A. Modify the Simulink model to include this saturation, and use the model to obtain plots of the responses to a unit-step command and a step disturbance of magnitude 0.1. Discuss the results.

11.62 Consider the liquid-level controller designed in Example 10.10.1, whose Simulink diagram is shown in Figure 10.10.1. Modify the model to include a Rate Limiter block to limit the rate of q_1, in front of the Saturation block. The limits on the rate should be ± 20. Use this model to obtain plots of the response of the height h_2 to a unit-step command and a unit-step disturbance. Compare these responses to those found in Example 10.10.1.

12

Compensator Design and the Bode Plot

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to do the following:

1. Design a series compensator using either the root locus or the frequency response plot.

2. Use the open-loop frequency response plot to design a controller.

3. Analyze the effect of time delays on system response.

4. Apply MATLAB and Simulink to the methods of this chapter.

Section 12.1 shows how to use the open-loop frequency response plots to design PID controllers and series compensators. It also introduces some additional performance criteria: the phase and gain margins, and the static error coefficients. The frequency response plots of the system's open-loop transfer function contain much information about the behavior of the closed-loop system. These plots are easily generated even for high-order systems, and they enable the proper control gain to be selected simply by adjusting the scale factor on the plot. The technique is especially useful, because it does not require the values of the characteristic roots. This is helpful for analyzing high-order systems and systems with dead time. The latter are especially difficult to treat by analysis of the characteristic roots, because they possess an infinite number of roots. ∎

12.1 SERIES COMPENSATION

If implemented in hardware, a *series* compensator is a physical device that is inserted in the control system at the output of the controller (see Figure 12.1.1). If implemented in software in a digital control system, the series compensator is an algorithm that creates the equivalent of a transfer function inserted in series with the transfer function of the main control algorithm. Series compensation is particularly well suited for design with the root locus or with the open-loop Bode plots. Usually, the system is first analyzed with only proportional control action included. The series compensator is then added to meet the performance specifications.

We may think of I action and D action as forms of series compensation. P action compensated with D action gives the transfer function $K_P(1 + T_D s)$. The term $(1 + T_D s)$ can be considered as a series compensator to the proportional controller. The D action adds an open-loop zero at $s = -1/T_D$. The PI control algorithm is described by

$$\frac{F(s)}{E(s)} = K_P \left(1 + \frac{1}{T_I s}\right) E(s) = \frac{K_P}{s} \left(s + \frac{1}{T_I}\right)$$

The integral action can thus be considered to add an open-loop pole at $s = 0$ and a zero at $s = -1/T_I$.

Three other commonly used series compensators are the *lead,* the *lag,* and the *lag-lead* compensators. Their names derive from the change they produce in the system phase angle, which we will discuss with Bode plot design in Section 12.2. These compensators can be easily realized with passive electrical RC networks or with active circuits using op amps. Devices using springs and dampers can implement these compensators mechanically. Many commercially available digital control packages include these compensators as a choice of control action; some in fact do not implement PID action, but use exclusively the lead, lag, and lag-lead compensators, which they call the "filter." We will see how they can be made to emulate PID action.

The transfer functions of the lead and the lag compensators have the form

$$G_c(s) = K\frac{s + a}{s + b} \tag{12.1.1}$$

For a lead compensator, $a < b$, whereas for a lag compensator, $a > b$. The transfer function of the lag-lead compensator has the form

$$G_c(s) = K\frac{s + a}{s + b}\frac{s + c}{s + d} \tag{12.1.2}$$

It is useful to compare these compensators with PID control (Table 12.1.1). First note that I action and D action always require an active circuit unless the control algorithm is being implemented in software as part of a digital control system. On the other hand, the lead, lag, and lag-lead compensators can be implemented with passive circuits.

Figure 12.1.1 Structure of a system with series compensation.
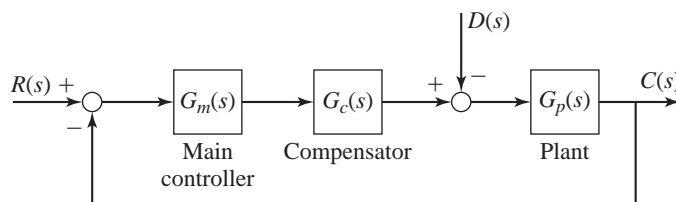
Table 12.1.1 Comparison of PID, lead, lag, and lag-lead compensators.

Compensator	Application	Transfer function	Advantages and limitations
P	Basis for all control action	K_P	1. Often results in nonzero steady-state error. 2. Easily implemented in electrical and nonelectrical systems.
PI	Reduces steady-state error	$K_P + \dfrac{K_I}{s}$	1. Increases system order by 1. 2. Requires active circuit. 3. May increase overshoot.
PD	Improves transient response	$K_P + K_D s$	1. Requires active circuit. 2. May increase overshoot. 3. May cause noise. 4. Better implemented by rate feedback.
PID	Reduces error and improves transient response	$K_P + \dfrac{K_I}{s} + K_D s$	Same as PI and PD.
Lead	Improves transient response	$K\dfrac{s+a}{s+b} \quad (a < b)$	1. Active circuit not required. 2. Increases system order by 1 if no cancellation.
Lag	Reduces steady-state error	$K\dfrac{s+c}{s+d} \quad (c > d)$	1. Active circuit not required. 2. Increases system order by 1 if no cancellation.
Lag-Lead	Reduces error and improves transient response	$K\dfrac{s+a}{s+b}\dfrac{s+c}{s+d}$	1. Same as lead and lag.

Another implementation problem with I action K_I/s is that it is difficult to construct a circuit that gives a pole at $s = 0$, and usually the circuit transfer function is an approximation that has a negative pole close to the origin. Note that PI action can be approximated with the lag compensator by choosing b to be close to 0.

Physical devices implementing D action are prone to amplify noise in the signals because their response is designed to be proportional to the rate of change of the input signal. The op-amp circuit for PD action presented in Chapter 10 was designed not to give pure PD action at higher frequencies for this reason. As we also saw in Chapter 10, theoretical models of D action produce an impulse in response to a step command, which is physically impossible. For these reasons, you should consider implementing the equivalent of D action by using rate feedback if possible. In such an arrangement, the derivative is computed for signals that are more slowly varying than the command input.

HARDWARE IMPLEMENTATION

A variety of lead, lag, and lag-lead compensator circuits have been developed, and three passive circuits are shown in Figures 12.1.2, 12.1.3, and 12.1.4.

When used as series compensators, the circuits shown must see a small impedance at the source (v_i) and a large impedance at the load (v_o). Sometimes isolating amplifiers are inserted to ensure the validity of this assumption. Taking these impedances to be

Figure 12.1.2 Passive lead compensator circuit.
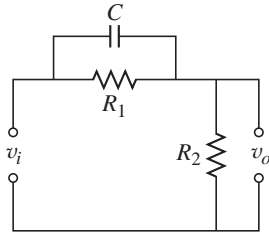
Figure 12.1.3 Passive lag compensator circuit.
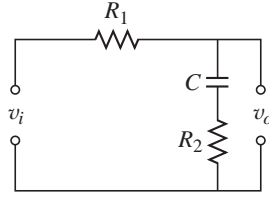
Figure 12.1.4 Passive lag-lead compensator circuit.
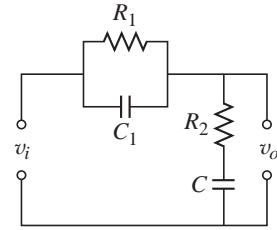
zero and infinity, respectively, we can derive the transfer functions with the methods in Chapter 6. For the lead compensator (Figure 12.1.2).

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 + R_1 R_2 Cs}{R_1 + R_2 + R_1 R_2 Cs} = \frac{1}{\mu_1}\frac{1 + \mu_1 T_1 s}{1 + T_1 s} = \frac{s + 1/\mu_1 T_1}{s + 1/T_1} \tag{12.1.3}$$

where

$$\mu_1 = \frac{R_1 + R_2}{R_2} > 1 \tag{12.1.4}$$

$$T_1 = \frac{R_1 R_2}{R_1 + R_2} C \tag{12.1.5}$$

Note that the circuit would be useless as a series compensator without the resistance R_1 because it would not pass dc signals. Comparing (12.1.1) with (12.1.3), we see that $K = 1$, $a = 1/\mu_1 T_1$, and $b = 1/T_1$. Note that the pole and zero are separated by the factor μ_1, so that $b = \mu_1 a$.

For the lag compensator (Figure 12.1.3),

$$\frac{V_o(s)}{V_i(s)} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \mu_2 T_2 s}{1 + T_2 s} = \mu_2 \frac{s + 1/\mu_2 T_2}{s + 1/T_2} \tag{12.1.6}$$

where

$$\mu_2 = \frac{R_2}{R_1 + R_2} < 1 \tag{12.1.7}$$

$$T_2 = (R_1 + R_2)C \tag{12.1.8}$$

Comparing (12.1.1) with (12.1.6), we see that $a = 1/\mu_2 T_2$, $b = 1/T_2$, and $K = \mu_2$, which gives a low-frequency gain. Note that the pole and zero are separated by the factor μ_2, so that $b = \mu_2 a$.

Lag and lead compensators placed in series give the transfer function

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{s + 1/\mu_1 T_1}{s + 1/T_1}\right)\mu_2\left(\frac{s + 1/\mu_2 T_2}{s + 1/T_2}\right) \tag{12.1.9}$$

A simpler approach is to use the single network shown in Figure 12.1.4. With the usual impedance assumptions, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{1 + (R_1 C_1 + R_2 C_2)s + R_1 C_1 R_2 C_2 s^2}{1 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + R_1 C_1 R_2 C_2 s^2}$$

$$= \left(\frac{s + 1/\mu_3 T_3}{s + 1/T_3}\right)\left(\frac{s + 1/\mu_4 T_4}{s + 1/T_4}\right) \tag{12.1.10}$$
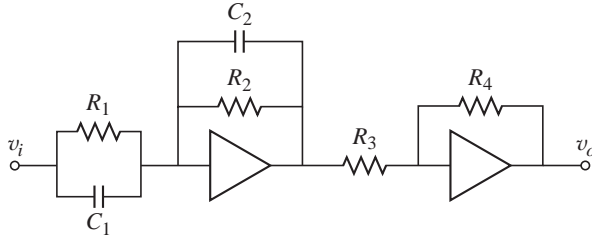
Figure 12.1.5 Active circuit for lead or lag compensation.

where

$$\mu_3 T_3 = R_1 C_1 \quad \mu_3 > 1 \tag{12.1.11}$$

$$\mu_4 T_4 = R_2 C_2 \tag{12.1.12}$$

$$T_3 + T_4 = R_1 C_1 + R_1 C_2 + R_2 C_2 \tag{12.1.13}$$

$$\mu_4 = \frac{1}{\mu_3} \tag{12.1.14}$$

Equations (12.1.11) through (12.1.13) contain four unknowns: R_1, R_2, C_1, and C_2. Once an analysis has determined values for T_3, T_4, μ_3, and μ_4, a convenient value for either C_1 or C_2 can be selected, and the equations solved for the three remaining unknowns.

An active circuit for lead or lag compensation is shown in Figure 12.1.5. Using the methods of Chapter 6, we can derive the following transfer function for the circuit:

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R_2 R_4}{R_1 R_3}\right) \frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} = \left(\frac{R_4 C_1}{R_3 C_2}\right) \frac{s + 1/R_1 C_1}{s + 1/R_2 C_2} \tag{12.1.15}$$

or

$$\frac{V_o(s)}{V_i(s)} = K_c \frac{s + 1/\mu_5 T_5}{s + 1/T_5} \tag{12.1.16}$$

where

$$T_5 = R_2 C_2 \tag{12.1.17}$$

$$\mu_5 = \frac{R_1 C_1}{R_2 C_2} \tag{12.1.18}$$

$$K_c = \frac{R_4 C_1}{R_3 C_2} \tag{12.1.19}$$

The circuit is a lead compensator if $R_1 C_1 > R_2 C_2$. It is a lag compensator if $R_1 C_1 < R_2 C_2$. Note that the passive lag compensator circuit has a low frequency gain of 1, whereas the op-amp compensator has a gain K_c.

An active circuit for lag-lead compensation is shown in Figure 12.1.6. Its transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{R_4 R_6}{R_3 R_5} \left[\frac{(R_1 + R_3)C_1 s + 1}{R_1 C_1 s + 1}\right] \left[\frac{R_2 C_2 s + 1}{(R_2 + R_4)C_2 s + 1}\right] \tag{12.1.20}$$

or

$$\frac{V_o(s)}{V_i(s)} = K_c \frac{(s + 1/\mu_6 T_6)(s + 1/\mu_7 T_7)}{(s + 1/T_6)(s + 1/T_7)} \tag{12.1.21}$$

Figure 12.1.6 Active circuit for lag-lead compensation.
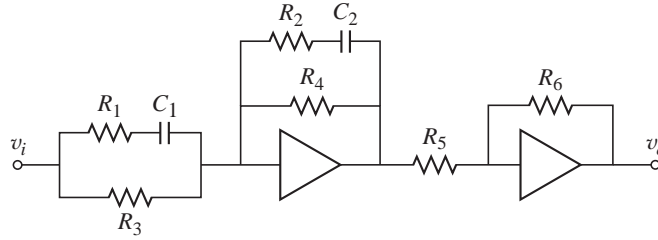
where

$$T_6 = R_1 C_1 \tag{12.1.22}$$

$$T_7 = (R_2 + R_4)C_2 \tag{12.1.23}$$

$$\mu_6 = \frac{R_1 + R_3}{R_1} > 1 \tag{12.1.24}$$

$$\mu_7 = \frac{R_2}{R_2 + R_4} < 1 \tag{12.1.25}$$

$$K_c = \frac{R_2 R_4 R_6}{R_1 R_3 R_5}\left(\frac{R_1 + R_3}{R_2 + R_4}\right) \tag{12.1.26}$$

The circuit has a low frequency gain of $R_4 R_6 / R_3 R_5$.

COMPENSATOR DESIGN BY ROOT PLACEMENT

Sometimes we have the flexibility of designing the compensator simply by specifying all the root locations. This can be done for an nth-order system if the last $n - 1$ coefficients of the characteristic equation are independent functions of the gains.

EXAMPLE 12.1.1

Lead and PID Compensation for the Plant $1/s^2$

■ Problem
The equation of motion of an object in pure rotation is $I\ddot{\theta} = T - T_d$, where I is the inertia, T is the torque applied by the controller to control the angular position θ, and T_d is a disturbance torque. Consider the case shown in Figure 12.1.7, where $I = 1$. Design lead and PID compensators for this system. The specifications are that the closed-loop system must have a dominant time constant no greater than 0.5 and a damping ratio no less than 0.707. Evaluate the resulting steady-state error for unit-step and unit-ramp commands and disturbances.

Figure 12.1.7 Control of a rotating object.
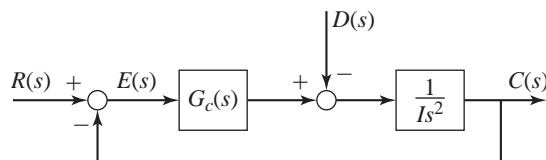
■ **Solution**

Using a lead compensator with P action gives the command transfer function

$$\frac{C(s)}{R(s)} = \frac{K_P(s+a)}{s^3 + bs^2 + K_P s + aK_P}$$

Values must be obtained for three parameters: K_P, a, and b. Note that we can select any three closed-loop roots we want because K_P, a, and b can be selected to obtain any values for the last three coefficients of the characteristic polynomial. That is, if the desired polynomial is $s^3 + a_2 s^2 + a_1 s + a_0$, then we select $b = a_2$, $K_P = a_1$, and $a = a_0/K_P$. Note, however, that if b is chosen small to emulate integral action, then we lose the flexibility to choose all three roots. In that case, a root locus analysis could be used to set the values of K_P and a.

With the PID compensator, the command transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + K_P s + K_I}$$

Values must be obtained for three parameters: K_D, K_P, and K_I. We can select any three closed-loop roots we want because these gains can be selected independently as follows. For the desired polynomial $s^3 + a_2 s^2 + a_1 s + a_0$, we select $K_D = a_2$, $K_P = a_1$, and $K_I = a_0$.

So both designs can meet the given specifications for the transient response. The error transfer functions are given in the following Table 12.1.2. From these we can evaluate the

Table 12.1.2 Comparison of lead and PID compensators for the plant $1/s^2$.

Compensator	P Action with lead	PID
$G_c(s)$	$K_P \dfrac{s+a}{s+b}$	$\dfrac{K_D s^2 + K_P s + K_I}{s}$
$\dfrac{C(s)}{R(s)}$	$\dfrac{K_P(s+a)}{s^3 + bs^2 + K_P s + aK_P}$	$\dfrac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + K_P s + K_I}$
$\dfrac{E(s)}{R(s)}$	$\dfrac{s^3 + bs^2}{s^3 + bs^2 + K_P s + aK_P}$	$\dfrac{s^3}{s^3 + K_D s^2 + K_P s + K_I}$
$\dfrac{E(s)}{D(s)}$	$\dfrac{s+b}{s^3 + bs^2 + K_P s + aK_P}$	$\dfrac{s}{s^3 + K_D s^2 + K_P s + K_I}$
Steady-state error		
Step command	0	0
Ramp command	0	0
Step disturbance	$\dfrac{b}{aK_P}$	0
Ramp disturbance	∞	$\dfrac{1}{K_I}$
Roots	$s = -2$, $s = -2 \pm 2j$	$s = -2$, $s = -2 \pm 2j$
Gains	$K_P = 16$, $a = 1$, $b = 6$	$K_P = 16$, $K_I = 16$, $K_D = 6$
Steady-state error		
Step disturbance	$\dfrac{3}{8}$	0
Ramp disturbance	∞	$\dfrac{1}{16}$
Step response		
Overshoot	32.8%	26.1%
Peak time	1.1	0.58
Settling time	2.43	2.49
10%–90% rise time	0.417	0.201

Figure 12.1.8 Step command response for Example 12.1.1.
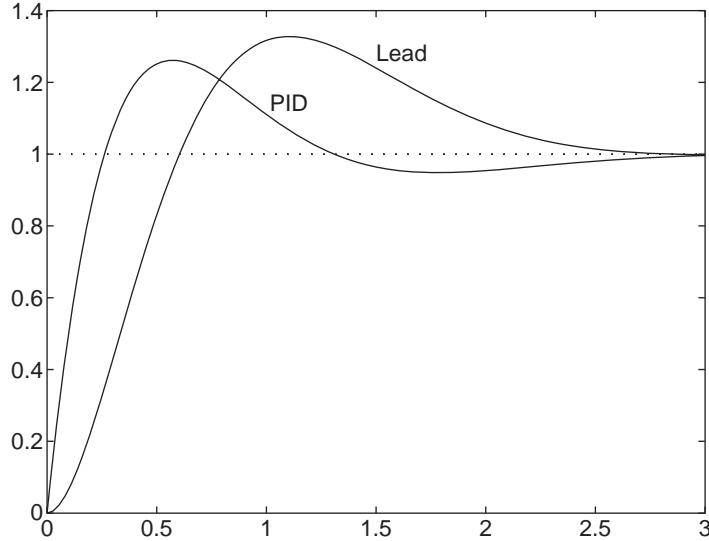
Figure 12.1.9 Ramp response for Example 12.1.1.
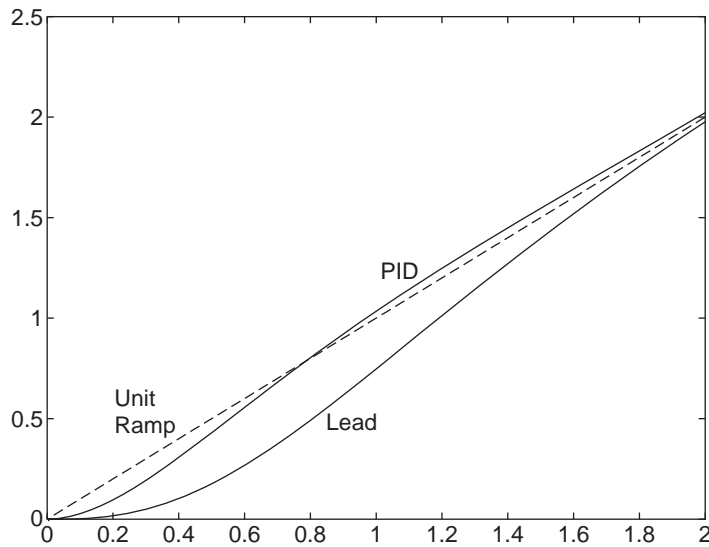
steady-state error for each design. Those errors are also given in the table. We see that the lead compensator does not do as well as the PID in rejecting the disturbance. Its response to a step disturbance can be made small by choosing b small, but, as we have seen, we then lose the flexibility to choose all three roots.

The two transient response specifications are satisfied by choosing the roots to be $s = -2$ and $s = -2 \pm 2j$. This gives the polynomial

$$(s + 2)\left[(s + 2)^2 + 2^2\right] = s^3 + 6s^2 + 16s + 16$$

For the lead compensator, this gives $K_P = 16$, $a = 1$, and $b = 6$. For the PID compensator, this gives $K_P = 16$, $K_I = 16$, $K_D = 6$. The numerical results for these values are given in the table. Figures 12.1.8, 12.1.9, and 12.1.10 show the step command response, the ramp command response, and the step disturbance error, respectively.

The PID has a smaller overshoot but a faster rise time. Because of this and its smaller errors, the performance of PID is better than the lead compensator in this example, assuming that the pole at $s = 0$ in the PID can be implemented physically or in a digital controller.

ROOT LOCUS DESIGN OF COMPENSATORS

The series compensators' usefulness can be briefly described as follows. When used in series with a proportional gain K_P, the lead compensator enables an increase in the speed of response. On the other hand, the lag compensator is used when the speed of response and damping of the closed-loop system are satisfactory, but the steady-state error is too large. As you might expect, the lag-lead compensator is used when both the transient and steady-state performance must be improved.

When the performance specifications, such as required time constant, damping ratio, and so forth are given in terms of root locations, but not all the root locations are known or can be freely specified, the root locus method is the preferred way of designing the compensator.

LEAD COMPENSATORS

The effects of the compensator in terms of time domain specifications (the characteristic roots) can be shown with the root locus plot. Consider the second-order plant $1/(s + \alpha)(s + \beta)$ with the distinct real roots $s = -\alpha, -\beta$. The root locus for this system with proportional control is shown in Figure 12.1.11a. The smallest dominant time constant obtainable is τ_1, marked in the figure. With lead compensation, the root locus becomes that shown in Figure 12.1.11b. The pole and zero introduced by the compensator reshape the locus so that a smaller dominant time constant can be obtained. This is done by choosing the proportional gain high enough to place the roots close to the asymptotes.

(a)   (b)
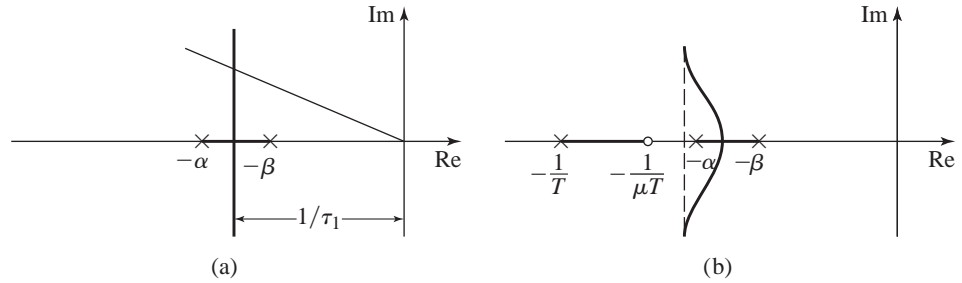
Designing a lead compensator with the root locus is done as follows.

1. From the time domain specifications (time constant, damping ratio, etc.), determine the required locations of the dominant closed-loop poles.

2. From the root locus plot of the uncompensated system, determine whether or not the desired closed-loop poles can be obtained by adjusting the open-loop gain. If not, determine the net angle associated with the desired closed-loop pole by drawing vectors to this pole from the open-loop poles and zeros. The difference between this angle and $-180°$ is the *angle deficiency*.

3. Locate the pole and zero of the compensator so that they will contribute the angle required to eliminate the deficiency. A method for doing this is presented in Example 12.1.2.

4. Compute the required value of the open-loop gain from the root locus plot.

5. Check the design to see if the specifications are met. If not, adjust the locations of the compensator's pole and zero.

EXAMPLE 12.1.2

Root Locus Design of a Lead Compensator

■ Problem

Consider the system shown in Figure 12.1.12a. The root locus plot with $G_c(s) = 1$ is given in Figure 12.1.12b. The transient specifications require that $\zeta = 0.707$ with a time constant of $\tau = 0.2$. No steady-state error specifications are given. This information implies that the closed-loop poles should be at $s = -5 \pm 5j$. This performance cannot be obtained with a gain change in the present system. Design a lead compensator to meet the specifications.

■ Solution

From (12.1.3), the open-loop transfer function of the compensated system can be expressed as

$$K_P G_c(s) G_p(s) = K_P \frac{s + a}{s + b} \frac{1}{s(s + 5)} = K_P \frac{s + 1/\mu T}{s + 1/T} \frac{1}{s(s + 5)} \tag{1}$$

We must pull the original locus to the left, so we place the pole and zero to the left of the pole at $s = -5$. Drawing vectors from the poles and zeros to the desired root location, we obtain Figure 12.1.12c. The angle of the uncompensated system at the desired root location is

$$\angle \left. \left| \frac{1}{s(s + 5)} \right| \right|_{s=-5+5j} = -90° - 135° = -225°$$
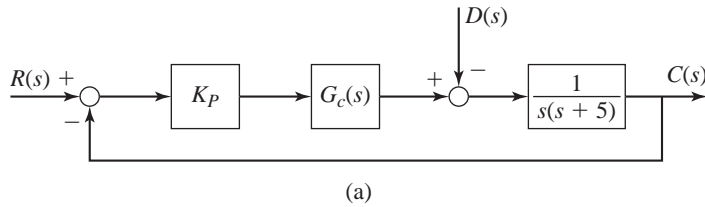
(a)

(b)

(c)
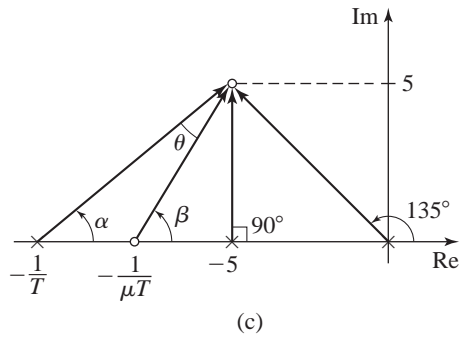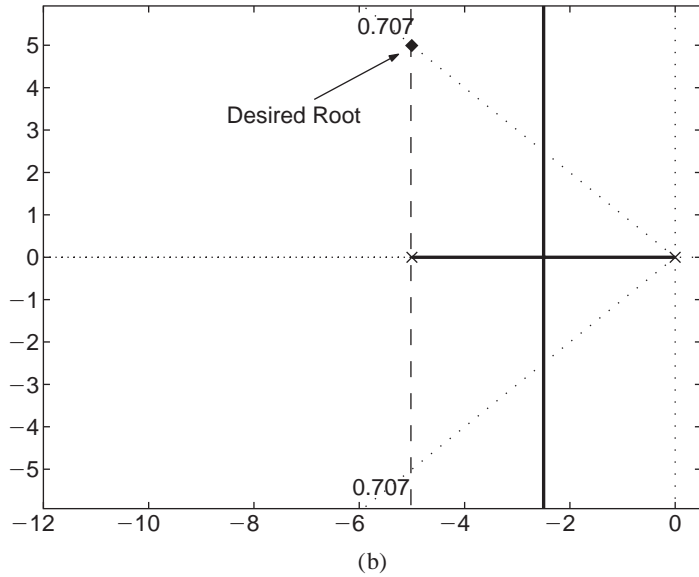
Thus, the angle deficiency is $-225° + 180° = -45°$, and the difference between the angle contributions of the pole and zero of the compensator must be $45°$. We therefore need a lead compensator to increase the phase angle by $45°$. This is the angle θ shown in the apex of the triangle in the figure.

Because $\beta = \alpha + 45°$, simple trigonometry applied to Figure 12.1.12c gives the required value for T. The tangents of α and β are

$$\tan \alpha = \frac{5}{1/T - 5} = \frac{5T}{1 - 5T}$$

$$\tan \beta = \frac{5}{1/\mu T - 5} = \frac{5\mu T}{1 - 5\mu T}$$

Figure 12.1.13 Root locus of the lead-compensated system of Example 12.1.2.
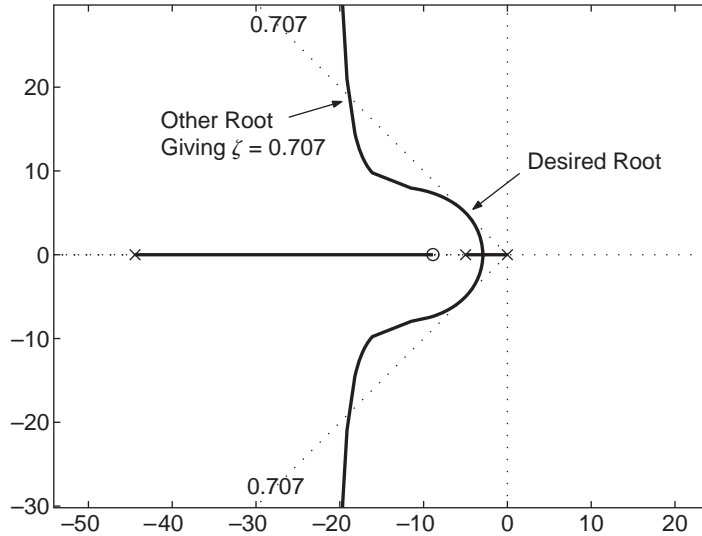
From the identity for the tangent of a sum, we have

$$\tan \beta = \tan(\alpha + 45°) = \frac{\tan \alpha + \tan 45°}{1 - \tan \alpha \, \tan 45°}$$

Eliminating α and β yields

$$50\mu T^2 - 10\mu T + 1 = 0 \qquad\qquad (2)$$

Because the inverse of μ appears in the numerator of the transfer function (1), we should select its value to be as small as possible in order to minimize the additional amount of gain required to cancel its effect. Keeping physical realizability in mind, we choose $\mu = 5$. This gives

$$250T^2 - 50T + 1 = 0$$

which has the solutions $T = 0.1775$ and $T = 0.0225$. The first solution results in the pole lying to the left of $s = -5$ and the zero lying to the right of $s = -5$, which contradicts the assumed geometry. So we use $T = 0.0225$. This puts the compensator pole at $s = -44.4$, and the zero at $s = -8.89$.

The open-loop transfer function of the compensated system is

$$K_P G_c(s) G_p(s) = \frac{K_P(s + 8.89)}{s(s + 44.4)(s + 5)}$$

Its root locus is shown in Figure 12.1.13. From this, the gain K_P required to place the roots at $s = -5 \pm 5j$ is $K_P = 227$. The third root is far to the left at $s = -39.3$ for this value of K_P, so its effect on the transient behavior is probably slight. This can be checked by analysis or simulation before the design is made final.

The unit-step response is shown in Figure 12.1.14. The overshoot is 7.7% with a peak time of 0.47 and a settling time of 0.75.

From (12.1.4) and (12.1.5), the resistances required with a 1-μF capacitance are $R_1 = 112.5$ kΩ and $R_2 = 28.1$ kΩ.
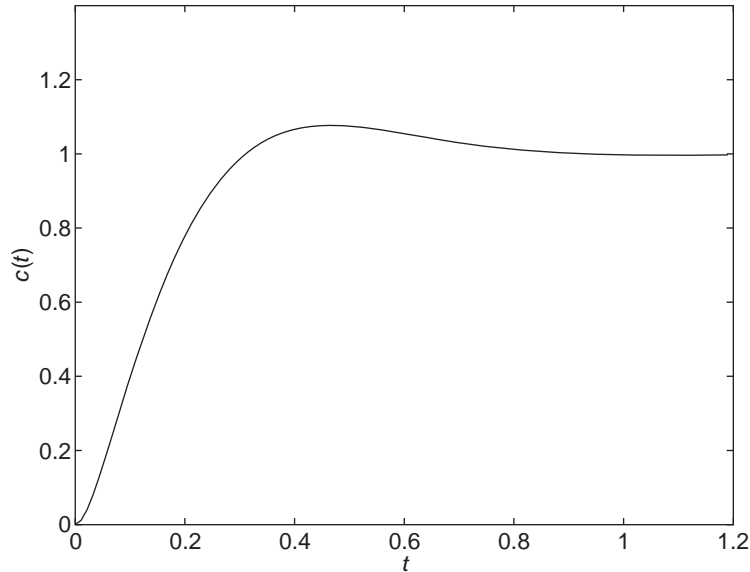
Figure 12.1.14 Unit-step response of the lead-compensated system of Example 12.1.2.

This design procedure does not always give a root locus like Figure 12.1.1b. For example, in Example 12.1.2, the choice of $\mu = 10$, rather than 5, results in the following solution of equation (2): $T = 0.1894$ and $T = 0.0106$. The latter solution puts the pole at -94.72 and the zero at -9.472. These result in the root locus shown in Figure 12.1.15. The solution giving $\zeta = 0.707$ and $\tau = 0.2$ corresponds to $K_P = 474$ with the roots at $s = -5 \pm 5j$ and $s = -89.72$. The step response has an overshoot of 7.1%, a peak time of 0.475, and a settling time of 0.753.

LAG COMPENSATORS

Consider proportional control of the plant $1/(s + \alpha)(s + \beta)$. Its root locus plot is shown in Figure 12.1.16a. Suppose that the desired damping ratio ζ and desired time
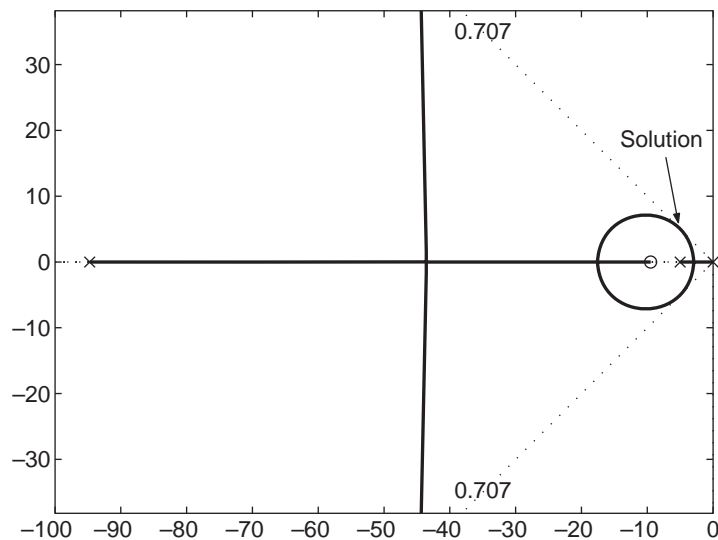
Figure 12.1.15 Another design solution for Example 12.1.2.
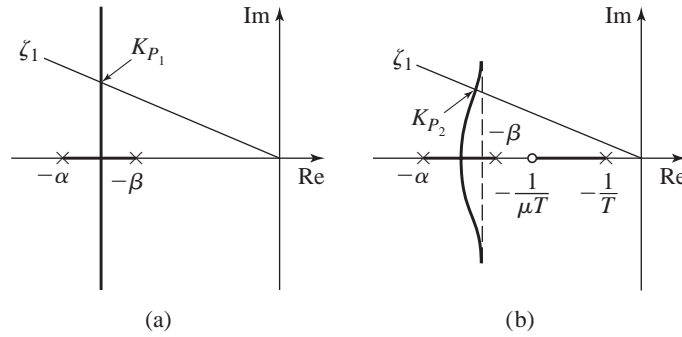
constant τ are obtainable with a proportional gain of K_{P_1} but the resulting steady-state
error $\alpha\beta/(\alpha\beta + K_{P_1})$ for a step input is too large. We need to increase the gain while
preserving the desired damping ratio and time constant. With the lag compensator, the
root locus is as shown in Figure 12.1.16b. By considering specific numerical values,
we can show that for the compensated system, roots with a damping ratio ζ_1 correspond
to a high value of the proportional gain. Call this value K_{P_2}. Thus, $K_{P_2} > K_{P_1}$, and the
steady-state error will be reduced.

The effect of the lag compensator on the time constant can be seen as follows. The
open-loop transfer function is

$$K_P G_c(s) G_p(s) = \frac{\mu K_P (s + 1/\mu T)}{(s + \alpha)(s + \beta)(s + 1/T)} \tag{12.1.27}$$

If the value of T is chosen large enough, the pole at $s = -1/T$ in (12.1.27) is approx-
imately cancelled by the zero at $s = -1/\mu T$ and the open-loop transfer function is
given approximately by

$$K_P G_c(s) G_p(s) = \frac{\mu K_P}{(s + \alpha)(s + \beta)} \tag{12.1.28}$$

Thus, the system's response is governed approximately by the complex roots corre-
sponding to the gain value K_{P_2}. By comparing Figure 12.1.16a with 12.1.16b, we see
that the compensation leaves the time constant relatively unchanged. From (12.1.28),
it can be seen that because $\mu < 1$, K_P can be selected as the larger value K_{P_2}. The ratio
of K_{P_1} to K_{P_2} is approximately given by the parameter μ.

Design by pole-zero cancellation can be difficult to accomplish, because a response
pattern of the system is essentially ignored. The pattern corresponds to the behavior
generated by the cancelled pole and zero, and this response can be shown to be beyond
the influence of the controller. In this example, the cancelled pole gives a stable response,
because it lies in the left-hand plane. However, another input not modeled here, such as
a disturbance, might excite the response and cause unexpected behavior. The designer
should therefore proceed with caution. None of the physical parameters of the system
are known exactly, so exact pole-zero cancellation is not possible. A root locus study
of the effects of parameter uncertainty and a simulation study of the response are often
advised before the design is accepted as final.

Based on this example, we can outline an approach to the design of a lag compen-
sator with the root locus as follows:

1. From the root locus of the uncompensated system, determine the gain K_{P_1} to
   place the roots at the locations required to meet the transient performance
   specifications.

2. Let K_{P_2} denote the value of the gain required to achieve the desired steady-state performance. The parameter μ is the ratio of these two gain values $\mu = K_{P_1}/K_{P_2} < 1$.

3. The value of T is then chosen large so that the compensator's pole and zero are close to the imaginary axis. This placement should be made so that the compensated locus is relatively unchanged in the vicinity of the desired closed-loop poles. This will be true if the angle contribution of the lag compensator is close to zero.

4. Locate the desired closed-loop poles on the compensated locus and set the open-loop gain so that the dominant roots are at this location (neglecting the existence of the compensator's pole and zero).

5. Check the design to see if the specifications are met. If not, adjust the locations of the compensator's pole and zero.

Root Locus Design of a Lag Compensator | **EXAMPLE 12.1.3**

■ **Problem**

Consider the system shown in Figure 12.1.17a. Suppose that we require a time constant of 0.4 and a damping ratio of $\zeta = 0.707$. These correspond to the roots $s = -2.5 \pm 2.5j$, which are obtainable with $K_P = 12.5$. The steady-state error for a unit ramp command is 0.4, however, which is considered too large. Design a compensator to reduce the error to 0.1.
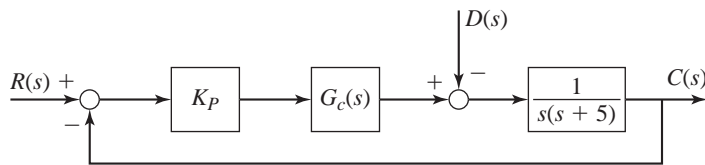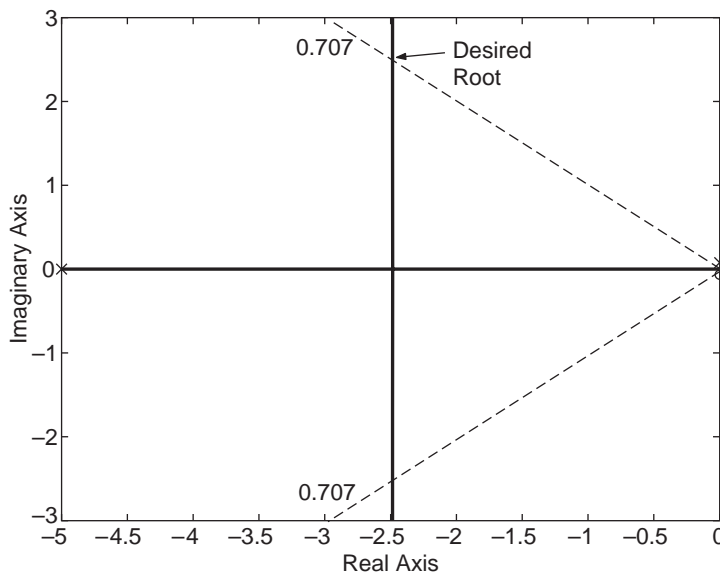
Figure 12.1.17 (a) System for Example 12.1.3. (b) Root locus plot of the lag-compensated system.

■ Solution

A lag compensator is indicated, because the transient response is acceptable but the steady-state error is too large. The gain K_P required to achieve the desired transient performance has already been established as $K_{P_1} = 12.5$. The second step is to determine the value of the parameter μ. For this system, the steady-state ramp error is $e_{ss} = 5/K_P$, and K_{P_2} is the value of K_P that gives $e_{ss} = 0.1$. Thus, $K_{P_2} = 5/0.1 = 50$, and the parameter $\mu = K_P/K_{P_2} = 12.5/50 = 0.25$. The compensator's pole and zero must be placed close to the imaginary axis, with the ratio of their distances being $1/0.25 = 4$. Noting that the plant has a pole at $s = -5$, we select locations well to the right of this pole, say, at $s = -0.01$ and $s = -0.04$ for the pole and zero, respectively. This gives $T = 100$.

The open-loop transfer function of the compensated system is thus

$$K_P G_c(s) G_p(s) = \frac{0.25 K_P (s + 0.04)}{s(s + 5)(s + 0.01)}$$

The root locus is shown in Figure 12.1.17b, where we have offset the pole and zero to make them visible. For the desired damping ratio of 0.707, the root locus shows that two roots are at $s = -2.5 \pm 2.5j$ if $0.25 K_P = 12.7$, or $K_P = 12.7/0.25 = 50.8$. The dominant root is at $s = -0.0405$, but its effect is diminished by the zero at $s = -0.04$. The steady-state error is thus $e_{ss} = 0.098$ and less than the required value of 0.1.

The step response of the final design has an overshoot of 5.75%, a peak time of $t_p = 1.27$, and a settling time of 1.89. With $T = 100$ and the capacitance $C = 1\,\mu F$, (12.1.7) and (12.1.8) give $R_1 = 7.5\,k\Omega$ and $R_2 = 2.5\,k\Omega$.

DESIGN OF LAG-LEAD COMPENSATORS

The lead and lag compensators are complementary to each other in that one improves the transient performance while the other improves steady-state performance. In situations where one of these fails to produce a satisfactory design, a lag-lead compensator can be tried.

The root locus approach to designing a lag-lead compensator is a combination of the approaches used for the lead and lag compensators.

1. Evaluate the uncompensated system by determining the desired locations of the dominant closed-loop poles from the transient performance specifications.
2. Improve the transient performance with a lead compensator. Calculate the phase angle deficiency of the uncompensated system. This deficiency must be supplied by the lead compensator.
3. Evaluate the steady-state error and if necessary, improve the steady-state performance with a lag compensator.
4. Evaluate the transient performance to see if the specifications are satisfied.
5. Evaluate the hardware requirements and check realizability.

EXAMPLE 12.1.4   Design of a Lag-Lead Compensator

■ Problem

The problem is to design a lag-lead compensator for the plant shown in Figure 12.1.18a. The specifications are

1. Damping ratio of the dominant roots should be $\zeta = 0.707$.
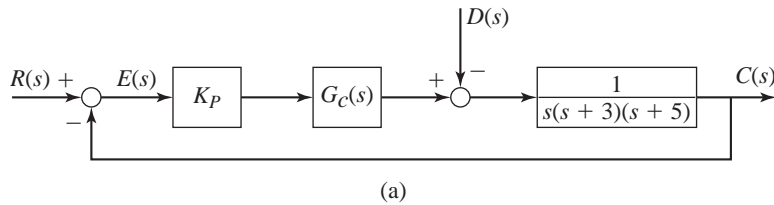
Figure 12.1.18 (a) System for Example 12.1.4. (b) Root locus plot of the uncompensated system.
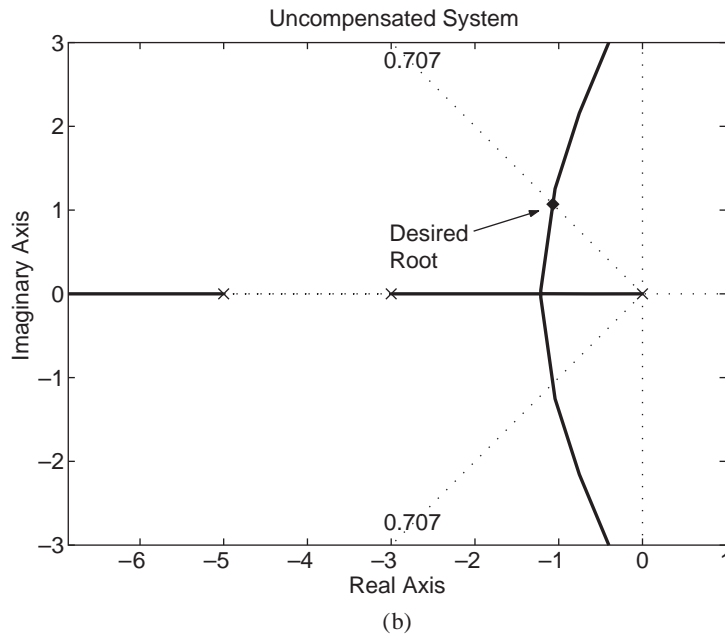
2. The time constant of the dominant roots should be no greater than 0.5.
3. The steady-state error for a unit-ramp disturbance should be no greater than 0.1.

The first two specifications will be satisfied with a dominant root pair $s = -2 \pm 2j$, but other roots are possible.

■ **Solution**

Step 1: Evaluate the uncompensated system. The root locus equation for the uncompensated system using P action is

$$1 + K_P \frac{1}{s(s+3)(s+5)} = 0$$

Figure 12.1.18b shows the root locus plot. From this plot we determine that $K_P = 13.6$ will give a dominant root pair of $s = -1.08 \pm 1.08j$, for which $\zeta = 0.707$. The time constant is 0.93 and the unit-ramp disturbance error is 1.03, so the uncompensated system does not meet the second and third specifications.

Table 12.1.3 summarizes the various designs.

Step 2: Improve the transient performance with a lead compensator. The form of the lead compensator is

$$G_c(s) = \frac{s+a}{s+b}$$

If we can cancel the plant pole at $s = -3$, the root locus will be shifted to the left, closer to the desired roots at $s = -2 \pm 2j$. Thus, we place the compensator zero at $s = -3$ and choose

Table 12.1.3 Design of a lag-lead compensator for the plant $1/s(s+3)(s+5)$.

Design	Uncompensated	Lead compensated	Lag-Lead compensated
Open-loop transfer function	$\dfrac{K_P}{s(s+3)(s+5)}$	$\dfrac{K_P}{(s+11.9)s(s+5)}$	$\dfrac{K_P(s+0.0578)}{(s+0.01)(s+11.9)s(s+5)}$
Dominant roots	$s = -1.08 \pm 1.08j$	$s = -2 \pm 2j$	$s = -1.9 \pm 1.9j$
K_P	13.6	103	101
ζ	0.707	0.707	0.707
τ	0.93	0.5	0.526
Steady-state ramp error	1.03	0.578	0.1

$a = 3$. The location of the compensator pole $s = -b$ is found from the geometry shown in Figure 12.1.19a. From the angle criterion

$$\angle N(s) - \angle D(s) = (2n+1)180° \quad n = 0, 1, 2, 3, \ldots$$

we have

$$-\theta - 33.6° - 135° = (2n+1)180°$$

which gives $\theta = (2n+1)180° - 168.6° = 11.4°$ by choosing $n = 0$. From the figure,

$$\tan \theta = \tan 11.4° = 0.202 = \frac{2}{b-2}$$

which gives $b = 11.9$. Thus the open-loop transfer function of the lead-compensated system is

$$K_P G_c(s) G_p(s) = \frac{K_P(s+3)}{(s+11.9)s(s+3)(s+5)} = \frac{K_P}{(s+11.9)s(s+5)}$$

The root locus plot for this system is shown in Figure 12.1.19b. From it we determine that a gain of $K_P = 103$ will give dominant roots at the desired location of $s = -2 \pm 2j$. Thus, this design meets the first two specifications. The ramp error, however, is 0.578, which is still too large.

Step 3: Improve the steady-state performance with a lag compensator. The form of the lag-lead compensator is

$$K_P G_c(s) = K_P \frac{s+3}{s+11.9} \frac{s+c}{s+d}$$

We place the pole of the lag compensator close to the origin and choose $d = 0.01$. The zero is separated from the pole by the ratio of the current steady-state error to the desired error; that is, $c = (0.578/0.1)d = 0.0578$. Thus, the open-loop transfer function of the lag-lead system is

$$K_P G_c(s) G_p(s) = \frac{K_P(s+0.0578)}{(s+0.01)(s+11.9)s(s+5)}$$

Part of the root locus plot for this system is shown in Figure 12.1.20, and it is very similar to the plot of the lead-compensated system. From it we determine that a gain of $K_P = 101$ will give dominant roots at $s = -1.9 \pm 1.9j$, which is close to the desired location of $s = -2 \pm 2j$. The time constant is 0.526, which is slightly larger than desired. The ramp error, however, is 0.1, which is exactly what was specified.

Further analysis of the root locus shown in Figure 12.1.20 reveals that we can obtain a time constant of 0.5 with a gain of $K_P = 96$, but the damping ratio would be $\zeta = 0.743$.

Step 4: Evaluate the transient performance. Simulation of the final design using $K_P = 101$ shows that the step response is satisfactory. The ramp response, however, reveals something perhaps unexpected (Figure 12.1.21). While the steady-state error is as predicted, the time to reach steady state is much longer than expected from the time constant of 0.526. The effect is due to the small root at $s = -0.0595$, which is actually the dominant root and has a time constant

Figure 12.1.19 (a) Geometry of the lead-compensated root locus for Example 12.1.4. (b) Root locus plot of the lead-compensated system.

(a)

(b)
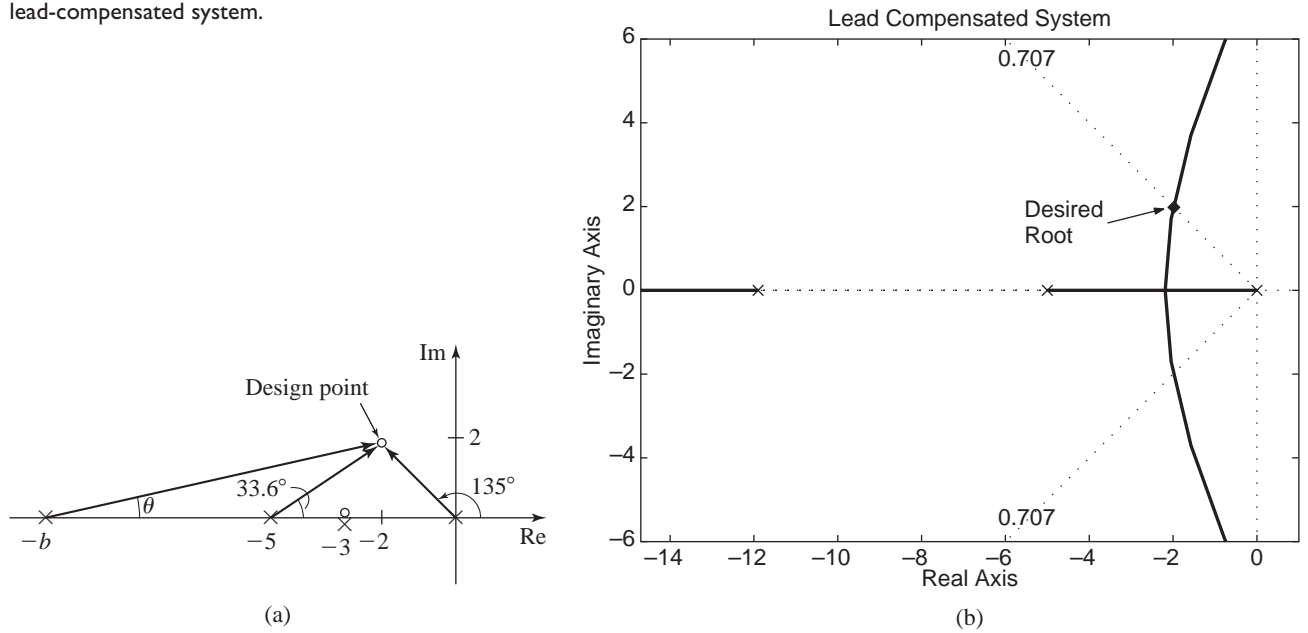
Figure 12.1.20 Root locus plot of the lag-lead system (the pole at -11.9 is not shown).
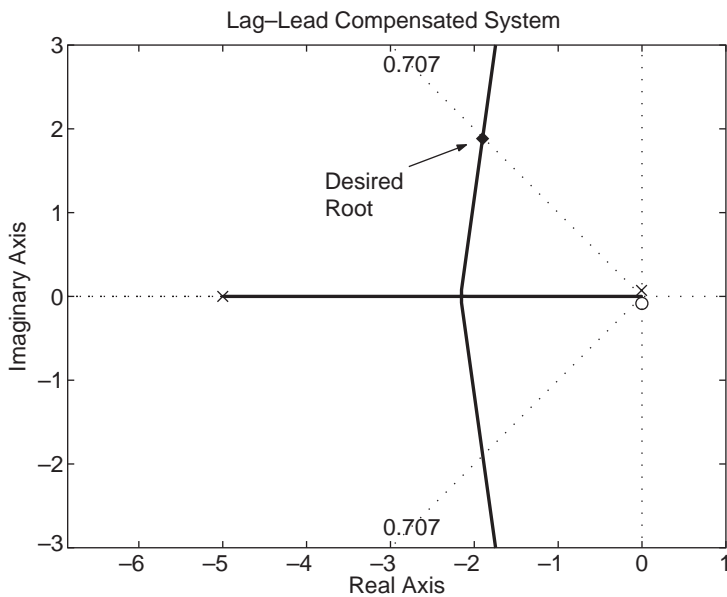
of $1/0.0595 = 16.8$. Figure 12.1.22 is an enlarged view of the root locus near the origin. It shows the dominant root path that ends at the zero at $s = -0.0578$.

This example shows the importance of checking any design with simulation before proceeding further.

Step 5: Evaluate the hardware requirements. Before implementing this design in hardware, we should check to see if it is possible to set the poles and zeros of the compensator as accurately

Figure 12.1.21 Error response to a unit-ramp disturbance for the lag-lead system of Example 12.1.4.
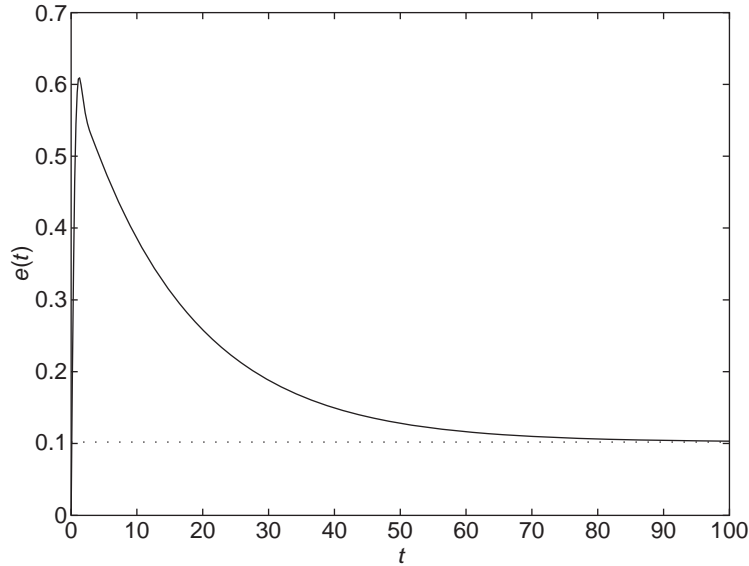
Figure 12.1.22 Enlarged view of the root locus for Example 12.1.4.
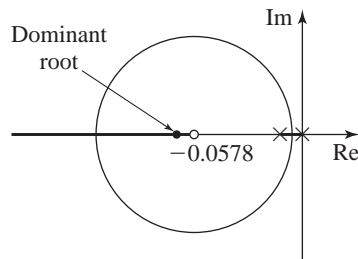
as indicated by our calculations. In particular, we should check the feasibility of obtaining the small values, namely, $d = 0.01$ and $c = 0.0578$.

You should also realize that it is physically impossible to set the lead compensator zero to cancel exactly the plant pole at $s = -3$. Further simulation should be done to check the effects of a not being exactly equal to 3.

We note in passing that the pole at $s = -3$ is not canceled in the disturbance transfer function, which is a fifth-order model.

SYSTEM TYPE AND STEADY-STATE ERROR

Consider a single-loop system whose open-loop transfer function is $G(s)H(s)$. The error signal is related to the input as follows:

$$E(s) = \frac{1}{1 + G(s)H(s)} R(s)$$

A system is of *type n* if $G(s)H(s)$ can be written as $s^{-n}F(s)$. We can relate the steady-state error to the type number for various kinds of command inputs. Assuming that the final value theorem can be applied, the steady-state error is

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

Table 12.1.4 Type number and steady-state error

Input	Steady-state error		
	Type 0	**Type 1**	**Type 2**
Unit step $1/s$	$1/(1 + C_0)$	0	0
Unit ramp $1/s^2$	∞	$1/C_1$	0
Unit parabola $1/s^3$	∞	∞	$1/C_2$

The *static error coefficient C_i* is defined as

$$C_i = \lim_{s \to 0} s^i G(s)H(s) \quad i = 0, 1, 2, \ldots \tag{12.1.29}$$

Often the steady-state performance specifications are given in terms of the static error coefficients. If the input is a unit step, we obtain the steady-state error

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + C_p}$$

where

$$C_p = \lim_{s \to 0} G(s)H(s) = C_0 \tag{12.1.30}$$

The constant C_p is the *static position error coefficient*. The name derives from servo-mechanism applications in which the output is a position. The larger C_p is, the smaller the error, and a unity-feedback system will have a nonzero steady-state error if no integration occurs in the forward path.

The *static velocity error coefficient* is obtained for a unit-ramp input as follows:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s^2} = \frac{1}{C_v}$$

where the velocity coefficient is

$$C_v = \lim_{s \to 0} s G(s)H(s) = C_1 \tag{12.1.31}$$

Note that the error here is not an error in velocity, but a position error that results when a unit-ramp input is applied. For type 0 systems, $C_v = 0$; for type 1 systems, C_v is finite but nonzero; for type 2 and higher, $C_v = \infty$. Thus, to eliminate the steady-state error in a unity feedback system with a ramp input, at least two integrations are required in the forward loop. For a type 1 system with unity feedback, the output velocity at steady state equals that of the input (the slope of the ramp), but an error exists between the desired and the actual positions. These results are summarized in Table 12.1.4.

12.2 DESIGN USING THE BODE PLOT

In comparison to using the root locus plot, there are several advantages to designing a system using the open-loop frequency response. Frequency response data are often easier to obtain experimentally, which is useful when it is difficult to develop a transfer function model of the plant and actuators from basic principles. The method is also easier to use for systems with dead-time elements, which we will see in this section. Finally, this technique is sometimes useful for examining response and instability in nonlinear systems. This topic is rather specialized and is treated in more advanced control system texts.

Figure 12.2.1 A single-loop control system.
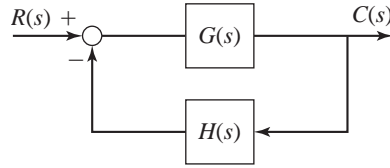
Figure 12.2.2 (a) Definition of gain and phase margins on the open-loop Bode plots. (b) Plots for an unstable system.
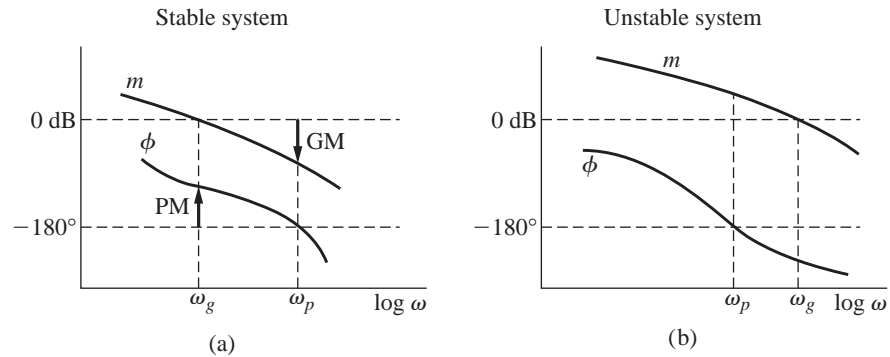
PHASE AND GAIN MARGINS

It is possible to determine the stability and transient response characteristics of a closed-loop system from the frequency response characteristics of the open-loop system. Referring to Figure 12.2.1, we define the *gain crossover frequency* ω_g to be the frequency at which the magnitude ratio (called the "gain") of $G(j\omega)H(j\omega)$ is 1, which corresponds to 0 dB. The *phase margin* (PM) is defined to be the difference between the phase angle at the frequency ω_g and $-180°$. Thus

$$\text{phase margin (PM)} = \angle[G(j\omega_g)H(j\omega_g)] + 180° \qquad (12.2.1)$$

The *phase crossover frequency* ω_p is the frequency at which the phase angle is $-180°$. The *gain margin* (GM) is the difference in decibels between the unity gain condition (0 dB) and the value of $|GH|$ in dB at the phase crossover frequency. Thus,

$$\text{gain margin (GM)} = 0 - |G(j\omega_p)H(j\omega_p)| = -|G(j\omega_p)H(j\omega_p)| \text{ dB} \qquad (12.2.2)$$

These definitions are illustrated on the Bode plots of the *open-loop* transfer function shown in Figure 12.2.2a. Note that GM is positive when measured *down,* but PM is positive when measured *up*.

In many applications, the plant has no poles in the right-half plane, and control algorithms typically do not have such poles, so for many applications the open-loop transfer function $G(s)H(s)$ has no such poles. If the open-loop transfer function $G(s)H(s)$ has no poles in the right-half plane, then the closed-loop system is *stable* if and only if the phase and gain margins are both positive. We will not prove this statement, but it has long been well established. The situation shown in Figure 12.2.2a thus corresponds to a stable system. Figure 12.2.2b represents an unstable system that lacks positive gain and phase margins.

The phase and gain margins are often used as safety margins in the design specifications. A typical set of such specifications is:

$$\text{gain margin} \geq 8 \text{ dB and phase margin} \geq 30° \qquad (12.2.3)$$

Another common set of specifications is

$$\text{gain margin} \geq 6 \text{ dB and phase margin} \geq 40° \tag{12.2.4}$$

In most situations, only one of these equalities can be met, and the other margin is allowed to be greater than its minimum value. It is not desirable to make the margins too large, because this might result in a sluggish system or one with a large steady-state error.

PHASE MARGIN VERSUS DAMPING RATIO

It is sometimes useful to be able to translate time-domain performance specifications, such as damping ratio or overshoot, into frequency-domain specifications, such as phase or gain margin. The open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{12.2.5}$$

in a unity-feedback system results in the following closed-loop transfer function.

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12.2.6}$$

which has the characteristic roots $s = -\zeta\omega_n \pm \omega_n j \sqrt{1 - \zeta^2}$. These roots describe the dominant root pair of a higher-order system having complex dominant roots. Thus, if we can relate the phase margin of $G(s)$ to either ζ or ω_n or both, then we can derive an approximate expression for the phase margin of a higher-order system having complex dominant roots.

It is shown in one of the chapter's homework problems that the phase margin of $G(s)$ given by (12.2.5) is

$$\text{PM} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \tag{12.2.7}$$

We saw in Chapter 9 that the maximum percent overshoot is the following function of ζ.

$$M_\% = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{12.2.8}$$

So a 10% maximum overshoot corresponds to $\zeta = 0.59$, and from (12.2.7), the phase margin is 59°. Consider, for example, a third-order system having the dominant roots $s = -5 \pm 6.84 j$ and the secondary root $s = -10$. The damping ratio of the dominant roots is $\zeta = 0.59$, and thus from (12.2.7) we can estimate the phase margin of this third order system to be 59°. The accuracy of (12.2.7) obviously depends on the separation between the dominant roots and the secondary roots.

BODE PLOT DESIGN FOR PID CONTROL

When using the open-loop frequency response method, it is convenient to write the PID algorithm with the proportional gain factored out, as

$$F(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s\right) E(s) \tag{12.2.9}$$

The proportional gain is selected last, because it simply moves the gain curve up or down without affecting the phase curve, and thus can be used to adjust the gain curve until the specifications for the gain and phase margins are satisfied.

Figure 12.2.3 (a) Bode plots for PD action.
(b) Bode plots for PI action.

(a)

(b)
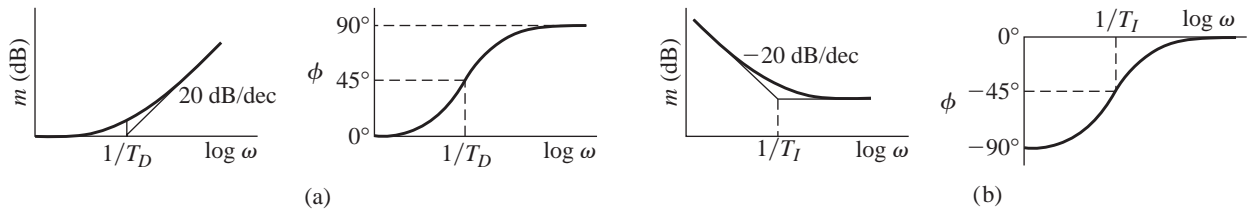
D action affects both the phase and gain curves. The increase in phase margin due to the positive phase angle introduced by D action is partly negated by the derivative gain, which reduces the gain margin. Increasing the derivative gain increases the speed of response, makes the system more stable, and enables a larger proportional gain to be used to improve the system performance. However, if the phase curve is too steep near $-180°$, it is difficult to use D action to improve the performance.

I action affects both the gain and phase curves. It can be used to increase the open-loop gain at low frequencies. However, it lowers the phase crossover frequency ω_p and thus reduces some of the benefits provided by D action. If required, the D action term is usually designed first, followed by I action and P action, respectively.

The PD algorithm is

$$F(s) = K_P(1 + T_D s)E(s) \tag{12.2.10}$$

The term $(1 + T_D s)$ can be considered as a series compensator to the proportional controller. Its Bode plots are shown in Figure 12.2.3a. From these, it can be seen that the usefulness of D action is that it adds phase shift at higher frequencies. It is thus said to give phase "lead." However, it also increases the gain at these frequencies, because the derivative term gives more response for rapidly changing signals.

The PI control algorithm is

$$F(s) = \frac{K_P}{s}\left(s + \frac{1}{T_I}\right)E(s) = \frac{K_P}{T_I s}(T_I s + 1)\,E(s) \tag{12.2.11}$$

The Bode plots are shown in Figure 12.2.3b. PI action adds gain at the lower frequencies but decreases the phase.

DESIGN APPROACH WITH THE BODE PLOT

Classical design methods based on the Bode plots obviously have a large component of trial and error because usually both the phase and gain curves must be manipulated to achieve an acceptable design. Given the same set of specifications, two designers can use these methods and arrive at different designs. An experienced designer, however, can often obtain a good design quickly with these techniques. Using a computer-plotting routine greatly speeds up the design process.

It is difficult to state a rigid set of rules to follow for designing a series compensator because of the variety of specifications and plant types that occur. However, the following considerations should be kept in mind.

1. To minimize the steady-state error, the open-loop gain should be kept as high as possible in the low-frequency range of the Bode plot.

2. At intermediate frequencies (near the gain crossover frequency), a slope of −20 dB/decade in the gain curve will help to provide an adequate phase margin.

3. At high frequencies, small gain is desirable to attenuate high-frequency disturbances, such as electronic noise, or mechanical vibrations induced by gear teeth, shaft elasticity, or hydraulic and pneumatic pressure fluctuations, and so forth.

Design of an Integral Control System | **EXAMPLE 12.2.1**

■ Problem

For the integral control system shown in Figure 12.2.4a, the parameter values are $A = 0.02$, $\tau_1 = 0.01$ sec, and $\tau_2 = 0.02$ sec. Adjust the integral gain to obtain an overshoot of no more than 10%.
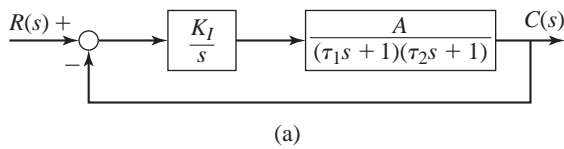
Figure 12.2.4 (a) An integral control system. (b) Bode plots for $K_I = 50$.
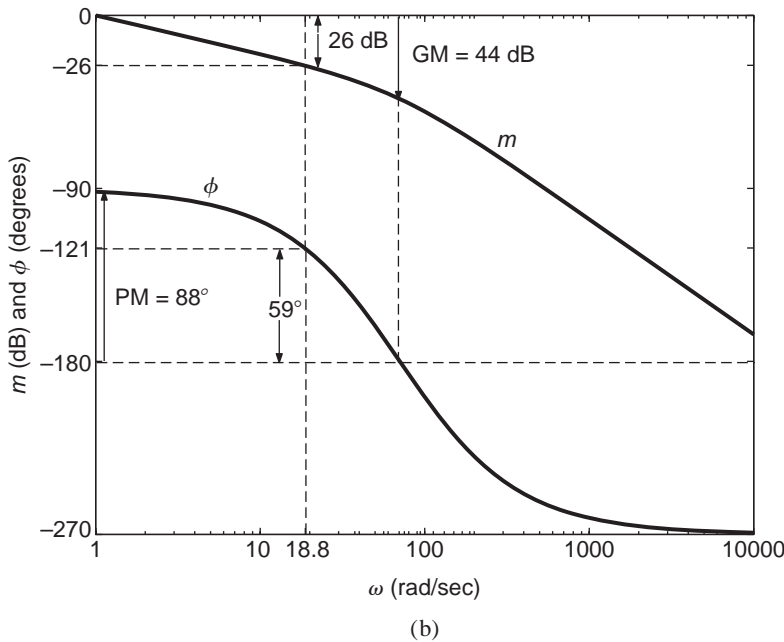
■ Solution

For the given values, the open-loop transfer function is

$$G(s) = 0.02K_I \frac{1}{s(0.01s + 1)(0.02s + 1)} \tag{1}$$

For an overshoot of 10%, assuming that the dominant roots of the final design are a complex pair, the damping ratio must be

$$\zeta = -\frac{\ln 0.1}{\sqrt{\pi^2 + \ln^2 0.1}} = 0.59$$

Using (12.2.7), we compute the required phase margin as follows, again assuming that the dominant roots of the final design are a complex pair.

$$\text{PM} = \tan^{-1} \frac{1.1823}{\sqrt{-0.6989 + 1.22}} = \tan^{-1} 1.6378 = 59°$$

The next step is to draw the open-loop Bode plots for an arbitrary value of K_I, say $K_I = 50$, which gives $0.02K_I = 1$. The Bode plots are shown in Figure 12.2.4b. The gain margin is 44 dB and the phase margin is 88° (these values can be determined easily with the MATLAB `margin` function; see Section 12.3). To achieve a phase margin of 59°, the phase curve must cross $59° - 180° = -121°$ at the new gain crossover frequency. By moving the cursor along the MATLAB plot, we can see that the phase equals $-121°$ at $\omega = 18.8$ rad/sec, which must be the new gain crossover frequency. To achieve this, the gain curve must be shifted up by 26.2 dB, which is accomplished by changing the value of K_I from $K_I = 50$ to $K_I = 50(10^{26.2/20}) = 1021$. This value gives a gain margin of 17.3 dB and a phase margin of 59°.

The final step is to check the results by simulation. The closed-loop transfer function is

$$T(s) = \frac{1021(0.02)}{0.0002s^3 + 0.03s^2 + s + 1021(0.02)}$$

The step response has an overshoot of 9%, less than required, and a 2% settling time of 0.208 sec. The closed-loop roots are $s = -114$ and $s = -18 \pm 23.9j$, so the assumption regarding complex dominant roots is correct.

Note that, although positive phase and gain margins are required for stability, large values of phase and gain margin may degrade performance. For example, with $K_I = 50$, the gain margin is 44 dB and the phase margin is 88°, but the closed-loop system has a 2% settling time of 3.82 sec, which is 18 times larger than with $K_I = 1021$. So large margins may give sluggish response.

Example 12.2.1 can also be solved with the root locus plot. From equation (1), the root locus equation is

$$1 + K\frac{1}{s(s + 100)(s + 50)} = 0$$

where $K = 100K_I$. Figure 12.2.5 shows the plot with the $\zeta = 0.59$ line indicated. The intersection of this line with the locus occurs where $K = 1.10 \times 10^5$, which corresponds to $K_I = 1010$. This is essentially the same value of K_I computed with the Bode plots, considering the limits of graphical accuracy.

Some designers prefer the root locus method over the Bode method. However, sometimes the system model is given in the form of experimentally determined open-loop frequency response curves or, as the next example shows, the model contains dead time. In such cases, the Bode plot approach is much more convenient to use.

DEAD-TIME ELEMENTS

Dead time is a time delay between an action and its effect. It occurs, for example, when a fluid flows through a pipe. Suppose the fluid velocity v is constant with time. The
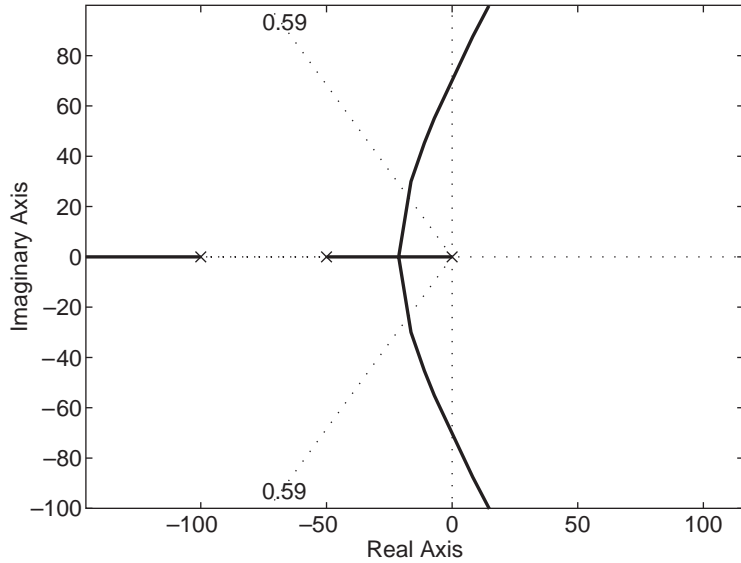
Figure 12.2.5 Root locus plot for Example 12.2.1.

(a)

(b)
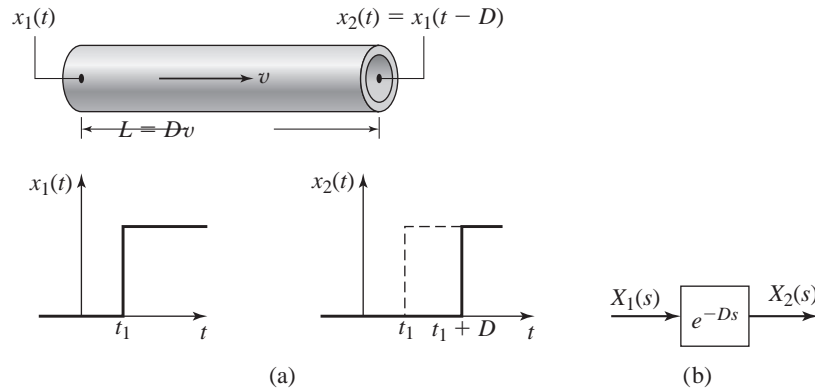
Figure 12.2.6 (a) A dead-time element. (b) Block diagram of a dead-time element.

pipe length is L, so it takes a time $D = L/v$ for the fluid to move from one end to the other. Let $x_1(t)$ denote the incoming fluid temperature and $x_2(t)$ the temperature of the fluid leaving the pipe. Suppose that the temperature of the incoming fluid suddenly increases at time t_1. If this is modeled as a step function, the result is shown in Figure 12.2.6a. If no heat energy is lost, then $x_2(t)$, the temperature at the output, is $x_1(t - D)$. Thus, a time D later, the output temperature suddenly increases.

A similar effect occurs for any change in $x_1(t)$, not just for a step change. In general, we can write $x_2(t) = x_1(t - D)$. The shifting property of the Laplace transform can be used to determine the response of a system with dead time. From the shifting theorem,

$$X_2(s) = e^{-Ds} X_1(s) \tag{12.2.12}$$

This result is shown in block diagram form in Figure 12.2.6b.

Dead time can be described as a "pure" time delay, in which no response at all occurs for a time D, as opposed to the time lag associated with the time constant of a response, for which $x_2(t) = (1 - e^{-t/\tau})x_1(t)$.

Some systems have an unavoidable time delay in the signal flow between components (Figure 12.2.7). The delay often results from the physical separation of the components and typically occurs as a delay D_1 between a change in the manipulated

Figure 12.2.7 A system having dead-time elements in the forward path and in the feedback loop.
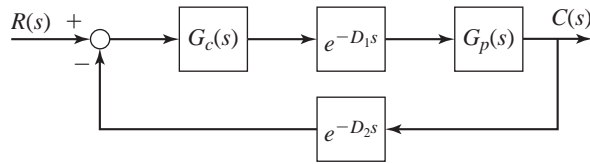
variable and its effect on the plant, or as a delay D_2 in the measurement of the output. Another, perhaps unexpected, source of dead time is the computation time required for a digital control computer to calculate the control algorithm. This can be a significant dead time in systems using inexpensive and slower microprocessors. It can be modeled as the delay D_1 between the controller and the actuator.

The presence of dead time means that the system does not have a characteristic equation of finite order. In fact, there are an infinite number of characteristic roots for a system with dead time. This can be seen by noting that the term e^{-Ds} can be expanded in an infinite series as

$$e^{-Ds} = \frac{1}{e^{Ds}} = \frac{1}{1 + Ds + (D^2 s^2/2) + \cdots} \tag{12.2.13}$$

For example, if $D_1 = 5$, $D_2 = 0$, $G_c(s) = K_P$, and $G_p(s) = 1/(s+4)$ in Figure 12.2.7, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P e^{-5s}}{s + 4 + K_P e^{-5s}}$$

The characteristic equation is $s + 4 + K_P e^{-5s} = 0$, which is a transcendental equation having an infinite number of roots.

The open-loop Bode plots are particularly useful for systems with dead-time elements. A delay in either the manipulated variable or the measurement will result in an open-loop transfer function of the form

$$G(s)H(s) = e^{-Ds} P(s) \tag{12.2.14}$$

For this case,

$$|G(j\omega)H(j\omega)| = |P(j\omega)|\left|e^{-j\omega D}\right| = |P(j\omega)| \tag{12.2.15}$$

because

$$\left|e^{-j\omega D}\right| = |\cos \omega D - j \sin \omega D| = \sqrt{\cos^2 \omega D + (-\sin \omega D)^2} = 1$$

The dead time therefore does not affect the open-loop gain curve. This makes the analysis of its effects easier to accomplish with the open-loop frequency response plot. However,

$$\angle[G(j\omega)H(j\omega)] = \angle[P(j\omega)e^{-j\omega D}] = \angle P(j\omega) + \angle e^{-j\omega D}$$
$$= \angle P(j\omega) - \omega D \tag{12.2.16}$$

where the angles are in radians and ω is the radian frequency. Thus, the dead time decreases the phase proportionally to the frequency ω.

The following example shows how these methods can be applied to a system containing dead time, whose plant transfer function cannot be obtained analytically.

A Control System with Dead Time | **EXAMPLE 12.2.2**

■ Problem

The frequency response for a particular plant $G_p(s)$ was determined experimentally, and the results are shown in Figure 12.2.8a, where the gain curve is denoted by m_1. It is intended to use proportional control for this plant with a unity feedback loop, as shown in Figure 12.2.8b. It is known that there will be dead time D between the controller action and its effect on the plant.

a. Design the controller to achieve the specifications given by GM ≥ 8 dB and PM ≥ 30° if $D = 0$.
b. How large can the dead time be before the system becomes unstable?
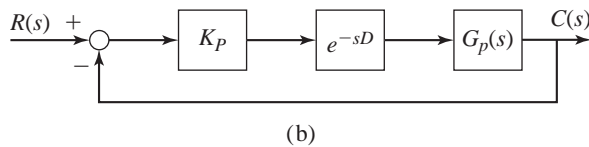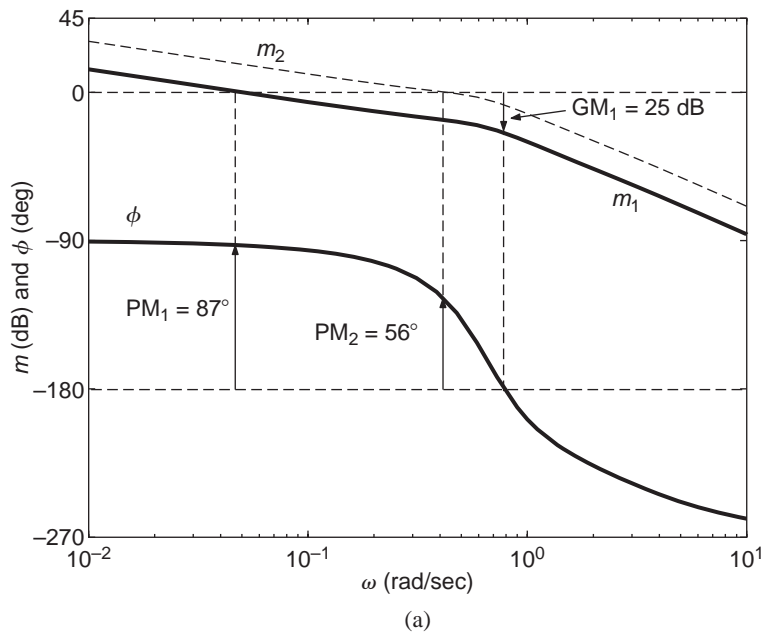
Figure 12.2.8 (a) Measured frequency response of a plant. (b) A proportional control system having dead time.

■ Solution

a. First neglect the dead time and design the controller. With $D = 0$, the open-loop transfer function is $K_P G_p(s)$. The plot in Figure 12.2.8a, which is for $K_P = 1$, shows that the phase margin is 87° and the gain margin is 25 dB. Increasing K_P above unity affects only the gain curve. This curve can be raised by $25 - 8 = 17$ dB without violating the specifications given for PM and GM. Thus, for GM = 8 dB, K_P must be such that $20 \log K_P = 17$, or $K_P = 10^{17/20} = 7.08$. With this value of K_P, the phase margin becomes 56°, and the new gain crossover frequency is $\omega_g = 0.41$ rad/sec. This can be seen by translating the gain curve upward by 17 dB. The new gain curve is labeled m_2 on the plot.

b. The dead time affects only the phase curve, and stability requires that the phase margin be positive. This occurs only if the dead time's contribution to the phase curve at the new gain crossover frequency is greater than $-56°$. The stability requirement thus is

$$0.41D \leq (56°)\frac{\pi}{180°} \quad \text{rad}$$

or

$$D \leq 2.38 \quad \text{sec}$$

By affecting the phase curve, however, the dead time changes the phase crossover frequency, and thus changes the gain margin. If we subtract 2.38ω from the phase curve shown in the plot, and recompute the margins, we find that the gain margin is now slightly negative. If we reduce D somewhat, we will obtain positive or zero margins. By trial and error, the value $D = 2.1$ gives a slightly positive GM and PM $= 0$. So the stability limit is $D \leq 2.1$ sec.

BODE DESIGN OF COMPENSATORS

In addition to shaping the gain curve, the phase curve must often be shaped by means of a series compensator in order to achieve phase and gain margin specifications, for example. The PID-type compensators do not always have sufficient flexibility to do this, because they have only an s term in the denominator, but the lead and lag compensators often do have enough flexibility.

The Bode plots of the lead and lag compensators are shown in Figures 12.2.9 and 12.2.10. When used in series with a proportional gain K_P, the lead compensator increases the phase angle and thus increases the phase margin. This enables the gain K_P/μ to be made larger than is possible without the compensator. The result is an increase in the closed-loop bandwidth and an increase in the speed of response.

On the other hand, the lag compensator is used when the speed of response and damping of the closed-loop system are satisfactory, but the steady-state error is too

Figure 12.2.9 Bode plots of a lead compensator.

(a)   (b)
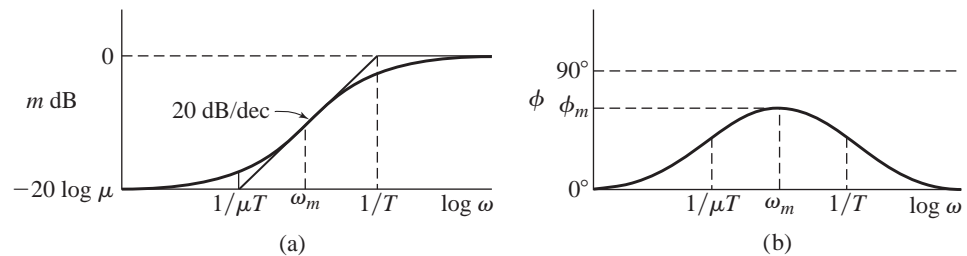
Figure 12.2.10 Bode plots of a lag compensator.

(a)   (b)
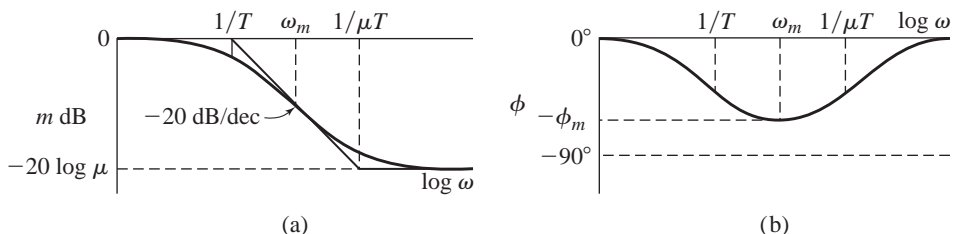
Figure 12.2.11 Bode plots of a lag-lead compensator.

(a)

(b)
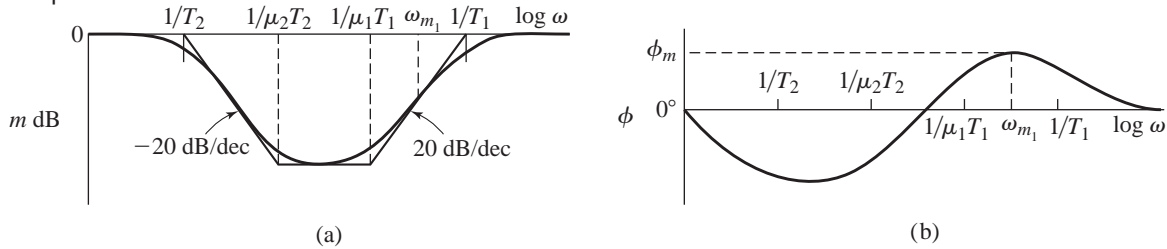
large. The lag compensator enables the gain to be increased without substantially changing the resonance frequency ω_r, and the resonance peak m_p of the closed-loop system.

The Bode plots for the lag-lead compensator are shown in Figure 12.2.11 for $T_2 > \mu_2 T_2 > \mu_1 T_1 > T_1$. The maximum phase shift ϕ_m occurs at $\omega_{m_1} = 1/T_1\sqrt{\mu_1}$, and the attenuation at this frequency is equal in magnitude but opposite in sign to that of the lead compensator alone. Thus, the lag-lead compensator can be more effective than the lead compensator. The plots also show that the compensator affects the gain and phase in only the intermediate frequency range from $\omega = 1/T_2$ to $1/T_1$.

DESIGN OF LEAD COMPENSATORS

The lead compensator has the transfer function

$$G_c(s) = \frac{s + 1/\mu T}{s + 1/T}$$

For any particular value of the parameter $\mu > 1$, the lead compensator can provide a maximum phase lead ϕ_m (see Figure 12.2.9). This value, and the frequency ω_m at which it occurs, can be found as a function of μ and T from the Bode plot. The frequency ω_m is the geometric mean of the two corner frequencies of the compensator. Thus,

$$\omega_m = \frac{1}{T\sqrt{\mu}} \qquad (12.2.17)$$

By evaluating the phase angle of the compensator at this frequency, we can show that

$$\sin \phi_m = \frac{\mu - 1}{\mu + 1} \qquad (12.2.18)$$

or

$$\mu = \frac{1 + \sin \phi_m}{1 - \sin \phi_m} \qquad (12.2.19)$$

These relations are useful in designing a lead compensator with the Bode plot.

We assume that the specifications include requirements for phase and gain margin as well as for the allowable steady-state error. The purpose of the lead compensator is to use the maximum phase lead of the compensator to increase the phase of the open-loop system near the gain crossover frequency while not changing the gain curve near that frequency. This is usually not entirely possible, because the gain crossover frequency

is increased in the process, and a compromise must be sought between the resulting increase in bandwidth and the desired values of the phase and gain margins.

A suggested design method is the following:

1. Set the gain K_P of the uncompensated system to meet the steady-state error requirement.
2. Determine the phase and gain margins of the uncompensated system from the Bode plot, and estimate the amount of phase lead ϕ required to achieve the margin specifications. The extra phase lead required can be used to estimate the value for ϕ_m to be provided by the compensator. Thus, μ can be found from (12.2.19).
3. Choose T so that ω_m from (12.2.17) is located at the gain crossover frequency of the compensated system. One way of doing this is to find the frequency at which the gain of the uncompensated system equals $-20 \log \sqrt{\mu}$. Choose this frequency to be the new gain crossover frequency. This frequency corresponds to the frequency ω_m at which ϕ_m occurs.
4. Construct the Bode plot of the compensated system to see if the specifications have been met. If not, the choice for ϕ_m needs to be evaluated and the process repeated. It is possible that a solution does not exist.

The attempt to design a lead compensator can be unsuccessful if the required value of μ is too large. This can occur with plants that are not stable or have a low relative stability with a rapidly decreasing phase curve near the gain crossover frequency. In the former case, the extra phase lead required can be too large. In the latter case, the phase angle at the compensated gain crossover frequency is much less than at the uncompensated gain crossover. Thus, the extra phase required can be excessive. The difficulty with a large value of μ is that the resistance and capacitance values that result might be incompatible or impossible to obtain physically. The usual range for μ is $1 < \mu < 20$. Additional phase lead can sometimes be obtained by cascading more than one lead compensator.

In spite of these potential difficulties, the lead compensator has a record of many successful applications.

EXAMPLE 12.2.3

Bode Design of a Lead Compensator

■ Problem
The system shown in Figure 12.2.12a has an acceptable steady-state error of 0.01 for a unit-ramp command if the gain is set to $K_P = 500$ and $G_c(s) = 1$. If this is done, however, the transient performance is lightly damped and thus unsatisfactory. Design a compensator to give a gain margin of at least 8 dB and a phase margin of at least $30°$.

■ Solution
The Bode plot of the open-loop uncompensated system with $K_P = 500$ is shown in Figure 12.2.12b. The phase margin is $12.8°$, and the gain margin is infinite, because the phase curve never falls below $-180°$. Thus, $30° - 12.8° = 17.2°$ must be added to the phase curve to meet the specifications, so we try a lead compensator. Instead of using the compensator to add the $17.2°$, we will try to add something more, say $20°$, because this method is approximate. From (12.2.19), we have

$$\mu = \frac{1 + \sin \phi_m}{1 - \sin \phi_m} = \frac{1 + \sin 20°}{1 - \sin 20°} = 2.04$$

From step 3, we compute

$$-20 \log \sqrt{\mu} = -3.096 \text{ dB}$$

From the plot we determine that the open-loop gain of the uncompensated system is -3.096 dB at approximately 26.5 rad/sec. Take this frequency to be ω_m, and solve for T from (12.2.17):

$$T = \frac{1}{26.5\sqrt{2.04}} = 0.0264$$

These values of μ and T give a compensator with the transfer function

$$G_c(s) = \frac{s + 1/\mu T}{s + 1/T} = \frac{s + 18.55}{s + 37.88} = 0.4897\frac{0.0539s + 1}{0.0264s + 1}$$

The open-loop transfer function of the compensated system is

$$K_P G_c(s)G(s)H(s) = 0.4897K_P \frac{0.0539s + 1}{0.0264s + 1}\frac{1}{s(s + 5)}$$

The original choice of $K_P = 500$ no longer gives required error because the compensator introduces the factor 0.4897. Thus, we must choose $K_P = 500/0.4897 = 1021$ to achieve an error of 0.01. With this value of K_P, the Bode plot of the compensated system is shown in Figure 12.2.13.

Figure 12.2.13 Bode plots of the compensated system.
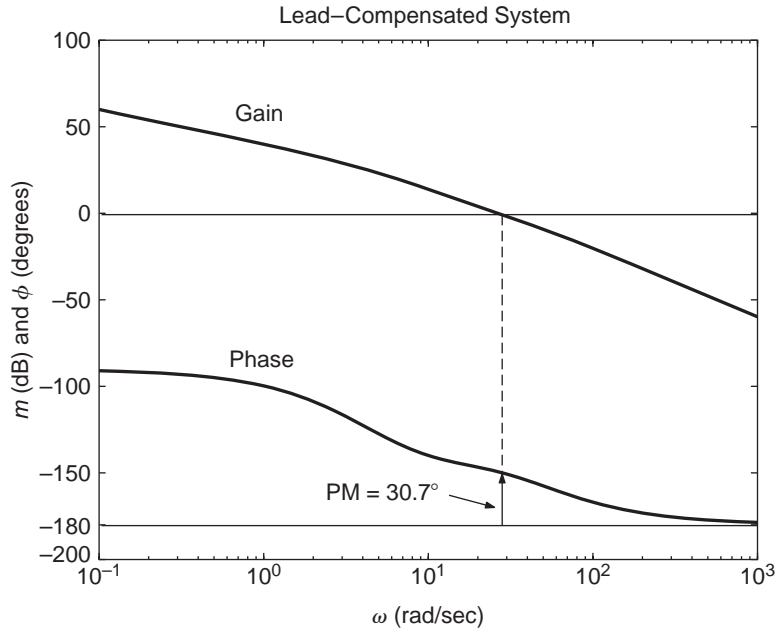
The phase margin is 30.7°, which slightly exceeds the requirement, and the gain margin is still infinite, so the specifications have been met. Now you see why we added some extra phase angle. This illustrates the trial-and-error nature of this method. The guides listed in the design steps are based on approximations of the real phase and gain curves of the compensator.

DESIGN OF LAG COMPENSATORS

If $\mu < 1$, the following transfer function represents a lag compensator.

$$G_c(s) = \mu \frac{s + 1/\mu T}{s + 1/T}$$

Lag compensation uses the high-frequency attenuation of the compensator to keep the phase curve unchanged near the gain crossover frequency while this frequency is lowered. A suggested design procedure is the following:

1. Set the open-loop gain K_P of the uncompensated system to meet the steady-state error requirements.

2. Construct the Bode plots for the uncompensated system, and determine the frequency at which the phase curve has the desired phase margin. Determine the number of decibels required at this frequency to lower the gain curve to 0 dB. Let this amount be $m' > 0$ dB and this frequency be ω'_g. Then μ is found from

$$\mu = 10^{-m'/20} \qquad (12.2.20)$$

3. The first two steps alter the phase curve. However, this curve will not be changed appreciably near ω'_g if T is chosen so that $\omega'_g \gg 1/\mu T$. A good choice is to place the frequency $1/\mu T$ one decade below ω'_g. Any larger separation might result in a system with a slow response.

4. Construct the Bode plots of the compensated system to see if all the specifications are met. If not, choose another value for T and repeat the process.

As with lead compensation, this procedure is one of trial and error and might not work. The physical elements must be realizable, so a common range for μ is $0.05 < \mu < 1$. The compensator introduces a lag angle that is not accounted for in the preceding procedure. To account for this effect, the designer might add $5°$ to $10°$ to the specified phase margin before starting the design process.

	Bode Design of a Lag Compensator	**EXAMPLE 12.2.4**

■ Problem
Consider the system shown in Figure 12.2.14a. When $K_P = 12.5$ and $G_c(s) = 1$, the characteristic roots are $s = -2.5 \pm 2.5j$. Suppose that the transient response given by these roots is acceptable, but we want the steady-state error due to a ramp input to be no greater than 0.01. Setting $K_P = 500$ will accomplish this but will alter the transient performance by giving lightly damped roots at $s = -2.5 \pm 22.22j$, for which $\zeta = 0.118$. Design a compensator to improve the system so that the error will be no greater than 0.01, PM $\geq 30°$, and GM ≥ 8 dB.
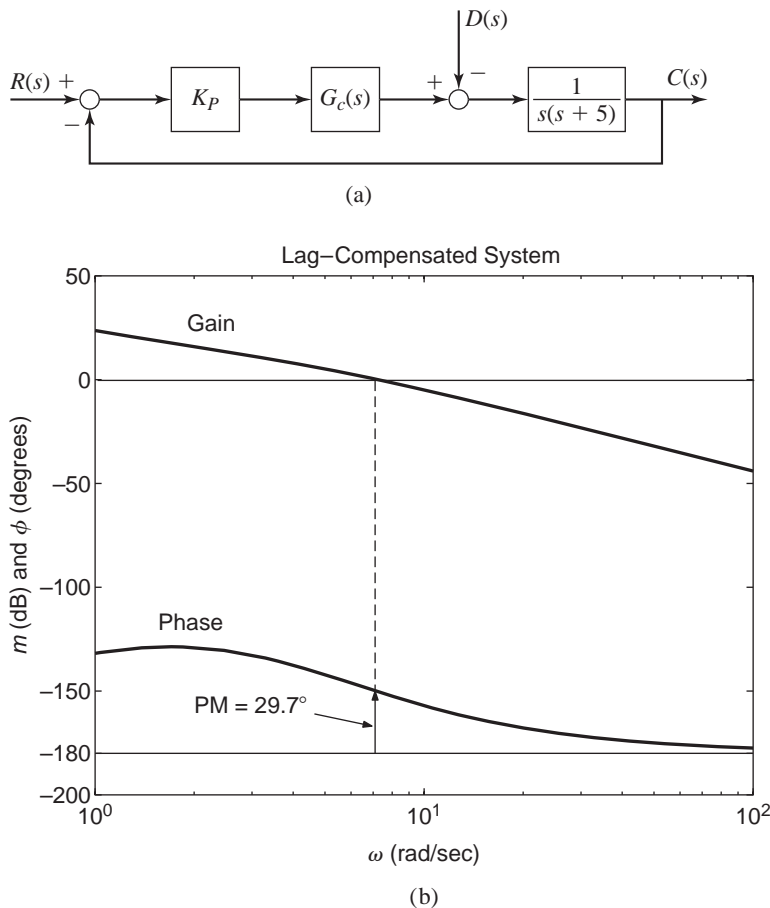
Figure 12.2.14 (a) System for Example 12.2.4. (b) Bode plots of the compensated system.

(a)

(b)

■ Solution

A lag compensator is indicated because the transient response is satisfactory. Set $K_P = 500$ to achieve the desired error and construct the Bode plots for the uncompensated open-loop system using $K_P = 500$. The plot is the same as Figure 12.2.12b. Keeping in mind the lag introduced by the compensator, we attempt to achieve a phase margin of 35° rather than the required 30°.

From the plot we determine that the uncompensated system would have a phase margin of 35° if the gain crossover occurs at $\omega = 7.2$. Thus, $\omega_g' = 7.2$. The gain at this frequency is 18 dB, so $m' = 18$, and from (12.2.20),

$$\mu = 10^{-18/20} = 0.126$$

For step 3, we choose T to place $\omega = 1/\mu T$ one decade below $\omega = 7.2$. Thus, $1/\mu T = 0.72$, which gives $T = 11.023$. The lag compensator that results is

$$G_c(s) = 0.126\frac{s + 0.72}{s + 0.0907}$$

The open-loop transfer function of the compensated system is

$$K_P G_c(s)G(s)H(s) = 0.126\frac{s + 0.72}{s + 0.0907}\frac{K_P}{s(s + 5)}$$

With $K_P = 500/0.126 = 3970$, the ramp error is the required value of 0.01. The resulting Bode plots are given in Figure 12.2.14b. The phase margin is 29.7°, which is approximately 30°, and the gain margin is infinite, so the specifications have essentially been met.

At a quick glance it might appear that Examples 12.2.3 and 12.2.4 are concerned with solving the same problem; namely, compensate the plant $1/s(s + 5)$ so that the ramp error is 0.01, PM $\geq$ 30°, and GM $\geq$ 8 dB. This, however, is not the case. The difference lies in what is taken as satisfactory in the performance of the uncompensated system and what must be improved. In Example 12.2.3, the transient response resulting from the gain $K_P = 500$ needed to obtain a ramp error of 0.01 was judged to be poor, and the lead compensator was used to improve the transient performance. In Example 12.2.4, acceptable transient response was obtained when $K_P = 12.5$, but the steady-state error was then unacceptable, and a lag compensator was therefore used.

Another way to view the differences between the two examples is to note that the compensation in Example 12.2.3 was obtained by shifting the phase curve, whereas in Example 12.2.4 it was obtained by shifting the gain curve.

12.3 MATLAB APPLICATIONS

In addtion to the `bode` function, MATLAB provides the `margin` function that computes the phase and gain margins. As we will see, the `tf` function can accept models with dead time, and MATLAB also provides the `pade` function for analyzing such models.

THE `margin` FUNCTION

A relevant function we have not yet introduced is the `margin` function. The `margin(sys)` function plots the open-loop Bode plots with the gain and phase margins marked with a vertical line, and their values displayed at the top of the plot, along with the values of the crossover frequencies.

The syntax `[GM,PM,wg,wp] = margin(sys)` computes the gain margin GM in absolute units, the phase margin PM in degrees, and wg, wp, which are the

crossover frequencies ω_g and ω_p. To obtain the gain margin in decibels, use `GMdB = 20*log10(GM)`. If there are several crossover points, `margin` returns the smallest margins (the gain margin nearest to 0 dB and the phase margin nearest to 0°).

The syntax `[GM,PM,wg,wp] = margin(mag,phase,w)` derives the gain and phase margins from the magnitude, phase, and frequency structures `mag`, `phase`, and `w` produced by the `bode` function. Interpolation is performed between the frequency points to estimate the values.

MODELS WITH DEAD TIME

To create a transfer function model having dead time D at its input, use the `'iodelay'` property with the `tf` function as follows:

```
sys = tf(num,den,'iodelay',D)
```

For example, to create the model

$$e^{-2s}\frac{5}{s^2 + 4s + 2}$$

type `sys = tf(5,[1,4,2],'iodelay',2)`.

The `feedback` function cannot accept time-delay models, but `bode`, `margin`, `step`, and others will. For example, to determine the margins for the previous model having dead time, type `[GM,PM,wg,wp] = margin(sys)`. This generates a warning that the system is unstable, and returns the values: GM = 0.7795, PM = $-47.4843°$, $\omega_g = 0.9325$ rad/sec, and $\omega_p = 1.2449$ rad/sec. To obtain GM in dB, type `20*log10(GM)`. The answer is -2.1634 dB.

Dead time in a system can cause oscillatory behavior or even instability in a system that is nonoscillatory and stable without dead time. For example, proportional control of the plant $1/(s+4)$ using $K_P = 9$ gives the closed-loop transfer function $9/(s+13)$, which is stable and has a nonoscillatory step response. If there is dead time between the plant and the controller, then the open-loop transfer function is

$$e^{-Ds}\frac{9}{s+4}$$

To obtain the closed-loop transfer function, note that the `feedback` function does not support dead time models (at this time), so we must first convert the dead-time model into discrete-time form. The function `c2d`, which stands for "continuous to discrete," converts a continuous-time model to discrete-time form. The syntax is `sysd = c2d(sysc,T,method)`, where `sysc` is the continuous-time model, `sysd` is the discrete-time model, `T` is the sample time, and `method` is a string specifying the method to be used. The default is `'zoh'` when `method` is omitted. This uses the zero-order hold method discussed in Chapter 10, Section 10.3.

To generate and compare the step responses of the models with and without dead time, use the following script file. The sample time T should be chosen to be a small fraction of the smallest estimated time constant of the closed-loop system. Here we use $T = 0.01$.

```
KP = 9;
sysA = tf(KP,[1,4],'iodelay',0.5);
sysB = c2d(sysA,0.01);
sysC = feedback(sysB,1);
sysD = tf(KP,[1, 4]);
sysE = feedback(sysD,1);
step(sysC,sysE,3)
```

Figure 12.3.1 (a) A control
system having dead time
between the controller and the
plant. (b) Responses of systems
with and without dead time
between the controller and the
plant. The response with a
dead time of 0.5 sec was
obtained with a discrete-time
approximation of the
dead-time element.

(a)

(b)
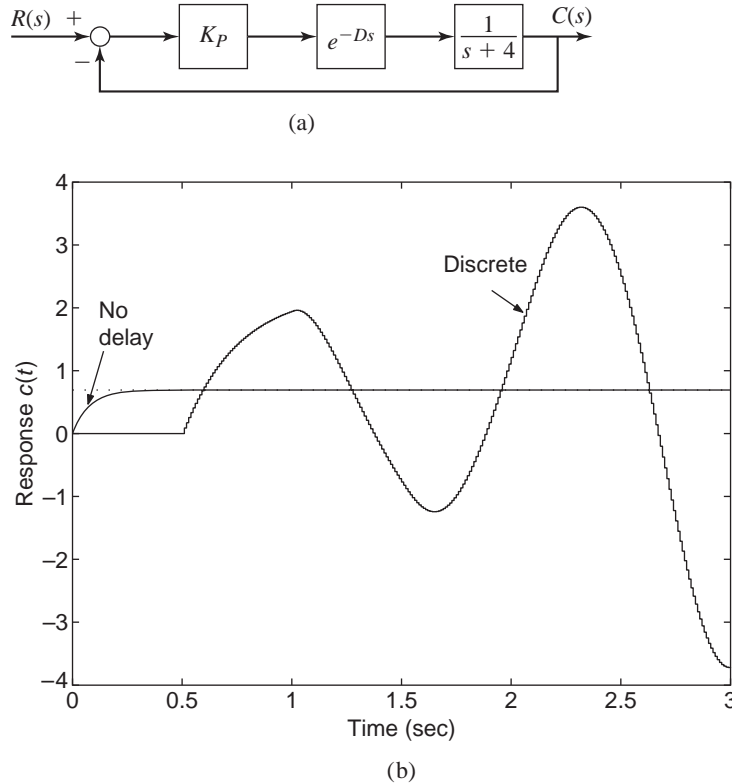
The step response is shown in Figure 12.3.1 after being edited with the Plot Editor.
Note that the dead time of $D = 0.5$ makes the system unstable and oscillatory.

MATLAB also provides the `pade` function for analyzing dead-time systems. The
Pade approximation to the dead-time element is

$$e^{-sD} = \frac{1}{[1 + (Ds/n)]^n}$$

The approximation improves as n becomes larger. The syntax `[num,den] =`
`pade(D,n)` returns the nth order transfer function approximation of the dead time
transfer function e^{-Ds}. For example, suppose the dead time is $D = 0.5$. The third-
order Pade approximation obtained with `[num,den] = pade(0.5, 3)` is `num =`
`[-1, 24, -240, 960]` and `den = [1, 24, 240, 960]`, which correspond to

$$e^{-0.5s} \approx \frac{-s^3 + 24s^2 - 240s + 960}{s^3 + 24s^2 + 240s + 960}$$

This approximation can then be converted to an LTI model, which can then be used
with the MATLAB LTI functions. For example, the transfer function and step response
of a proportional control system with the plant $1/(s+4)$ with $K_P = 9$ and a time-delay
of 0.5 sec between the controller and the plant can be obtained with the following
script file.

```
KP = 9;
[num,den] = pade(0.5, 3);
sys1 = tf(num,den);
```
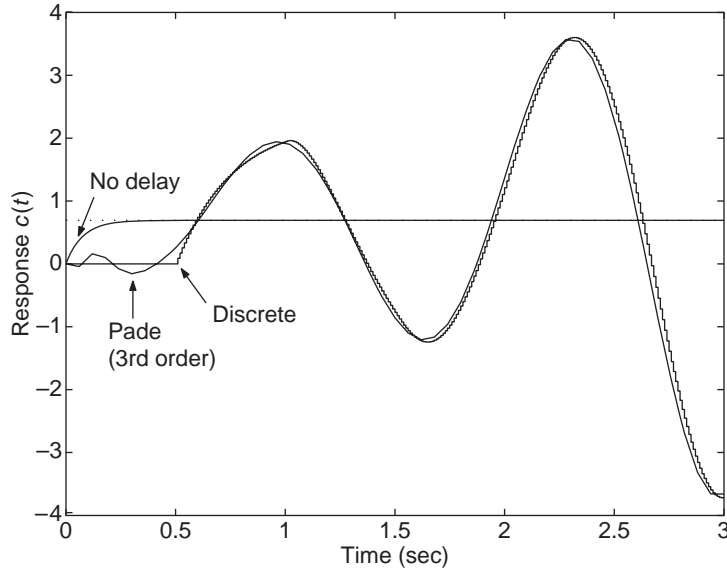
```
sys2 = tf(KP,[1, 4]);
sys3 = series(sys1,sys2);
sys4 = feedback(sys3,1);
step(sys4)
```

Figure 12.3.2 compares the step response using the third-order Pade approximation with that obtained by the discrete-time approximation. If more accuracy is needed, a higher-order approximation should be used. The advantage of using the Pade approximation is that it produces a continuous-time transfer function that, unlike the discrete-time approximation, can always be used with other continuous-time models. Its disadvantage is that it might produce very high-order models with resulting numerical inaccuracy.

SIMULATING THE PROCESS REACTION METHOD

Once the gains have been computed from the Ziegler-Nichols process reaction method, it is difficult to tune the controller by simulation because a transfer function or differential equation model of the plant is not available. Often, however, a model having a dead time can be used as an approximation. From a plot of the data given in Section 11.6, the steady-state response is 93, and we estimate that the settling time of the process is $12 - L = 12 - 1.5 = 10.5$ min, and thus the time constant is $\tau = 10.5/4 = 2.63$ min. Thus, a possible model for a transfer function description of the plant is

$$G_p(s) = e^{-Ls} \frac{93}{\tau s + 1} \tag{12.3.1}$$

The step response of this model, however, has an infinite slope at $t = L$. Perhaps a better model is

$$G_p(s) = e^{-Ls} \frac{93}{ms^2 + cs + 1} \tag{12.3.2}$$

After some experimentation, we find that a good approximation is given by the second model with $\zeta = 0.707$ and $\tau = 2.63$, which give $m = 3.44$ and $c = 2.61$. Figure 12.3.3 shows the response of this model and the original response data.
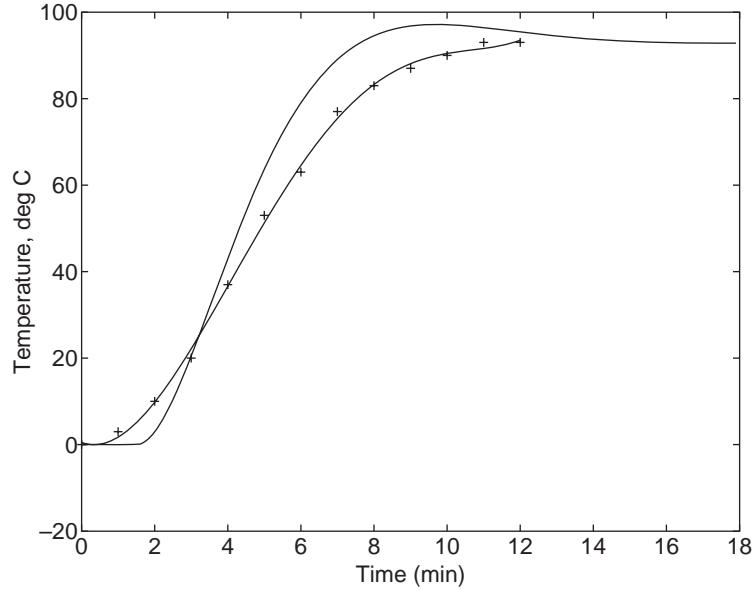
The methods of this section can then be used to obtain the closed-loop transfer function for simulation and tuning of the control gains. Another way to do this is to use Simulink. We will show how this is done in Section 12.4.

12.4 SIMULINK APPLICATIONS

Simulink is especially well suited to analyzing models with dead time. This section illustrates some of these applications.

SIMULATION OF SYSTEMS WITH DEAD TIME

Systems having dead-time elements are easily simulated in Simulink. The block implementing the dead-time transfer function e^{-Ds} is called the "Transport Delay" block. Figure 12.4.1 shows a Simulink model for proportional control of the plant $1/(s + 4)$ using $K_P = 9$. This system was analyzed in Section 12.3, with the response shown in Figure 12.3.1 for a dead time of $D = 0.5$ s. You will see this response in the Scope block of the Simulink model.

PROCESS REACTION SIMULATION

In Section 12.3 we found that a good model of the measured open-loop response is

$$G_p(s) = e^{-1.5s} \frac{93}{3.44s^2 + 2.61s + 1}$$

Figure 12.4.1 Simulink model
using a Transport Delay block.
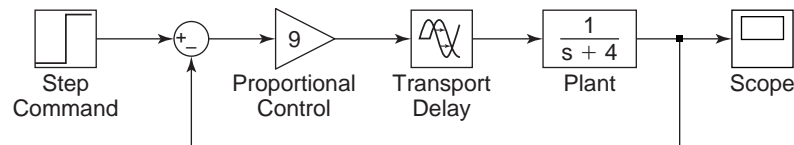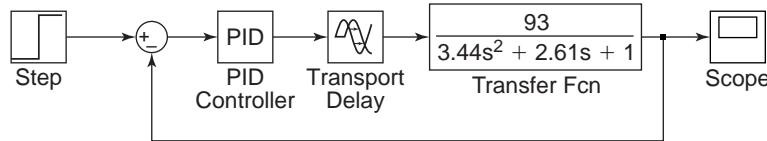
Figure 12.4.2 shows a Simulink model incorporating this model of the plant in a PID control system. The delay in the Transport Delay block has been set to 1.5.

The new block in this model is the PID Controller block. It is useful because if we were to implement the PID controller with a Transfer Function block, we would get an error message warning that the order of the numerator must be no greater than the order of the denominator. This is caused by the term $K_D s^2$ in the numerator. The PID block is specifically designed to handle this situation.

Simulation of the model using the gain values $K_P = 0.0524$, $K_I = 0.0170$, and $K_D = 0.0404$ computed from the Ziegler-Nichols process reaction method shows that the system is unstable. Further simulations show that it can be made stable by reducing the gains. A set giving good response is found to be one-fourth of the previous gain values; namely, $K_P = 0.0524/4$, $K_I = 0.017/4$, and $K_D = 0.0404/4$.

12.5 CHAPTER REVIEW

Series compensation involves inserting an additional control element in series with the main controller. Section 12.1 discusses several types of series compensators, the lead, lag, and lag-lead, and shows how to use the root locus to design them. Compensation often leads to a satisfactory design if the PID action is capable of satisfying most but not all of the performance specifications.

Section 12.2 shows how to use the open-loop frequency response plots to design PID controllers and series compensators using the phase and gain margins and the static error coefficients. The frequency response plots of the system's open-loop transfer function are easily generated even for high-order systems, and they enable the proper control gain to be selected simply by adjusting the scale factor on the plot. The technique is especially useful for analyzing systems with dead time.

Now that you have finished this chapter, you should be able to do the following.

1. Use the open-loop frequency response with phase and gain margin specifications to design a controller.
2. Design a series compensator using either root locus or frequency response methods.
3. Apply MATLAB and Simulink with the methods of this chapter.

PROBLEMS

Section 12.1 Series Compensation

12.1 Control of the attitude θ of a missile by controlling the fin angle ϕ, as shown in Figure P12.1, involves controlling an inherently unstable plant. Consider the specific plant transfer function

$$G_p(s) = \frac{\Theta(s)}{\Phi(s)} = \frac{1}{5s^2 - 6}$$
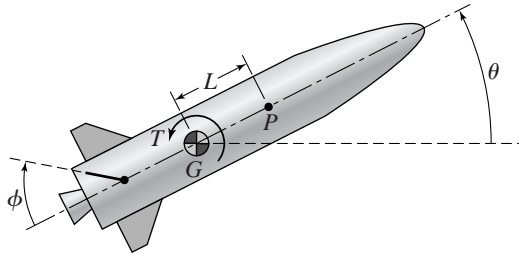
Design a PD compensator for this plant. The dominant roots of the closed loop system must have $\zeta = 0.707$ and $\omega_n = 0.5$.

12.2 Figure P12.2 shows a pneumatic positioning system, where the displacement x is controlled by varying the pneumatic pressure p_1. Assume that the pressure p_2 is constant, and consider the specific plant

$$G_p(s) = \frac{X(s)}{P_1(s)} = \frac{K}{s^2 + 2s}$$

With $K = 4$, the damping ratio is $\zeta = 0.5$, the natural frequency is $\omega_n = 2$ rad/s, and the steady-state ramp error is 0.5.

a. Design an electrical compensator to obtain $\omega_n = 4$ while keeping $\zeta = 0.5$. Obtain the compensator's resistances if $C = 1 \ \mu F$.

b. Suppose that with $K = 4$, the original system gives a satisfactory transient response, but the ramp error must be decreased to 0.05. Design a compensator to do this.

Figure P12.2
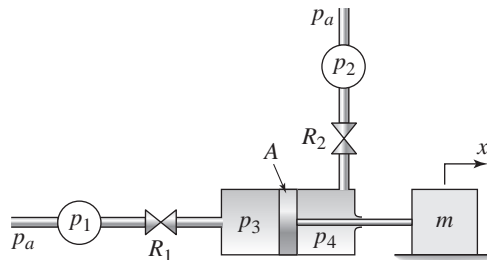
Figure P12.3
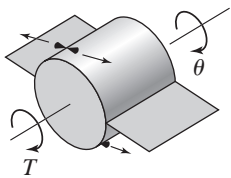
12.3 It is desired to control the angular displacement θ of a space vehicle by controlling the applied torque T supplied by thrusters (Figure P12.3). The plant model is

$$G_p(s) = \frac{\Theta(s)}{T(s)} = \frac{5}{s^2}$$

Design a compensator for this plant. The system must have a settling time no greater than 4 sec and a steady-state error of zero for a step command, and the dominant roots of the closed-loop system must have $\zeta \geq 0.45$.

12.4 When proportional control is applied to the following plant using a gain of $K_P = 1$, the closed-loop roots are satisfactory, but the static velocity error coefficient must be increased to $C_v = 5$/sec.

$$G_p(s) = \frac{1}{s^3 + 3s^2 + 2s}$$

Design a compensator for this plant that keeps the closed-loop roots near their original locations but increases C_v to 5.

12.5 The speed ω_1 of the load is to be controlled with the torque T_d acting through a fluid coupling (Figure P12.5). Design a compensator for the specific plant

$$G_p(s) = \frac{\Omega_1(s)}{T_d(s)} = \frac{1}{s^2 + s}$$

The static velocity error coefficient must be $C_v = 10$/sec, the dominant roots of the closed-loop system must have $\zeta = 0.5$ and $\omega_n = 2$.
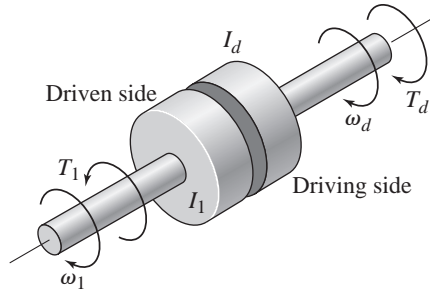
Figure P12.5

12.6 The block diagram of a speed control system is shown in Figure P12.6. For a particular system with proportional control, $G_1(s) = K_P$, the open-loop transfer function is

$$G(s) = \frac{K_P}{s(s+2)}$$

With $K_P = 4$, the damping ratio is $\zeta = 0.5$, the natural frequency is $\omega_n = 2$ rad/sec, and $C_v = 2$/sec. Design a compensator that will give a static velocity error coefficient of $C_v = 20$/sec, a phase margin of at least $40°$, and a gain margin of at least 6 dB.
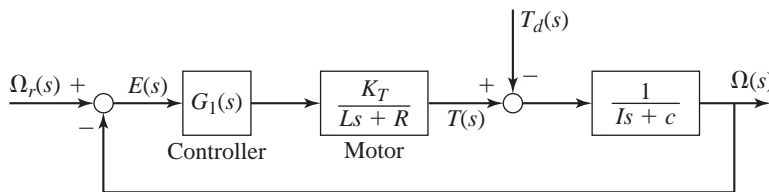
Figure P12.6

12.7 The block diagram of a position control system is shown in Figure P12.7. For a particular system with proportional control, $G_1(s) = K_P$, the open-loop transfer function is

$$G(s) = \frac{2.5K_P}{s(s+2)(0.25s+2)}$$

Design a compensator to give $s = -2 \pm j2\sqrt{3}$ and $C_v = 80$/sec.

12.8 It is desired to control the angular displacement θ of a space vehicle by controlling the applied torque T supplied by thrusters (Figure P12.8a). The plant model is

$$G_p(s) = \frac{\Theta(s)}{T(s)} = \frac{10}{s^2}$$

Figure P12.7

(a)

(b)

Figure P12.8

(a)

(b)
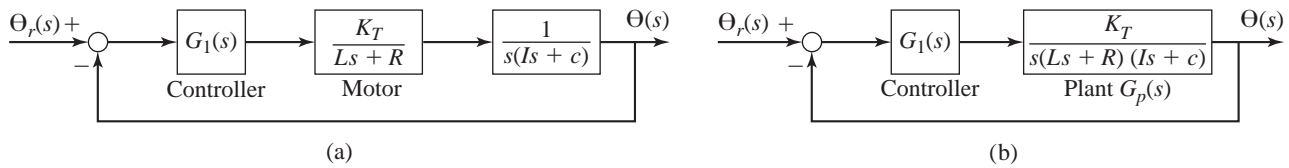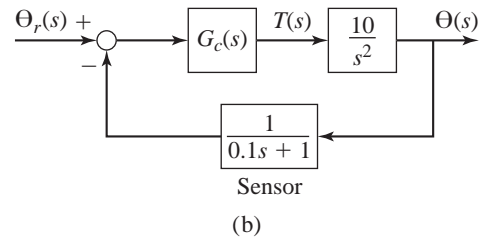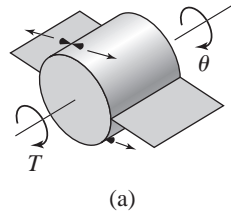
The feedback sensor that measures the displacement has a time constant of 0.1 sec (Figure P12.8b). Design a compensator so that the closed loop system will have a time constant of 1 sec and a damping ratio of 0.5.

12.9 The plant transfer function for the angular displacement θ of an inertia I subjected to a control torque T is (see Figure P12.8a)

$$G_p(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Is^2}$$

Suppose that $I = 5$ and that the output of the controller is the torque T. Use the root locus to investigate whether or not a controller transfer function of the following form will give a settling time of no more than 2 sec and an overshoot of no more than 5%.

$$G_c(s) = K\frac{s+a}{s+b}$$

Obtain suitable values of K, a, and b. What happens if we try to "cancel" one of the poles at the origin by placing the zero very close to the origin?

12.10 Consider a plant whose open-loop transfer function is

$$G(s)H(s) = \frac{1}{s\left[(s+2)^2+9\right]}$$

The complex poles near the origin give only slightly damped oscillations that are considered undesirable. Insert a gain K_c and a compensator $G_c(s)$ in series to speed up the closed-loop response of the system. Consider the following for $G_c(s)$:

a.   The lead compensator

b.   The lag compensator

c.   The so-called reverse-action compensator

$$G_c(s) = \frac{1 - T_1 s}{T_2 s + 1}$$

Obtain the root locus plots for the compensated system using each compensator. Use the compensator gain K_c as the locus parameter. Without computing specific values for the compensator parameters, determine which compensator gives the best response.

12.11 a) The equations of motion of the inverted pendulum model were derived in Example 2.4.4 in Chapter 2. Linearize these equations about $\phi = 0$, assuming that $\dot{\phi}$ is very small. b) Obtain the linearized equations for the following values: $M = 10$ kg, $m = 50$ kg, $L = 1$ m, $I = 0$, and $g = 9.81$ m/s^2. c) Use the linearized model developed in part (b) to design a series compensator to stabilize the pendulum angle near $\phi = 0$. It is required that the 2% settling time be no greater that 4 s and that the response be nonoscillatory. This means that the dominant root should be real and no greater than -1. No restriction is placed on the motion of the base. Assume that only ϕ can be measured.

Section 12.2 Design Using the Bode Plot

12.12 A certain unity feedback system has the following open-loop system transfer function.

$$G(s) = \frac{5K}{s^3 + 6s^2 + 5s}$$

Obtain the Bode plots and compute the phase and gain margins for

a. $K = 2$

b. $K = 20$

c.   Use the Bode plots to determine the upper limit on K for the system to be stable. Which is the limiting factor: the phase margin or the gain margin?

12.13 Figure P12.13 shows a pneumatic positioning system, where the displacement x is controlled by controlling the applied pneumatic pressure p_1. Assume that the pressure p_2 is constant, and consider the specific plant

$$G_p(s) = \frac{X(s)}{P_1(s)} = \frac{1}{100s^2 + s}$$

with the following series PD compensator is used to control the pressure.

$$G_c(s) = 2 + 19s$$

Obtain the Bode plots for this system, and determine the phase and gain margins.
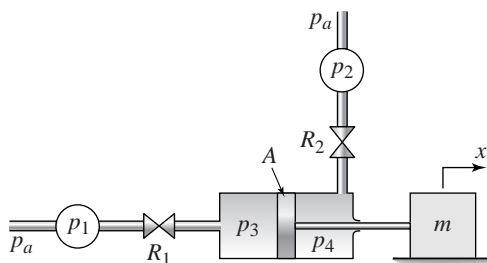
12.14 The height h_2 in Figure P12.14 can be controlled by adjusting the flow rate q_1. Consider the specific plant

$$G_p(s) = \frac{H_2(s)}{Q_1(s)} = \frac{25}{5s^2 + 6s + 5}$$

with the following series PI compensator is used to control it.

$$G_c(s) = \frac{7s + 64}{s}$$

Obtain the Bode plots for this system, and determine the phase and gain margins.

Figure P12.14
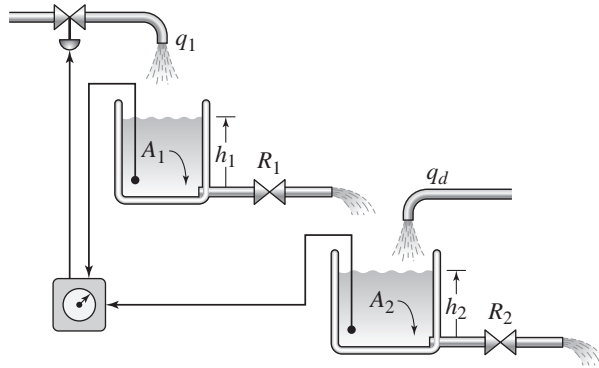
12.15 Rolling motion of a ship can be reduced by using feedback control to vary the angle of the stabilizer fins, much like ailerons are used to control aircraft roll. Figure P12.15 is the block diagram of a roll control system in which the roll angle is measured and used with proportional control action. Determine the phase and gain margins of the system if (a) $K_P = 1$, (b) $K_P = 10$, and (c) $K_P = 100$. Determine the stability properties for each case. If the system is unstable, what effect will this have on the ship roll?

Figure P12.15
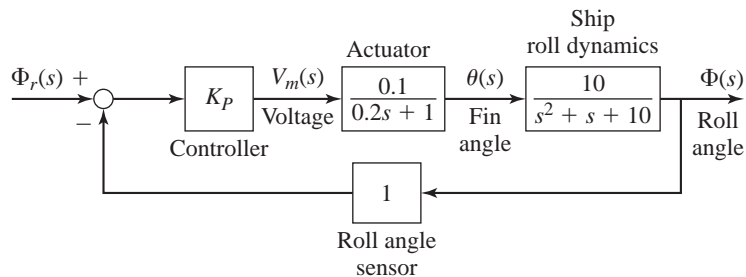
12.16 The open-loop transfer function of a certain unity feedback system is

$$G(s) = K\frac{10s + 1}{2s^3 + 3s^2 + s}$$

Compute the value of the gain K so that the phase margin will be $60°$.

12.17 The following transfer functions are the forward transfer function $G(s)$ and the feedback transfer function $H(s)$ for a system whose closed-loop transfer

function is

$$\frac{G(s)}{1 + G(s)H(s)}$$

For each case determine the system type number, the static position and velocity coefficients, and the steady-state errors for unit-step and unit-ramp inputs.

a.
$$G(s)H(s) = \frac{20}{s}$$

b.
$$G(s)H(s) = \frac{20}{5s + 1}$$

c.
$$G(s)H(s) = \frac{7}{s^2}$$

12.18 Remote control of systems over great distance, such as required with robot space probes, may involve relatively large time delays in sending commands and receiving data from the probe. Consider a specific system using proportional control, where the total dead time is $D = D_1 + D_2 = 100$ sec (Figure P12.18). The plant is

$$G_p(s) = \frac{1}{100s + 1}$$

How large can the gain K_P be without the system being unstable?
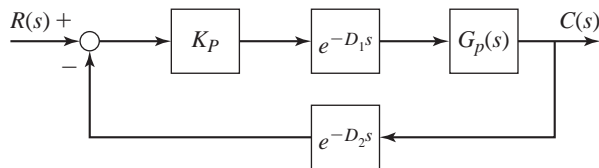
Figure P12.18

12.19 Hot-air heating control systems for large buildings may involve significant dead time if there is a large distance between the furnace and the room being heated (Figure P12.19). Proportional control applied to the specific plant

$$G_p(s) = \frac{1}{0.1s + 1}$$

has the gain $K_P = 10$. The time units are minutes. How large can the dead time D between the controller and the plant be if the phase margin must be no less than $40°$?

12.20 The block diagram of a position control system is shown in Figure P12.20. Design a compensator for the particular plant

$$G_p(s) = \frac{1}{s(s^2 + 3s + 2)}$$

so that the static velocity error coefficient will be $C_v = 5$/sec, the gain margin will be no less than 10 dB, and the phase margin no less than $40°$.

Figure P12.19

Figure P12.20

(a)   (b)
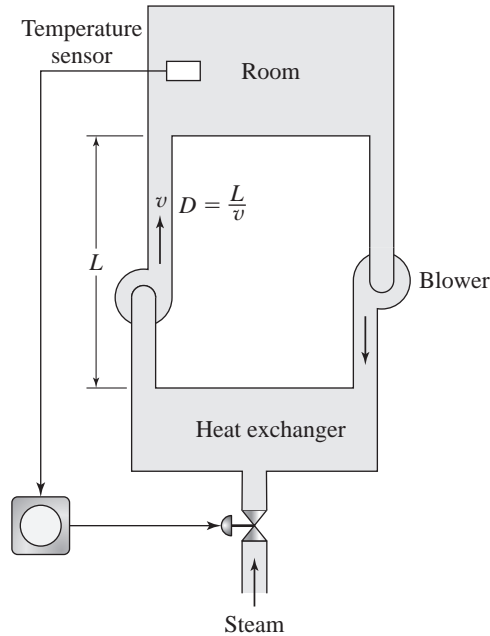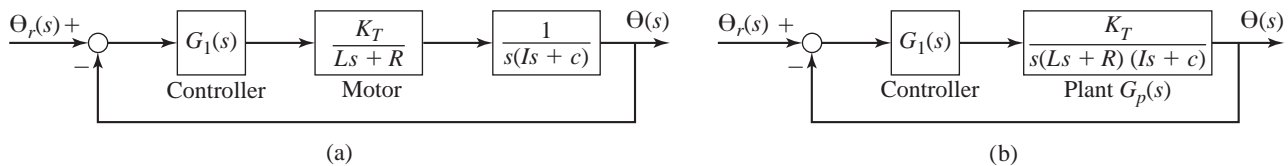
12.21 The speed ω_2 of the load is to be controlled with the torque T acting through a fluid coupling (see Figure P12.5). Design a compensator for the specific plant

$$G_p(s) = \frac{\Omega_2(s)}{T(s)} = \frac{4}{s^2 + 2s}$$

so that the static velocity error coefficient will be $C_v = 20$/sec, the gain margin will be no less than 10 dB, and the phase margin no less than $50°$.

12.22 Design a compensator for the plant

$$G_p(s) = \frac{2}{s^2 + 2s}$$

so that the static velocity error coefficient will be $C_v = 20$/sec and the phase margin at least $45°$.

12.23 Figure P12.2 shows a pneumatic positioning system, where the displacement x is controlled by controlling the applied pneumatic pressure p_1. Assume that the pressure p_2 is constant, and consider the specific plant

$$G_p(s) = \frac{X(s)}{P_1(s)} = \frac{1}{s^2 + s}$$

Design a compensator for the plant so that $C_v = 20/\text{sec}$, the gain margin will be no less than 10 dB, and the phase margin no less than $50°$.

12.24 The block diagram of a position control system is shown in Figure P12.7. Design a compensator for the particular plant

$$G_p(s) = \frac{5}{s(s+5)(s+1)}$$

that will give a static velocity error coefficient of $C_v = 50/\text{sec}$ and closed loop roots with a damping ratio of $\zeta = 0.5$.

12.25 The block diagram of a position control system is shown in Figure P12.7. Design a compensator for the particular plant

$$G_p(s) = \frac{1}{s(s^2+3s+2)}$$

so that the static velocity error coefficient will be $C_v = 10/\text{sec}$, the gain margin will be no less than 10 dB, and the phase margin no less than $50°$.

12.26 Consider a unity-feedback system having the open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

Derive the following expression for this system's phase margin.

$$\text{PM} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

12.27 Two independent feedback control systems are needed to position the pen of a flatbed x-y plotter. Consider only the control system for the x axis. An optical encoder provides feedback of the pen's location. Assume that the dynamics of the motor and pen assembly are described by the following plant transfer function.

$$G_p(s) = \frac{X(s)}{V(s)} = \frac{0.005}{s(0.1s+1)(0.001s+1)}$$

where x is the pen location in meters and v is the motor voltage. Design a compensator to achieve zero steady-state error for a step input and a 2% settling time of no more than 1 s, with an overshoot of no more than 4%.

12.28 Automatic guided vehicles are used in factories and warehouses to transport materials. They require a guide path in the floor and a control system for sensing the guide path and adjusting the steering wheels. Figure P12.28 is a block diagram of such a control system. Obtain the transfer function $G_c(s)$ so that the step response has an overshoot no greater than 20% with a 2% settling time of no more than 1 s, and a steady-state unit-ramp error of no more than 0.1 m.
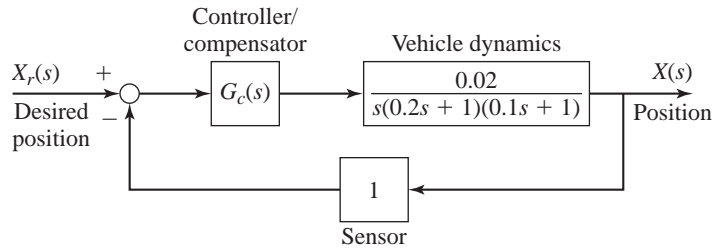
Section 12.3 MATLAB Applications

12.29 With the PI gains set to $K_P = 6$ and $K_I = 50$ for the plant

$$G_p(s) = \frac{1}{s+4}$$

the time constant is $\tau = 0.2$ and the damping ratio is $\zeta = 0.707$.
a.   Compute the gain and phase margins.
b.   Suppose there is dead time $D = 0.1$ between the controller and the plant. Compute the gain and phase margins, and plot the unit-step response.
c.   How large can the dead time be without the system becoming unstable?

12.30 In Example 12.1.4 a lag-lead compensator was designed by canceling the plant pole at $s = -3$ with a compensator zero. Suppose the plant model is slightly inaccurate and the plant pole is really at $s = -3.2$. Evaluate the resulting unit-step response and unit-ramp response in terms of the performance specifications.

Section 12.4 Simulink Applications

12.31 Refer to Example 12.1.1, in which a PID controller was designed to control the plant $1/s^2$. Using the gain values calculated in that example, construct a Simulink model of the system using the PID controller block and including the disturbance torque shown in Figure 12.1.7. Use the model to obtain plots of the responses to a unit-step command and a unit-step disturbance.

Vibration Applications

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to

1. Analyze the performance of spring and damping elements for isolating a mass from base excitation.

2. Analyze the performance of spring and damping elements for minimizing transmission of force caused by rotating unbalance.

3. Design a vibration absorber.

4. Analyze the modal behavior of systems having multiple degrees of freedom.

5. Design an elementary active vibration control system.

6. Analyze nonlinear vibration response using numerical methods.

As we have seen many times, the transfer function method enables us to model dynamic elements with an algebraic description and provides a concise graphical description of a system's frequency response, which characterizes the response to periodic inputs. In this chapter we treat four important vibration applications of transfer functions and frequency response. Section 13.1 covers a common source of vibration, that due to motion of the supporting structure; this is called *base excitation*. Section 13.2 treats another common source of vibration, that due to *rotating unbalance,* which occurs when the center of mass of a rotating machine does not coincide with the center of rotation. The section discusses how to design spring and damping elements for minimizing the transmission of force to a supporting structure from a machine having rotating unbalance. This topic is called *vibration isolation*. Section 13.3 shows how to reduce the vibration amplitude of a machine by using a *dynamic vibration absorber,* which consists of an attached stiffness and mass whose motion "absorbs" the vibration.

Systems having multiple degrees of freedom, such as multiple masses connected by elastic or damping elements, can exhibit complicated behavior. Their free response is the sum of certain behavior patterns called *modes*. In Section 13.4 we will see how the mode concept helps us to understand a system's response.

The vibration isolators and absorbers treated in Section 13.2 and 13.3 are *passive* systems. Section 13.5 introduces *active* systems that use actuators and sensors to control vibration. Section 13.6 illustrates the differences between the responses of linear and nonlinear models. Section 13.7 shows how the root locus plot can be useful in vibrations analysis. ■

13.1 BASE EXCITATION

The most common sources of inputs to vibrating systems are motion of a base support (called base excitation or sometimes *seismic* excitation) and rotating unbalance. We treat base excitation in this section and rotating unbalance in Section 13.2.

A common example of base excitation is caused by a vehicle's motion along a bumpy road surface. This motion produces a displacement input to the suspension system via the wheels. The primary purpose of a vehicle suspension is to maintain tire contact with the road surface, and the secondary purpose is to minimize the motion and force transmitted to the passenger compartment. The suspension is an example of a *vibration isolation system* designed isolate the source of vibration from other parts of the machine or structure. In addition to metal springs and hydraulic dampers, vibration isolators consisting of highly damped materials like rubber provide stiffness and damping between the source of vibration and the object to be protected. For example, the rubber motor mounts of an automobile engine are used to isolate the automobile's frame from the effects of the motor's rotating unbalance. Cork, felt, and pneumatic springs are also used as isolators.

Sometimes we want to reduce the effects of a force transmitted to the supporting structure, and sometimes we want to reduce the output displacement caused by an input displacement. Thus we speak of *force isolation* and *displacement isolation*. We begin our study of isolation system design by analyzing the displacement transmitted through base motion.

DISPLACEMENT TRANSMISSIBILITY

Figure 13.1.1 Base excitation where the motion $y(t)$ of the base produces the motion $x(t)$ of the mass m.

The motion of the mass shown in Figure 13.1.1 is produced by the motion $y(t)$ of the base. This system is a model of many common displacement isolation systems. Assuming that the mass displacement x is measured from the rest position of the mass when $y = 0$, the weight mg is canceled by the static spring force. The force transmitted to the mass by the spring and damper is denoted f_t and is given by

$$f_t = c(\dot{y} - \dot{x}) + k(y - x) \tag{13.1.1}$$

This gives the following equation of motion.

$$m\ddot{x} = f_t = c(\dot{y} - \dot{x}) + k(y - x)$$

or

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{13.1.2}$$

The transfer function is

$$\frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k} \tag{13.1.3}$$

This transfer function is called the *displacement transmissibility* and can be used to analyze the effects of the base motion $y(t)$ on $x(t)$, the motion of the mass. Notice that this transfer function has numerator dynamics, so its frequency response plots will be different than those of the transfer function $1/(ms^2 + cs + k)$, which was studied in Chapter 8.

FORCE TRANSMISSIBILITY

From (13.1.1)

$$F_t(s) = (cs + k)\,[Y(s) - X(s)] \tag{13.1.4}$$

Substituting for $X(s)$ from (13.1.3) gives

$$F_t(s) = (cs + k)\left[Y(s) - \frac{cs + k}{ms^2 + cs + k}Y(s)\right] = (cs + k)\frac{ms^2}{ms^2 + cs + k}Y(s)$$

Thus, the second desired ratio is

$$\frac{F_t(s)}{Y(s)} = (cs + k)\frac{ms^2}{ms^2 + cs + k} \tag{13.1.5}$$

This is the ratio of the transmitted force to the base motion. It is customary to use instead the ratio $F_t(s)/kY(s)$, which is a dimensionless quantity representing how the base displacement y affects the force transmitted to the mass. Thus,

$$\frac{F_t(s)}{kY(s)} = \frac{cs + k}{k}\frac{ms^2}{ms^2 + cs + k} \tag{13.1.6}$$

The ratio $F_t(s)/kY(s)$ is called the *force transmissibility*. It can be used to compute the transmitted force $f_t(t)$ that results from a specified base motion $y(t)$.

We can express the transmissibility relations in terms of the damping ratio ζ and the natural frequency ω_n. Recall that $\omega_n^2 = k/m$ and $\zeta = c/2\sqrt{km}$, which can be manipulated to show that $c/m = 2\zeta\omega_n$. Divide the numerator and denominator of (13.1.3) by m:

$$\frac{X(s)}{Y(s)} = \frac{(c/m)s + k/m}{s^2 + (c/m)s + k/m} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{13.1.7}$$

Similarly, we can express the force transmissibility in terms of ζ and ω_n by dividing the numerator and denominator of (13.1.6) by m as follows:

$$\frac{F_t(s)}{kY(s)} = \frac{(c/m)s + k/m}{k}\frac{ms^2}{s^2 + (c/m)s + k/m} = \frac{2\zeta\omega_n s + \omega_n^2}{\omega_n^2}\frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{13.1.8}$$

A common application of the transmissibility expressions is to analyze the steady-state response $x(t) = X\sin(\omega t + \phi)$ of a sinusoidal input $y(t) = Y\sin\omega t$, having an amplitude Y and frequency ω. In this case, we can derive the expressions for the frequency transfer functions by substituting $s = j\omega$ into (13.1.7) and (13.1.8). The

result for (13.1.7) is

$$\frac{X(j\omega)}{Y(j\omega)} = \frac{2\zeta\omega_n\omega j + \omega_n^2}{-\omega^2 + 2\zeta\omega_n\omega j + \omega_n^2}$$

Dividing the numerator and denominator by ω_n^2 gives

$$\frac{X(j\omega)}{Y(j\omega)} = \frac{2\zeta(\omega/\omega_n)j + 1}{1 - \omega^2/\omega_n^2 + 2\zeta(\omega/\omega_n)j} = \frac{2\zeta rj + 1}{1 - r^2 + 2\zeta rj}$$

where the frequency ratio is $r = \omega/\omega_n$.

The *magnitude* of the transmissibility is

$$\left|\frac{X(j\omega)}{Y(j\omega)}\right| = \frac{X}{Y} = \sqrt{\frac{4\zeta^2 r^2 + 1}{(1 - r^2)^2 + 4\zeta^2 r^2}} \tag{13.1.9}$$

This expression can be used to calculate the amplitude X of the steady-state motion caused by a sinusoidal input displacement of amplitude Y.

Similarly, we can express the frequency transfer function for the force transmissibility as follows. From (13.1.8) with $s = j\omega$,

$$\frac{F_t(j\omega)}{kY(j\omega)} = \frac{2\zeta\omega_n\omega j + \omega_n^2}{\omega_n^2} \frac{-\omega^2}{-\omega^2 + 2\zeta\omega_n\omega j + \omega_n^2} = (2\zeta rj + 1)\frac{-r^2}{1 - r^2 + 2\zeta rj}$$

and thus

$$\left|\frac{F_t(j\omega)}{kY(j\omega)}\right| = \frac{F_t}{kY} = r^2\sqrt{\frac{4\zeta^2 r^2 + 1}{(1 - r^2)^2 + 4\zeta^2 r^2}} \tag{13.1.10}$$

This expression can be used to calculate the steady-state amplitude F_t of the transmitted force caused by a sinusoidal input displacement of amplitude Y. Comparing (13.1.9) and (13.1.10) we see that

$$\frac{F_t}{kY} = r^2\frac{X}{Y} \tag{13.1.11}$$

and

$$F_t = r^2 kX \tag{13.1.12}$$

These relations enable us to calculate F_t/kY and F_t easily if we have previously computed X/Y and X.

TRANSMISSIBILITY PLOTS

It is instructive to plot X/Y and F_t/kY versus the frequency ratio r, for various values of the damping ratio ζ. These plots are shown in Figures 13.1.2 and 13.1.3. Consider first the displacement transmissibility plot in Figure 13.1.2. When r is near 1, the forcing frequency ω is near the system's resonance frequency, and the curve is at a maximum. This means that the maximum base motion is transferred to the mass when r is near 1. Note also that the transmissibility can be greater than 1, which indicates that the base motion can be amplified. In fact, the transmissibility is greater than 1 if $r < \sqrt{2}$, and is less than 1 when $r \geq \sqrt{2}$. If $r \geq \sqrt{2}$, the displacement transmissibility decreases as r is increased. Note also that for a specific value of r, the transmissibility decreases as ζ is increased, if $r \leq \sqrt{2}$.

Figure 13.1.3 shows the plots of F_t/kY versus r, for several values of ζ. When $r \geq \sqrt{2}$, the force transmissibility does not necessarily decrease as r is increased. For
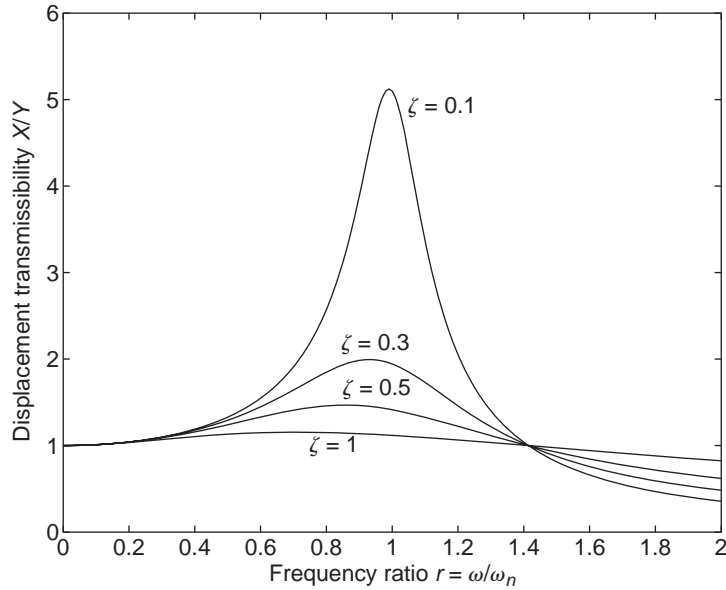
Figure 13.1.2 Displacement transmissibility versus r for base excitation.
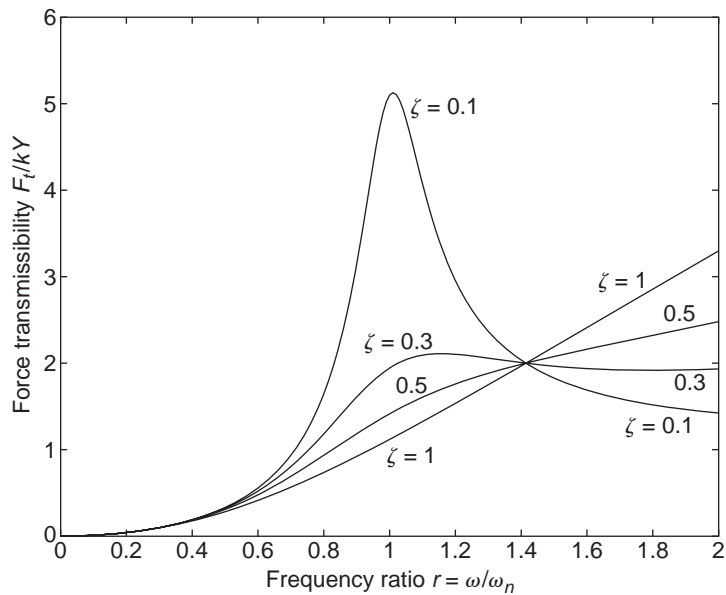
Figure 13.1.3 Force transmissibility versus r for base excitation.

example, if $\zeta = 1$, the force transmissibility increases with r. If ζ is small, say $\zeta = 0.1$, the force transmissibility decreases with r, if $r \geq \sqrt{2}$. This is in contrast with the behavior of the displacement transmissibility, which always decreases with r for any value of ζ, as long as $r \geq \sqrt{2}$.

The formulas and plots for force and displacement transmissibility can be used to design isolators to protect objects from unwanted vibration. The plots of X/Y and F_t/Y versus r show that the values of X/Y and F_t/Y are very sensitive to the value of ζ when r is near 1. Therefore, because the value of c is usually difficult to estimate from data, and thus ζ is difficult to compute accurately, you should be careful in using the formulas for X/Y and F_t/Y to design isolators when r is near 1.

Figure 13.1.4 Quarter-car, rigid-tire model of a vehicle suspension.

Table 13.1.1 Transmissibility formulas for base excitation.

Model
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad y(t) = Y \sin \omega t$$

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad r = \frac{\omega}{\omega_n}$$

Steady-state response
$$x(t) = X \sin(\omega t + \phi)$$

Displacement transmissibility
$$\frac{X}{Y} = \sqrt{\frac{4\zeta^2 r^2 + 1}{(1 - r^2)^2 + 4\zeta^2 r^2}}$$

Force transmissibility
$$\frac{F_t}{kY} = r^2 \frac{X}{Y}$$
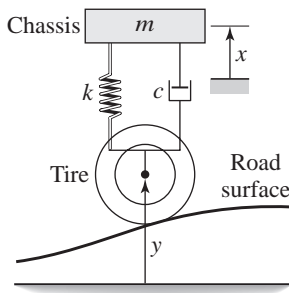
Table 13.1.1 summarizes these results.

An example of base excitation occurs when a car drives over a rough road. Figure 13.1.4 shows a quarter-car representation, where the stiffness k is the series combination of the tire and suspension stiffnesses. The equation of motion is given by (13.1.2). Although road surfaces are not truly sinusoidal in shape, we can nevertheless use a sinusoidal profile to obtain an approximate evaluation of the performance of the suspension at various speeds.

EXAMPLE 13.1.1

Vehicle Suspension Response

■ **Problem**

Suppose the road profile is given (in feet) by $y(t) = 0.05 \sin \omega t$, where the amplitude of variation of the road surface is 0.05 ft, and the frequency ω depends on the vehicle's speed and the road profile's period. Suppose the period of the road surface is 30 ft. Compute the steady-state motion amplitude and the force transmitted to the chassis, for a car traveling at speeds of 30 and 60 mph. The car weighs 3200 lb. The effective stiffness, which is a series combination of the tire stiffness and the suspension stiffness, is $k = 3000$ lb/ft. The damping is $c = 300$ lb-sec/ft.

■ **Solution**

For a period of 30 feet and a vehicle speed of v (mi/hr), the frequency ω is

$$\omega = \left(\frac{5280}{30}\right)\left(\frac{1}{3600}\right)(2\pi)\, v = 0.3072v \quad \text{rad/sec}$$

Thus, $\omega = 0.3072(30) = 9.215$ rad/sec for $v = 30$ mi/hr, and $\omega = 18.43$ rad/sec for $v = 60$ mi/hr.

For the car weighing 3200 lb, the quarter-car mass is $m = 800/32.2$ slug. Its natural frequency is $\omega_n = \sqrt{k/m} = \sqrt{3000/(800/32.2)} = 10.99$ rad/sec. Its frequency ratio at 30 mi/hr is $r = \omega/\omega_n = 9.215/10.99 = 0.839$, and at 60 mi/hr it is $r = 1.68$. Its damping ratio is

$$\zeta = \frac{300}{2\sqrt{3000(800/32.2)}} = 0.549$$

Now substitute these values of r and ζ and $Y = 0.05$ ft into the following expressions, obtained from (13.1.9) and (13.1.12).

$$X = Y\sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}}$$

$$F_t = r^2 k X$$

This gives the following table.

v (mi/hr)	r	X(ft)	F_t (lb)
30	0.839	0.072	148
60	1.68	0.041	342

Note that the motion amplitude is smaller at the higher speed because $r > 1.4$ at this speed. Examine Figure 13.1.2 to understand why this occurs.

13.2 ROTATING UNBALANCE

A common cause of sinusoidal forcing in machines is the unbalance that exists to some extent in every rotating machine. The unbalance is caused by the fact that the center of mass of the rotating part does not coincide with the center of rotation. Let m be the total mass of the machine and m_u the rotating mass causing the unbalance. Consider the entire unbalanced mass m_u to be lumped at its center of mass, a distance ϵ from the center of rotation. This distance is the *eccentricity*. Figure 13.2.1a shows this situation. The main mass is thus $(m - m_u)$ and is assumed to be constrained to allow only vertical motion. The motion of the unbalanced mass m will consist of the vector combination of its motion relative to the main mass $(m - m_u)$ and the motion of the main mass. For a constant speed of rotation ω, the rotation produces a radial acceleration of m_u equal to $\epsilon\omega^2$. This causes a force to be exerted on the bearings at the center of rotation. This force has a magnitude $m_u \epsilon \omega^2$ and is directed radially outward. The vertical component of this rotating unbalance force is, from Figure 13.2.1b,

$$f_r = m_u \epsilon \omega^2 \sin \omega t \tag{13.2.1}$$
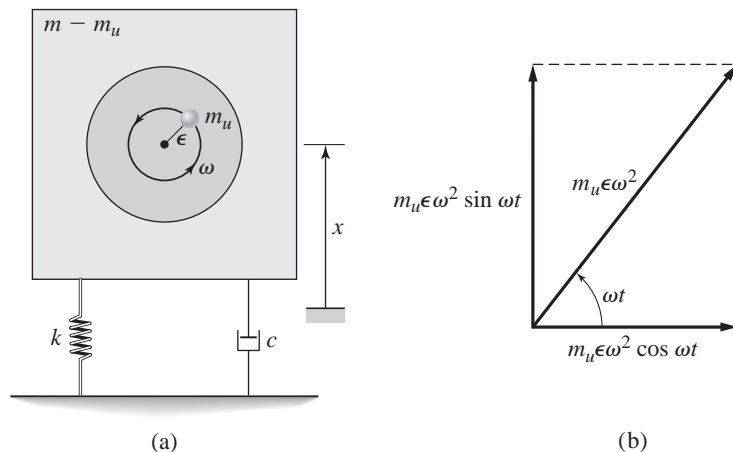
Figure 13.2.1 A machine having rotating unbalance.

VIBRATION ISOLATION AND ROTATING UNBALANCE

In many situations involving an unbalanced machine, we are interested in the force that is transmitted to the base or foundation. The equation of motion of a mass-spring-damper system, like that shown in Figure 13.2.1a, with an applied force $f_r(t)$ is

$$m\ddot{x} + c\dot{x} + kx = f_r(t) \tag{13.2.2}$$

where x is the displacement of the mass from its rest position. The force transmitted to the foundation is the sum of the spring and damper forces, and is given by

$$f_t = kx + c\dot{x} \tag{13.2.3}$$

The *force transmissibility* of this system is the ratio $F_t(s)/F_r(s)$, which represents the ratio of the force f_t transmitted to the foundation to the applied force f_r. The most common case of such an applied force is the rotating unbalance force. From (13.2.1), we see that the amplitude F_r of the rotating unbalance force is

$$F_r = m_u \epsilon \omega^2 \tag{13.2.4}$$

The transfer function is

$$\frac{X(s)}{F_r(s)} = \frac{1}{ms^2 + cs + k} \tag{13.2.5}$$

Thus,

$$\frac{X(j\omega)}{F_r(j\omega)} = \frac{1}{k - m\omega^2 + c\omega j}$$

and from (13.2.4),

$$X(j\omega) = \frac{m_u \epsilon \omega^2}{k - m\omega^2 + c\omega j}$$

Divide the numerator and denominator by the mass m and use the fact that $\omega_n = \sqrt{k/m}$, $c/m = 2\zeta\omega_n$, and $r = \omega/\omega_n$ to obtain

$$X(j\omega) = \frac{1}{m}\frac{m_u \epsilon r^2}{1 - r^2 + 2\zeta rj}$$

and

$$X = |X(j\omega)| = \frac{m_u \epsilon}{m}\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{13.2.6}$$

FORCE TRANSMISSIBILITY

From (13.2.3),

$$F_t(s) = (k + cs)X(s) \tag{13.2.7}$$

Substituting $X(s)$ from (13.2.5) into (13.2.7) gives

$$F_t(s) = \frac{k + cs}{ms^2 + cs + k}F_r(s)$$

Thus the force transmissibility is

$$\frac{F_t(s)}{F_r(s)} = \frac{k + cs}{ms^2 + cs + k} \tag{13.2.8}$$

The frequency transfer function is

$$\frac{F_t(j\omega)}{F_r(j\omega)} = \frac{k + c\omega j}{-m\omega^2 + c\omega j + k}$$

from which we obtain

$$\frac{F_t(j\omega)}{F_r(j\omega)} = \frac{1 + 2\zeta(\omega/\omega_n)j}{1 - \omega^2/\omega_n^2 + 2\zeta(\omega/\omega_n)j} = \frac{1 + 2\zeta rj}{1 - r^2 + 2\zeta rj}$$

The magnitude of this expression is

$$\left|\frac{F_t(j\omega)}{F_r(j\omega)}\right| = \frac{F_t}{F_r} = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \tag{13.2.9}$$

This is the force transmissibility expression for a sinusoidal applied force. It is the ratio of the amplitude F_t of the transmitted force to the steady-state amplitude F_r of the applied force. In what follows, we will use the symbol T_r to represent the transmissibility ratio F_t/F_r. Thus,

$$T_r = \frac{F_t}{F_r} = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \tag{13.2.10}$$

Note that although the expression for force transmissibility given by (13.2.10) is identical to that of *displacement* transmissibility X/Y, given by (13.1.9), the expressions arise from different physical applications.

FORCE TRANSMISSIBILITY PLOT

The force transmissibility T_r is plotted versus the frequency ratio r in Figure 13.2.2, for several values of ζ. Note that all the curves pass through the point $T_r = 1, r = \sqrt{2}$. To obtain good vibration isolation, that is, to decrease the force transmitted to the foundation, we need to make T_r small. Near resonance, when r is near 1, T_r is highly dependent on the value of ζ. When $r < \sqrt{2}$, *increasing* ζ will decrease T_r and thus
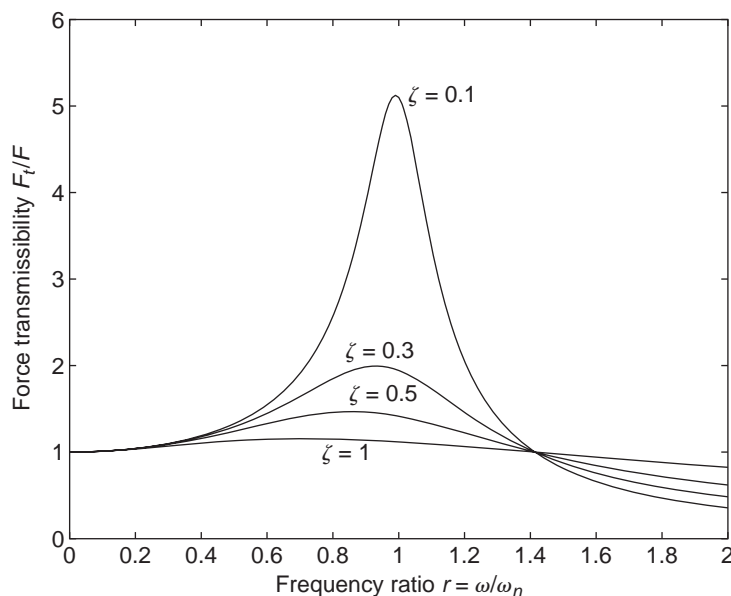
Figure 13.2.2 Force transmissibility versus r for force excitation.

Table 13.2.1 Formulas for rotating unbalance.

Model

$m\ddot{x} + c\dot{x} + kx = m_u \epsilon \omega^2 \sin \omega t = F_r \sin \omega t$

m_u = unbalanced mass ϵ = eccentricity

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad r = \frac{\omega}{\omega_n}$$

Displacement

$$X = \frac{m_u \epsilon}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Force transmissibility

$$T_r = \frac{F_t}{F_r} = \sqrt{\frac{1 + 4\zeta^2 r^2}{(1-r^2)^2 + 4\zeta^2 r^2}}$$

improve the vibration isolation. For $r \geq \sqrt{2}$, T_r is not so highly dependent on ζ, and T_r decreases as ζ decreases. Thus, for $r \geq \sqrt{2}$, we can improve the vibration isolation by *decreasing* ζ. We want to have some damping, however, because when the machine is started, its speed increases from zero and eventually passes through the resonance region near $r = 1$, and large forces can result if the machine's speed does not pass through the resonance region quickly enough. Some damping helps to limit the force buildup near resonance.

If the applied force is due to rotating unbalance and has an amplitude $F_r = m_u \epsilon \omega^2$, we can solve (13.2.10) for the amplitude of the transmitted force as follows:

$$F_t = m_u \epsilon \omega^2 \sqrt{\frac{1 + 4\zeta^2 r^2}{(1-r^2)^2 + 4\zeta^2 r^2}} \tag{13.2.11}$$

Table 13.2.1 summarizes these results.

EXAMPLE 13.2.1

Foundation Force Due to Rotating Unbalance

■ Problem
A system having a rotating unbalance, like that shown in Figure 13.2.1, has a total mass of $m = 20$ kg, an unbalanced mass of $m_u = 0.05$ kg, and an eccentricity of $\epsilon = 0.01$ m. The machine rotates at 1150 rpm. Its vibration isolator has a stiffness of $k = 2 \times 10^4$ N/m. Compute the force transmitted to the foundation if the isolator's damping ratio is (a) $\zeta = 0.1$ and (b) $\zeta = 0.5$.

■ Solution
First convert the machine's speed to radians per second.

$$\omega = 1150 \text{ rpm } = \frac{1150(2\pi)}{60} = 120 \text{ rad/s}$$

Then

$$m_u \epsilon \omega^2 = 0.05(0.01)(120)^2 = 7.25 \text{ N}$$

and the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{k/m}} = \frac{120}{\sqrt{2 \times 10^4/20}} = 3.81$$

We can calculate the transmissibility ratio from (13.2.10).

$$T_r = \sqrt{\frac{1 + 58.1\zeta^2}{182.7 + 58.1\zeta^2}}$$

The force transmitted to the foundation is

$$F_t = m_u\epsilon\omega^2 T_r = 7.25 T_r$$

a. If $\zeta = 0.1$, $T_r = 0.093$ and $F_t = 7.25(0.093) = 0.67$ N.
b. If $\zeta = 0.5$, $T_r = 0.28$ and $F_t = 7.25(0.28) = 2$ N. The more highly damped isolator in this case transmits more force to the foundation because $r > \sqrt{2}$.

If $r \geq \sqrt{2}$ and if ζ is small, then the formula (13.2.10) for T_r can be replaced with the approximate formula

$$T_r = \frac{1}{r^2 - 1} \qquad (13.2.12)$$

The approximate formula can be easily solved for r as a function of T_r, as follows:

$$r^2 = \frac{1 + T_r}{T_r} \qquad (13.2.13)$$

| Isolation of a Motor | **EXAMPLE I3.2.2** |

■ Problem
Often motors are mounted to a base with an isolator consisting of an elastic pad. The pad serves to reduce the motor's rotating unbalance force transmitted to the base. A particular motor weighs 10 lb and runs at 3200 rpm. Neglect damping in the pad and calculate the pad stiffness required to provide a 90% reduction in the force transmitted from the motor to the base.

■ Solution
A 90% force reduction corresponds to a transmissibility ratio of $T_r = 0.1$. Using the approximate formula (13.2.13), we obtain

$$r^2 = \frac{1 + T_r}{T_r} = \frac{1.1}{0.1} = 11$$

From the definition of r and the fact that $\omega_n = \sqrt{k/m}$, we have,

$$r^2 = \frac{\omega^2}{\omega_n^2} = \omega^2 \frac{m}{k}$$

Thus,

$$k = \frac{\omega^2}{r^2}m = \frac{[3200(2\pi)/60]^2}{11}\frac{10}{32.2} = 3170 \text{ lb/ft}$$

| Shaft Design with Rotating Unbalance | **EXAMPLE I3.2.3** |

■ Problem
Rotating machines such as pumps and fans must have their shafts supported by bearings, and often are enclosed in a housing. Suppose that the rotor (the rotating element) of a specific machine has mass of 300 kg and a measured unbalance of $m_u\epsilon = 0.5$ kg · m. The machine will be run at

a speed of 3500 rpm, and there is a clearance of 2 mm between the shaft and the housing. The shaft length from the bearings is $L = 0.05$ m. Assuming the shaft is steel, compute the minimum required shaft diameter. Model the shaft as a cantilever beam supported by the bearings, and neglect any damping in the system.

■ **Solution**

First convert the speed into rad/s: 3500 rpm $= 3500(2\pi)/60 = 367$ rad/s. From (13.2.6) with $\zeta = 0$,

$$X = \frac{m_u\epsilon}{m} \frac{r^2}{|1 - r^2|}$$

We are given that $m = 300$ kg. With $m_u\epsilon = 0.5$ and $X = 0.002$ m, we have

$$0.002 = \frac{0.5}{300} \frac{r^2}{|1 - r^2|}$$

Solve this for r^2 assuming that $r^2 > 1$:

$$0.002 = \frac{0.5}{300} \frac{r^2}{r^2 - 1}$$

which gives $r^2 = 6$ and thus

$$\omega_n^2 = \frac{\omega^2}{r^2} = \frac{(367)^2}{6} = 2.2387 \times 10^4$$

But $\omega_n^2 = k/m = k/300$. Thus $k = 300(2.2387 \times 10^4) = 6.716 \times 10^6$ N/m.

From the formula for a cantilever spring,

$$k = \frac{3EI_A}{L^3}$$

where the area moment of inertia for a cylinder is $I_A = \pi d^4/64$. Solving for the diameter d in terms of k, we obtain

$$d^4 = \frac{64kL^3}{3\pi E} = \frac{64(6.716 \times 10^6)(0.05)^3}{3\pi(2.07 \times 10^{11})} = 2.75 \times 10^{-8}$$

which gives a minimum shaft diameter of $d = 12.9$ mm.

ISOLATOR DESIGN

Commercially available isolators consist of a mount and an elastic material. In vibration isolator design, we use the formulas in Table 13.2.1 to compute the required values for the material's damping c and stiffness k. Designing the isolator also must take into account any requirements or constraints on size, shape, and weight imposed on the mount by the particular application. The designer then must look at vendor catalogs for existing mounts and materials that have values of c and k near the required values. If none can be found, there is often enough latitude to recompute another set of c and k values (that is, usually there will be more than one isolator design that will meet the specifications). Of course, other factors normally considered in engineering design, such as cost, ease of installation, reliability, and availability, must also be considered.

The transmissibility formulas in Table 13.2.1 assume that the input (either force or displacement) is sinusoidal and that the motion has reached steady state. In many applications, such as with vehicle suspensions, the input does not have a constant frequency,

and might even be somewhat random. In light of this, the forcing frequency used for the analysis can be chosen to represent a bound on the range of actual frequencies to be encountered, or might represent the frequency of the input that has the greatest effect on the system (for example, the one that has the largest amplitude). Often the engineer must design an isolator to provide isolation over a range of frequencies, and in such cases, the analysis must consider the isolator's performance of a range of frequency ratios r.

Applications where the input is not random but not sinusoidal must be handled with more advanced methods based on transient response analysis. Sometimes the transient requirements conflict with the steady-state requirements. For example, with base excitation where the base motion is a step function (a suddenly applied constant displacement), the isolator must provide protection over a wide range of frequencies, whereas to protect against sinusoidal inputs the isolator's stiffness and damping are chosen to give a small ζ and $r > \sqrt{2}$. Thus, we often find that an isolator that provides good protection against sinusoidal inputs will often provide poor protection against sudden transient inputs (called *shocks*) and vice versa.

Usually the mass is specified by the machine or object we are trying to isolate, and we choose the isolator's stiffness so that $r > \sqrt{2}$. If that is difficult to do, noting that $r = \omega\sqrt{m/k}$, we see that adding mass to the machine increases r and thus can enable an acceptable design to be achieved.

13.3 VIBRATION ABSORBERS

A *vibration absorber* is used to reduce vibration amplitude in situations where the disturbance has a constant frequency. Vibration absorbers are often found on devices that run at constant speed. These include saws, sanders, shavers, and devices powered by ac motors, because such motors are usually designed to operate at constant speed.

The absorber is a device consisting of another mass and stiffness element, which are attached to the main mass to be protected from vibration. The new system consisting of the main mass and the absorber mass has a fourth-order model, and thus the new system has two natural frequencies. If we know the frequency of the disturbing input and the natural frequency of the original system, we can select values for the absorber's mass and stiffness so that the motion of the original mass is very small, which means that its kinetic and potential energies will be small. To achieve this small motion, the energy delivered to the system by the disturbing input must be "absorbed" by the absorber's mass and stiffness. Thus the resulting absorber motion may be large. Because the principle of the absorber depends on the absorber's motion, such devices are sometimes called *dynamic* vibration absorbers. Another term for vibration absorber is a *tuned mass damper.*

EXAMPLES OF VIBRATION ABSORBERS

Figure 13.3.1 illustrates a vibration absorber that is used on some passenger cars to reduce the vibration of the exhaust pipe. It consists of a cantilever beam about 3 inches long and clamped to the pipe. The cylinder at the beam end is the absorber mass.

Power lines and structural cables often have vibration absorbers, called Stockbridge dampers, that are shaped somewhat like elongated dumbbells (Figure 13.3.2). These protect the lines and cables from excessive vibration caused by the wind. Wind blowing across the cables produces shedding vortices that cause the cables to vibrate.

Figure 13.3.1 Vibration absorber for a tailpipe.

Figure 13.3.2 A Stockbridge damper.

Figure 13.3.3 Vibration absorber for a tall building.
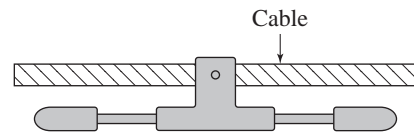
Many modern buildings and bridges have vibration absorbers to counteract the effects of wind and earthquakes. These are quite large; for example, the one in the Citicorp building in New York City uses a 400-ton concrete block that is allowed to slide horizontally on an oil bearings (see Figure 13.3.3). The tower has a natural frequency 0.15 Hz. The springs were designed to allow the mass to move up to 55 inches.

Some bridges and buildings use a pendulum damper. The designer must properly select the pendulum inertia and the pendulum length. The Taipei 101, located in Taipei, Taiwan, has 101 floors and a tuned mass damper weighing 728 tons. It consists of a steel sphere suspended as a pendulum from the 92nd to the 88th floor.

Other vibration absorber designs are being developed. For example, a liquid column damper is used in the Comcast Center in Philadelphia. It consists of two columns filled with water and connected at their bases. As the building sways, one water column rises and the other falls as the water moves between the two columns (think of water "sloshing" in a trough).

Design of vibration absorbers remains an exciting field, especially because of the new capabilities provided by computer control (see Section 13.5). In this section, we present the fundamentals of the basic absorber design.

ABSORBER ANALYSIS

Figure 13.3.4 illustrates a simple vibration absorber. The absorber consists of a mass m_2 and stiffness element k_2 that are connected to the main mass m_1. The disturbing input
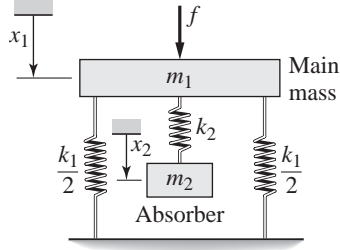
Figure 13.3.4 A system containing a vibration absorber.

is the applied force $f(t)$, which might be due to a rotating unbalance, for example. The equations of motion for the system are

$$m_1\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) + f$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2)$$

where x_1 and x_2 are measured from the rest positions of the masses. Applying the Laplace transform with zero initial conditions, and collecting the $X_1(s)$ and $X_2(s)$ terms, gives

$$\left(m_1 s^2 + k_1 + k_2\right) X_1(s) - k_2 X_2(s) = F(s)$$

$$-k_2 X_1(s) + \left(m_2 s^2 + k_2\right) X_2(s) = 0$$

Solve the second equation for $X_2(s)$ and substitute the result into the first equation to eliminate $X_2(s)$. Then solve for the transfer function $T_1(s) = X_1(s)/F(s)$ to obtain

$$T_1(s) = \frac{X_1(s)}{F(s)} = \frac{m_2 s^2 + k_2}{\left(m_1 s^2 + k_1 + k_2\right)\left(m_2 s^2 + k_2\right) - k_2^2} \tag{13.3.1}$$

Similarly, solve for the transfer function $T_2(s) = X_2(s)/F(s)$ to obtain

$$T_2(s) = \frac{X_2(s)}{F(s)} = \frac{k_2}{\left(m_1 s^2 + k_1 + k_2\right)\left(m_2 s^2 + k_2\right) - k_2^2} \tag{13.3.2}$$

If the applied force $f(t)$ is sinusoidal with a frequency ω, we can apply the frequency transfer functions to design a vibration absorber. These transfer functions are obtained by substituting $s = j\omega$ into the transfer functions $T_1(s)$ and $T_2(s)$ to obtain

$$T_1(j\omega) = \frac{k_2 - m_2\omega^2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$= \frac{1}{k_1} \frac{1 - (m_2/k_2)\omega^2}{\left[1 + k_2/k_1 - (m_1/k_1)\omega^2\right]\left[1 - (m_2/k_2)\omega^2\right] - k_2/k_1}$$

Define the frequency ratios r_1 and r_2 to be

$$r_1 = \frac{\omega}{\omega_{n_1}} = \frac{\omega}{\sqrt{k_1/m_1}} \tag{13.3.3}$$

$$r_2 = \frac{\omega}{\omega_{n_2}} = \frac{\omega}{\sqrt{k_2/m_2}} \tag{13.3.4}$$

Then we can write $T_1(j\omega)$ as follows:

$$T_1(j\omega) = \frac{1}{k_1} \frac{1 - r_2^2}{(1 + k_2/k_1 - r_1^2)(1 - r_2^2) - k_2/k_1} \tag{13.3.5}$$

Similarly we obtain

$$T_2(j\omega) = \frac{k_2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$= \frac{1}{k_1} \frac{1}{(1 + k_2/k_1 - r_1^2)(1 - r_2^2) - k_2/k_1} \tag{13.3.6}$$

Define

$$b = \frac{\omega_{n_2}}{\omega_{n_1}}$$

$$\mu = \frac{m_2}{m_1}$$

and note that

$$\frac{k_2}{k_1} = \frac{m_2}{m_1} \frac{m_1}{k_1} \frac{k_2}{m_2} = \mu \left(\frac{\omega_{n_2}}{\omega_{n_1}} \right)^2 = \mu b^2$$

and

$$r_1^2 = \left(\frac{\omega}{\omega_{n_1}} \right)^2 = \left(\frac{\omega_{n_2}}{\omega_{n_1}} \frac{\omega}{\omega_{n_2}} \right)^2 = b^2 r_2^2$$

Thus, $T_1(j\omega)$ and $T_2(j\omega)$ become

$$T_1(j\omega) = \frac{1}{k_1} \frac{1 - r_2^2}{(1 + \mu b^2 - b^2 r_2^2)(1 - r_2^2) - \mu b^2}$$

$$= \frac{1}{k_1} \frac{1 - r_2^2}{b^2 r_2^4 - [1 + (1 + \mu)b^2]r_2^2 + 1} \tag{13.3.7}$$

$$T_2(j\omega) = \frac{1}{k_1} \frac{1}{b^2 r_2^4 - [1 + (1 + \mu)b^2]r_2^2 + 1} \tag{13.3.8}$$

Note that there are no imaginary terms in these expressions because there is no damping in the model. Thus, both frequency transfer functions are real numbers.

ABSORBER DESIGN METHOD

Because $X_1(s) = T_1(s)F(s)$, we have $X_1(j\omega) = T_1(j\omega)F(j\omega)$. Therefore, if the applied force is $f(t) = F \sin \omega t$, the steady-state motion of mass m_1 will be given by $x_1(t) = X_1 \sin(\omega t + \phi_1)$, where

$$X_1 = |T_1(j\omega)| F$$

For m_1 to be motionless requires that $X_1 = 0$, which can be achieved if $|T_1(j\omega)| = 0$. From (13.3.7) we see that $|T_1(j\omega)| = 0$ if $1 - r_2^2 = 0$, which implies that $r_2 = \pm 1$. Because r_2 cannot be negative by definition, we see that the absorber design equation is given by $r_2 = 1$; that is,

$$r_2 = \frac{\omega}{\omega_{n_2}} = 1 \tag{13.3.9}$$

or

$$\omega_{n_2} = \sqrt{\frac{k_2}{m_2}} = \omega \qquad (13.3.10)$$

Thus, the mass m_1 will be motionless if we select an absorber having the same natural frequency ω_{n_2} as the frequency ω of the applied force. If this is done, the absorber is said to be "tuned" to the input frequency.

If $r_2 = 1$, the expression (13.3.8) for $T_2(j\omega)$ becomes

$$T_2(j\omega) = \frac{1}{k_1}\frac{1}{b^2 - 1 - (1 + \mu)b^2 + 1} = -\frac{1}{k_2}$$

Thus, if the absorber is designed so that $r_2 = 1$, then

$$X_2(j\omega) = -\frac{1}{k_2}F(j\omega) \qquad (13.3.11)$$

and the amplitude of the absorber's motion will be

$$X_2 = |X_2(j\omega)| = \frac{1}{k_2}F \qquad (13.3.12)$$

Thus the absorber's motion at steady state will be

$$x_2(t) = X_2 \sin(\omega t + \phi_2) = \frac{F}{k_2}\sin(\omega t + \pi) = -\frac{F}{k_2}\sin \omega t$$

If $r_2 = 1$, then $X_1 = 0$. Thus we can see that the absorber's spring force acting on the main mass is

$$k_2(x_2 - x_1) = k_2 x_2 = -F \sin \omega t \qquad (13.3.13)$$

Thus, if the absorber is tuned to the input frequency and its motion has reached steady state, the force acting on the absorber's mass has the same magnitude F as the applied force but is in the opposite direction. This causes the net force acting on the main mass to be zero; therefore, it does not move.

Equation (13.3.12) shows that in practice, the allowable clearance for the absorber's motion X_2 puts a limit on the allowable range of the absorber's stiffness k_2. Note also that the absorber's stiffness element k_2 must be able to support the force F and the compression or extension X_2 it causes.

The absorber design formulas are summarized in Table 13.3.1.

Table 13.3.1 Design formulas for absorber design.

Definitions

$F \sin \omega t = $ force to be suppressed

$m_2 = $ absorber mass

$k_2 \ = $ absorber stiffness

$X_2 \ = $ allowable clearance for the absorber motion

Design Equations

$$k_2 = \frac{F}{X_2}$$

$$m_2 = \frac{k_2}{\omega^2}$$

EXAMPLE 13.3.1

Design of a Vibration Absorber

■ Problem

A certain machine with supports has a measured natural frequency of 5 Hz. The machine will be subjected to a rotating unbalance force having an amplitude of 2 lb and a frequency of 4 Hz. Design a dynamic vibration absorber for this machine. The available clearance for the absorber's motion is 0.5 in.

■ Solution

The frequency of the applied force is $\omega = 2\pi(4) = 8\pi$ rad/sec. The absorber's design requires that $\omega_{n_2} = \omega = 8\pi$. Thus

$$\omega_{n_2} = \sqrt{\frac{k_2}{m_2}} = 8\pi$$

Solve for the mass:

$$m_2 = \frac{k_2}{64\pi^2}$$

The maximum allowable clearance is 0.5 in. $= 1/24$ ft. Using (13.3.12), we obtain

$$\frac{1}{24} = \frac{F}{k_2} = \frac{2}{k_2}$$

or $k_2 = 48$ lb/ft. Thus, the absorber's mass must be

$$m_2 = \frac{k_2}{64\pi^2} = \frac{48}{64\pi^2} = 0.076 \text{ slug}$$

Successful application of a vibration absorber depends on the disturbing frequency being known and constant. If the absorber's natural frequency is not exactly equal to the input frequency, the main mass will oscillate, and the amplitude of oscillation depends on the difference between the input frequency and the absorber's frequency. Another difficulty with absorber design is that the combined system now has two natural frequencies. If the input frequency is close to one of the natural frequencies, resonance will occur. Finally, the model used here does not include damping. If appreciable damping is present, motion of the main mass may not be small. It is advisable to perform a sensitivity analysis of the combined system to determine the effects of uncertainty in the input frequency.

EXAMPLE 13.3.2

Sensitivity Analysis of Absorber Design

Suppose the main mass in the system in Example 13.3.1 has the value $m_1 = 0.8$ slug. Evaluate the sensitivity of the absorber design to variations in the input frequency.

■ Solution

The absorber values from the previous example are $k_2 = 48$ lb/ft and $m_2 = 0.076$ slug. The natural frequency of the machine with its supports, in radians per second, is $\omega_{n_1} = 2\pi(5) = 10\pi$ rad/sec. Thus, the stiffness is

$$k_1 = \omega_{n_1}^2 m_1 = (10\pi)^2 0.8 = 790 \text{ lb/ft}$$
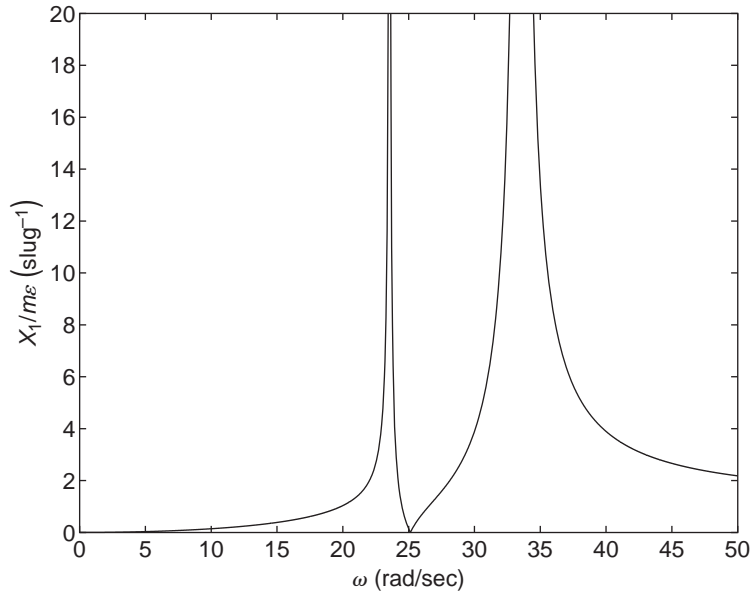
Figure 13.3.5 Frequency response of a vibration absorber.

and

$$r_1 = \frac{\omega}{\omega_{n_1}} = \frac{\omega}{10\pi}$$

$$r_2 = \frac{\omega}{\omega_{n_2}} = \frac{\omega}{8\pi}$$

Substituting these values into (13.3.5) with $F = m\epsilon\omega^2$, we obtain

$$\frac{X_1}{m\epsilon} = \left| \frac{\omega^2(1 - \omega^2/64\pi^2)}{(1.061 - \omega^2/100\pi^2)(1 - \omega^2/64\pi^2) - 0.061} \right| \frac{1}{790}$$

Figure 13.3.5 shows a plot of $X_1/m\epsilon$ versus ω. The plot shows that resonance occurs when the input frequency ω is near one of the two natural frequencies, which can be found from the roots of the denominator of $X_1/m\epsilon$. These frequencies are 23.54 and 33.53 rad/sec. The plot indicates the sensitivity of the design to changes in ω. From this plot, we can see that the motion amplitude of the main mass will be large if the input frequency is less than approximately 95% of its design value of $8\pi = 25.13$ rad/sec.

DAMPING IN VIBRATION ABSORBERS

The analysis in this section is based on a model with no damping. This has two implications. First of all, the equations $X_1 = |T_1(i\omega)|F$ and $X_2 = |T_2(i\omega)|F$ are strictly true only if the system is *stable* and at steady state. Because our absorber model has no damping, it is not stable, but is *neutrally stable*. Nevertheless, these equations are widely used because the inclusion of damping complicates the mathematics. The assumption made is that in practice every real system will have some damping. In addition, the *transient* response has not been accounted for, and this could affect the acceptability of absorber design.

The second implication is that the existence of damping in the real system could also affect the acceptability of absorber design. There has been much analysis of the design of absorbers with damping intentionally added, but this is a very complicated topic, and the results are not easily presented in a concise form.

Figure 13.3.6 Response of an undamped vibration absorber.
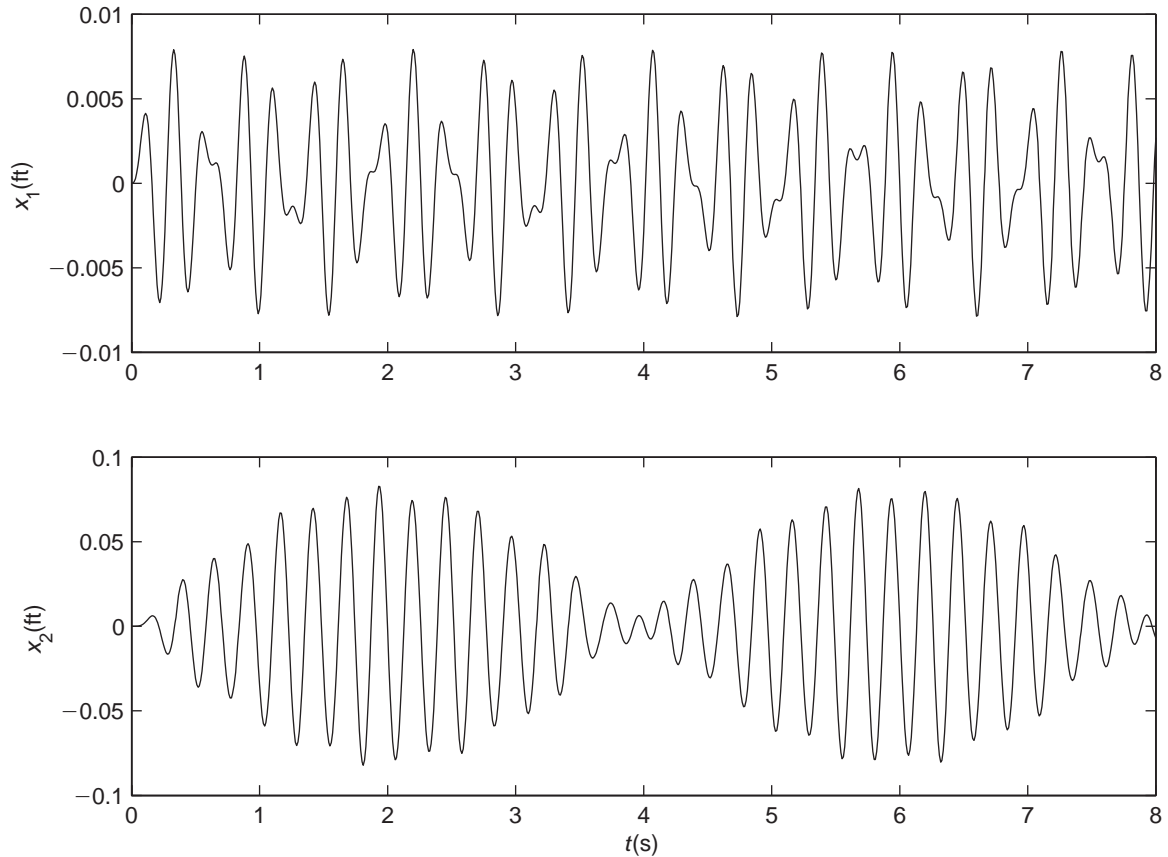
To show the effects of damping, we now include a damper between the mass m_1 and the support. The transfer functions are

$$\frac{X_1(s)}{F(s)} = \frac{m_2 s^2 + k_2}{(m_1 s^2 + cs + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2} \tag{13.3.14}$$

$$\frac{X_2(s)}{F(s)} = \frac{k_2}{(m_1 s^2 + cs + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2} \tag{13.3.15}$$

Let us use the values calculated in Examples 13.3.1 and 13.3.2. These are $m_1 = 0.8$ slug, $m_2 = 0.076$ slug, $k_1 = 790$ lb/ft, and $k_2 = 48$ lb/ft. The forcing function is $f(t) = 2 \sin 8\pi t$. The response without damping ($c = 0$), is shown in Figure 13.3.6. Note that beating occurs and so the masses do not oscillate at a single frequency, the frequency of the forcing function. This is due to the two natural frequencies, which are 23.54 and 33.53 rad/s. Note that the amplitude of the main mass m_1 is not zero, and the amplitude of the absorber mass m_2 is greater than the allowable clearance of 0.042 ft (0.5 in.), so we have obtained results different than those predicted by the steady-state analysis. The reason is that the steady-state analysis is based on the undamped model, which is neutrally stable.

Figure 13.3.7 Response of a damped vibration absorber.
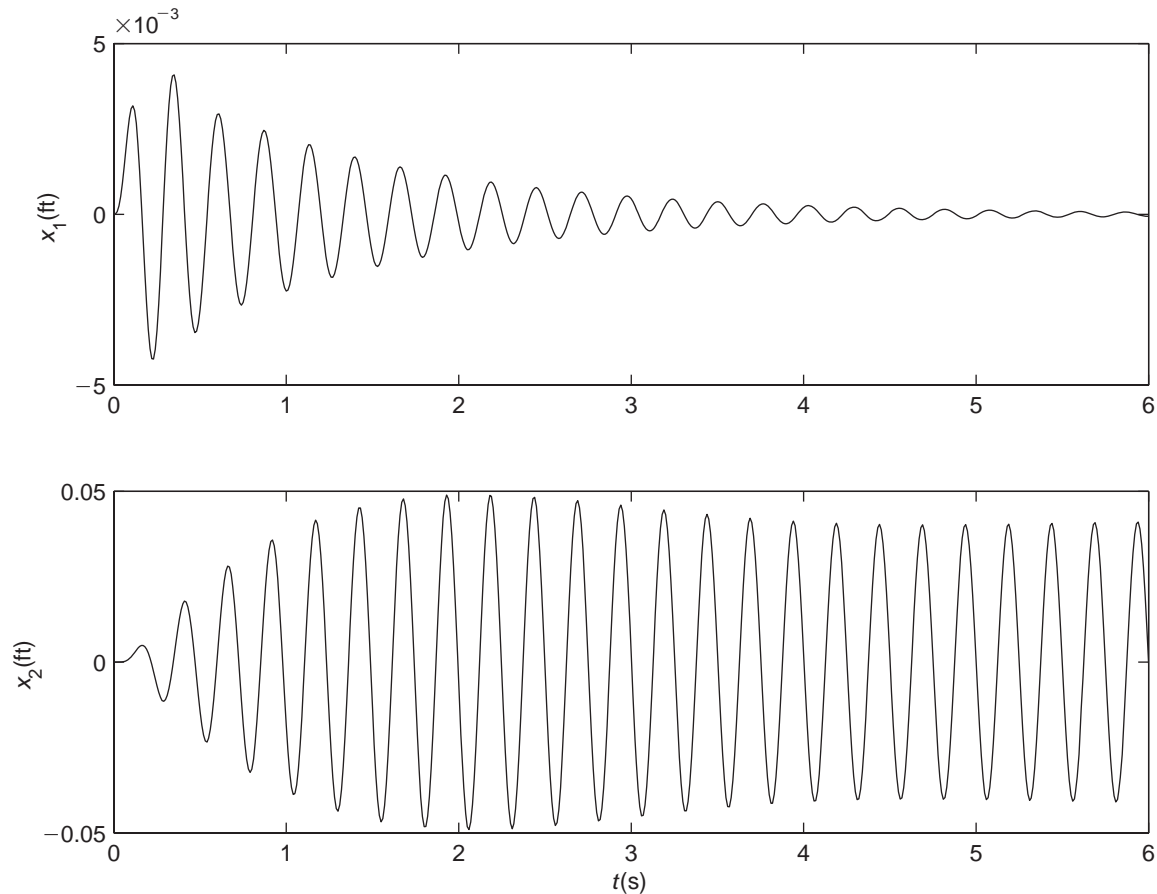
If we add damping, say $c = 10$ lb·sec/ft, the total response is like that shown in Figure 13.3.7. Now we see that the absorber is oscillating at the forcing frequency, and its amplitude at steady state is 0.042 ft as predicted. The main mass is motionless at steady state, also as predicted.

From Figure 13.3.7, you might erroneously conclude that damping is causing the main mass m_1 to be motionless at steady state, but this is not true. Figure 13.3.8 shows the response for the case with damping, but where the absorber mass is one-fourth of its design value. Now we see that the main mass is not motionless but oscillates with an amplitude of approximately 0.005 ft.

13.4 MODES OF VIBRATING SYSTEMS

Systems having multiple degrees of freedom, such as multiple masses connected by elastic or damping elements, can exhibit complicated behavior. Their free response is the sum of certain behavior patterns called *modes*. Knowledge of these modes enables us to understand better the response of such systems.

Consider an undamped two-mass system like that shown in Figure 13.4.1, where x_1 and x_2 are the displacements of the masses. The characteristic roots will be imaginary;

Figure 13.3.8 Response of a damped vibration absorber having a nonoptimal mass value.
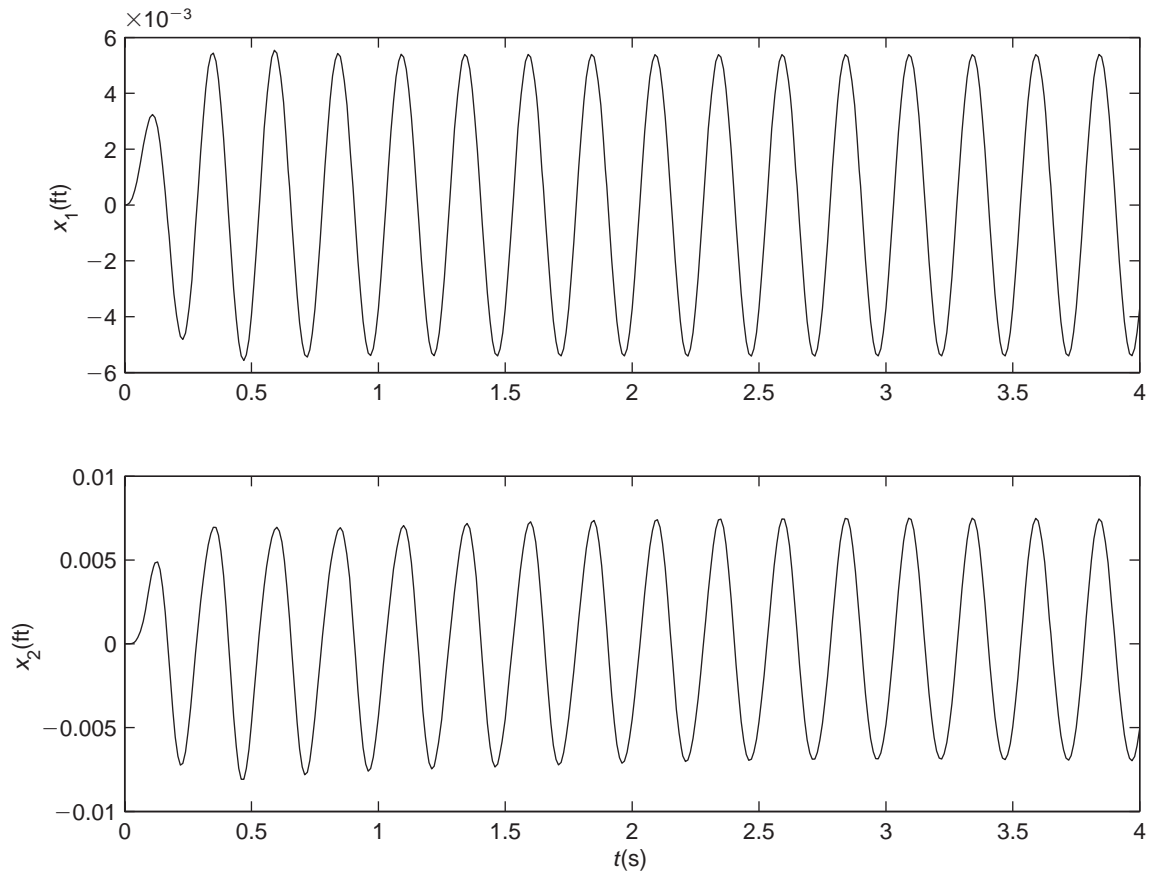
Figure 13.4.1 A vibrating system.
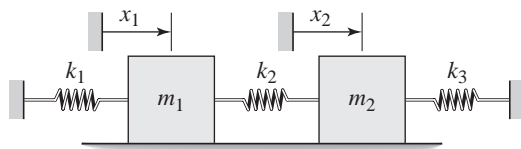
$s = \pm j b_1$ and $s = \pm j b_2$. The free response for x_1 has the form

$$x_1(t) = A_{11}e^{jb_1 t} + A_{12}e^{-jb_1 t} + A_{13}e^{jb_2 t} + A_{14}e^{-jb_2 t}$$

where A_{11} and A_{12} are complex conjugates, as are A_{13} and A_{14}. Thus the imaginary parts in the expression for $x_1(t)$ will cancel, leaving $x_1(t)$ to be real. This expression can be put into the form

$$x_1(t) = B_1 \sin(b_1 t + \phi_1) + B_2 \sin(b_2 t + \phi_2) \tag{13.4.1}$$

Similarly, the response of x_2 has the form

$$x_2(t) = r_1 B_1 \sin(b_1 t + \phi_3) + r_2 B_2 \sin(b_2 t + \phi_4) \tag{13.4.2}$$

where r_1 and r_2 are called the *mode ratios* for the first and second modes. The first mode is a motion in which $x_2(t) = r_1 x_1(t)$. In the second mode, $x_2(t) = r_2 x_1(t)$. The displacement in each mode is sinusoidal with a fixed amplitude. From (13.4.1) and (13.4.2), we see that the complete motion in general is a linear combination of the two modes.

The velocity in each mode is cosinusoidal and easily found from the displacements. For example, if the velocity of the first mass is $v_1 = \dot{x}_1$, then from (13.4.1),

$$v_1 = \dot{x}_1 = b_1 B_1 \cos(b_1 t + \phi_1) + b_2 B_2 \cos(b_2 t + \phi_2)$$

Even though this is a fourth-order system, we need be concerned with only two modes—the modes dealing with displacements. The other two modes describe the velocities, which in an undamped vibrating system are easily found, as already shown.

When there is no damping in the model, it is easier to perform the modal analysis using the second-order reduced model for each mass in the system. The undamped model can be arranged so that the mode ratios are real and are ratios of displacements only, and thus are easier to interpret. This is one reason why damping is often neglected when making a modal analysis of a vibratory system. If the damping is slight, the characteristic roots and modes will be almost the same as those of the undamped model. Even if the damping is not small, the insight gained from the undamped analysis is often quite useful for design purposes. If damping is to be accounted for in the modal analysis, the state variable form is easier to use.

Modes of Two Masses in Translation | **EXAMPLE 13.4.1**

■ Problem
Find and interpret the mode ratios for the system shown in Figure 13.4.1, for the case $m_1 = m_2 = m$, $k_1 = k_3 = k$, and $k_2 = 2k$.

■ Solution
The equations of motions for the system are

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = k_2(x_1 - x_2) - k_3 x_2$$

Substitute $x_1(t) = A_1 e^{st}$ and $x_2(t) = A_2 e^{st}$ into the preceding differential equations, cancel the e^{st} terms, and collect the coefficients of A_1 and A_2 to obtain

$$\left(m_1 s^2 + k_1 + k_2\right) A_1 - k_2 A_2 = 0 \tag{1}$$

$$-k_2 A_1 + \left(m_2 s^2 + k_2 + k_3\right) A_2 = 0 \tag{2}$$

To have nonzero solutions for A_1 and A_2, the determinant of the above equations must be zero. Thus,

$$\begin{vmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 + k_3 \end{vmatrix} = 0$$

Expanding this determinant gives

$$\left(m_1 s^2 + k_1 + k_2\right)\left(m_2 s^2 + k_2 + k_3\right) - k_2^2 = 0 \tag{3}$$

Using $m_1 = m_2 = m$, $k_1 = k_3 = k$, and $k_2 = 2k$, we obtain

$$\left(ms^2 + 3k\right)^2 - 4k^2 = 0$$

or

$$m^2 s^4 + 6kms^2 + 5k^2 = 0$$

This can be simplified to

$$s^4 + 6\alpha s^2 + 5\alpha^2 = 0$$

where $\alpha = k/m$.

This polynomial has four roots because it is fourth order. We can solve it for s^2 using the quadratic formula because it is quadratic in s^2 (there is no s term or s^3 term). To see why this is true, let $u = s^2$. Then the preceding equation becomes

$$u^2 + 6\alpha u + 5\alpha^2 = 0 \tag{4}$$

which has the solutions $u = -\alpha$ and $u = -5\alpha$. Thus $s = \pm j\sqrt{\alpha}$ and $s = \pm j\sqrt{5\alpha}$. The two modal frequencies are thus $\omega_1 = \sqrt{\alpha}$ and $\omega_2 = \sqrt{5\alpha}$.

The mode ratio can be found from either equation (1) or (2). Choosing the former, we obtain

$$\frac{A_1}{A_2} = \frac{2k}{ms^2 + 3k} = \frac{2\alpha}{s^2 + 3\alpha}$$

The mode ratio A_1/A_2 can be thought of as the ratio of the amplitudes of x_1 and x_2 in that mode. For the first mode, $s^2 = -\alpha$ and $A_1/A_2 = 1$. For the second mode, $s^2 = -5\alpha$ and $A_1/A_2 = -1$. Thus, in mode 1 the masses move in the same direction with the same amplitude. This oscillation has a frequency of $\omega_1 = \sqrt{k/m}$. In mode 2, the masses move in the opposite direction but with the same amplitude. This oscillation has a higher frequency of $\omega_2 = \sqrt{5k/m}$.

The specific motion depends on the initial conditions, and in general is a combination of both modes. If the masses are initially displaced an equal distance in the same direction and then released, only the first mode will be stimulated. Only the second mode will be stimulated if the masses are initially displaced an equal distance but in opposite directions.

For the system treated in Example 13.4.1, the modes were symmetric because the system is symmetric about its middle. In general, this is not the case, however.

EXAMPLE 13.4.2 | Nonsymmetric Modes

■ Problem
Find and interpret the mode ratios for the system shown in Figure 13.4.1, for the case $m_1 = m$, $m_2 = 3m$, $k_1 = k$, and $k_2 = k_3 = 2k$.

■ Solution
For this case, equation (3) in Example 13.4.1 becomes

$$3u^2 + 13\alpha u + 8\alpha^2 = 0$$

where $u = s^2$ and $\alpha = k/m$. From the quadratic formula we obtain $u = -0.743\alpha$ and $u = -3.591\alpha$. Thus the two modal frequencies are $\omega_1 = \sqrt{0.743\alpha} = 0.862\sqrt{k/m}$ and $\omega_2 = \sqrt{3.591\alpha} = 1.89\sqrt{k/m}$. From equation (1) of Example 13.4.1 the mode ratios are computed as

$$\frac{A_1}{A_2} = \frac{2\alpha}{s^2 + 3\alpha} = 0.886$$

for mode 1, and

$$\frac{A_1}{A_2} = \frac{2\alpha}{s^2 + 3\alpha} = -3.39$$

for mode 2.

Thus in mode 1 the masses move in the same direction with the amplitude of mass m_1 equal to 0.886 times the amplitude of mass m_2. This oscillation has a frequency of $\omega_1 = 0.862\sqrt{k/m}$. In mode 2, the masses move in the opposite direction with amplitude of mass m_1 equal to 3.39 times the amplitude of mass m_2. This oscillation has a higher frequency of $\omega_2 = 1.89\sqrt{k/m}$.

To stimulate the first mode, displace mass m_1 0.866 times the initial displacement of mass m_2, in the same direction. To stimulate the second mode, displace mass m_1 3.39 times the initial displacement of mass m_2, but in the opposite direction.

In practice, we usually need not solve for the mode amplitudes in terms of the initial conditions, because sufficient insight for design purposes can be gained from the mode ratios alone.

MODES OF A SYSTEM IN ROTATION AND TRANSLATION

The mode concept is also useful for understanding the motion of systems having more than one degree of freedom, for example, combined rotation and translation. A representation of a vehicle suspension suitable for modeling the vehicle's bounce and pitch motions is shown in Figure 13.4.2a, which is a side view of the vehicle's body showing the front and rear suspensions. Assume that the car's motion is constrained to a vertical translation x of the mass center and rotation θ about a single axis that is perpendicular to the page. The body's mass is m and its moment of inertia about the mass center is I_G. As usual, x and θ are the displacements from the equilibrium position corresponding to $y_1 = y_2 = 0$. The displacements $y_1(t)$ and $y_2(t)$ can be found knowing the vehicle's speed and the road surface profile.

Assuming that x and θ are small, we can derive the equations of motion for the bounce motion x and pitch motion θ as follows. The small displacement assumption implies that the suspension forces are nearly perpendicular to the centerline of the mass m, and thus are nearly vertical (see Figure 13.4.2b). To draw the free body diagram,

Figure 13.4.2 (a) Simple model of vehicle bounce and pitch. (b) Free body diagram.

(a)   (b)
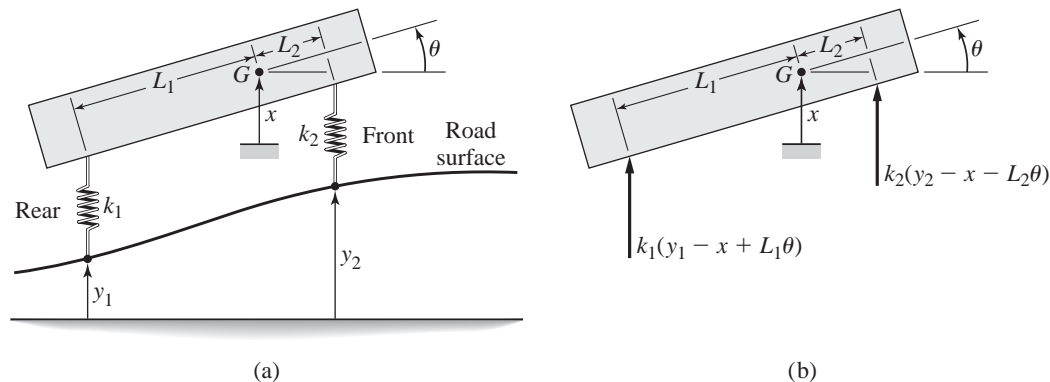
assume arbitrarily that

$$y_1 > x - L_1\theta$$

$$y_2 > x + L_2\theta$$

The force equation in the vertical direction gives

$$m\ddot{x} = k_1(y_1 - x + L_1\theta) + k_2(y_2 - x - L_2\theta) \tag{13.4.3}$$

The moment equation is

$$I_G\ddot{\theta} = -k_1(y_1 - x + L_1\theta)L_1 + k_2(y_2 - x - L_2\theta)L_2 \tag{13.4.4}$$

We can set the inputs $y_1(t)$ and $y_2(t)$ to zero because we need only the free response to find the modes. Make the substitutions $x(t) = A_1e^{st}$ and $\theta(t) = A_2e^{st}$ into the preceding differential equations, cancel the e^{st} terms, and collect the coefficients of A_1 and A_2 to obtain

$$(ms^2 + k_1 + k_2)A_1 + (k_2L_2 - k_1L_1)A_2 = 0 \tag{13.4.5}$$

$$(k_2L_2 - k_1L_1)A_1 + \left(I_Gs^2 + k_1L_1^2 + k_2L_2^2\right)A_2 = 0 \tag{13.4.6}$$

To have nonzero solutions for A_1 and A_2, the determinant of the above equations must be zero. Thus

$$\begin{vmatrix} ms^2 + k_1 + k_2 & (k_2L_2 - k_1L_1) \\ (k_2L_2 - k_1L_1) & I_Gs^2 + k_1L_1^2 + k_2L_2^2 \end{vmatrix} = 0$$

Expanding this determinant gives

$$(ms^2 + k_1 + k_2)\left(I_Gs^2 + k_1L_1^2 + k_2L_2^2\right) + (k_2L_2 - k_1L_1)^2 = 0 \tag{13.4.7}$$

or

$$mI_Gs^4 + \left[m\left(k_1L_1^2 + k_2L_2^2\right) + I_G(k_1 + k_2)\right]s^2 + k_1k_2(L_1 + L_2)^2 = 0 \tag{13.4.8}$$

This polynomial has four roots but is quadratic in s^2.

EXAMPLE 13.4.3 | Pitch and Bounce Modes of a Vehicle

■ Problem
Determine the mode shapes and mode frequencies of the vehicle shown in Figure 13.4.2.

■ Solution
The mode ratio can be found from either (13.4.5) or (13.4.6). Choosing the former, we obtain

$$\frac{A_1}{A_2} = \frac{k_1L_1 - k_2L_2}{ms^2 + k_1 + k_2} \tag{1}$$

The mode ratio A_1/A_2 can be thought of as the ratio of the amplitudes of x and θ in that mode. From Figure 13.4.3a we find that

$$\tan\theta = \frac{x}{D}$$

and for small angles θ,

$$D \approx \frac{x}{\theta} = \frac{A_1}{A_2} \tag{2}$$

Figure 13.4.3 (a) Node location. (b) Bounce center. (c) Pitch center.

(a)   (b)   (c)
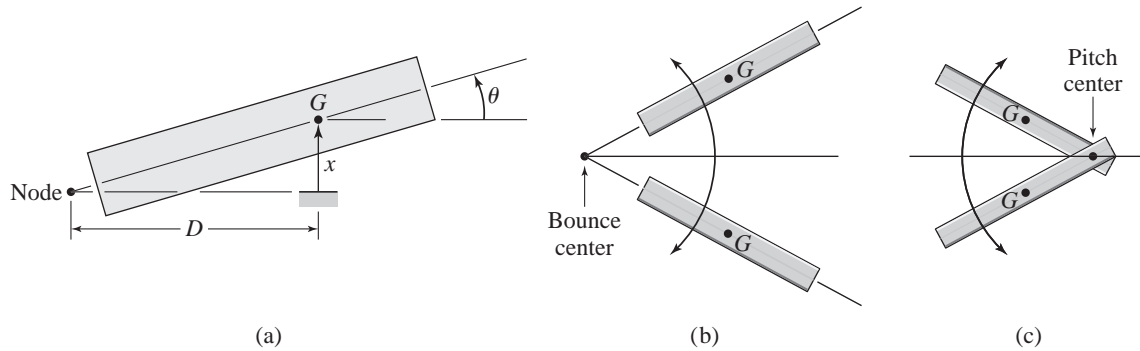
The distance D locates a point called a "node" or "motion center" at which no motion occurs (that is, a passenger located at a node would not move if the vehicle were moving in the corresponding mode). Thus, there are two nodes, one for each mode.

Equation (1) shows that A_1, the amplitude of x, will be zero if

$$k_1 L_1 - k_2 L_2 = 0 \tag{3}$$

In this case, equation (2) shows that $D = 0$. Thus, no coupling exists between the bounce motion x and the pitch motion θ, and the node for each mode is at the mass center. As we will discuss shortly, this condition will result in poor ride quality. Note also, that if equation (3) is satisfied, the characteristic equation (13.4.7) can be factored as follows:

$$(ms^2 + k_1 + k_2)\left(I_G s^2 + k_1 L_1^2 + k_2 L_2^2\right) = 0$$

or

$$ms^2 + k_1 + k_2 = 0 \tag{4}$$

$$I_G s^2 + k_1 L_1^2 + k_2 L_2^2 = 0 \tag{5}$$

Each of these equations has a pair of imaginary roots. Thus both modes are oscillatory, and the modal frequencies are

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}}$$

$$\omega_2 = \sqrt{\frac{k_1 L_1^2 + k_2 L_2^2}{I_G}}$$

Locating Node Points | **EXAMPLE 13.4.4**

■ **Problem**

Evaluate the mode shapes and mode frequencies of the vehicle shown in Figure 13.4.2 for the case: $k_1 = 1.6 \times 10^4$ N/m, $k_2 = 2.5 \times 10^4$ N/m, $L_1 = 1.5$ m, $L_2 = 1.1$ m, $m = 730$ kg, and $I_G = 1350$ kg $\cdot$ m^2.

■ **Solution**

Dividing the characteristic equation (13.4.8) by mI_G and using the given values for the constants, we obtain

$$s^4 + 105.24s^2 + 2744 = 0$$

The quadratic formula gives the roots $s^2 = -57.61, -47.63$. Thus, the four characteristic roots are

$$s = \pm 7.59j, \ \pm 6.901j$$

These correspond to frequencies of 1.21 Hz and 1.1 Hz.

The mode ratio can be found from equation (1) of Example 13.4.3.

$$\frac{A_1}{A_2} = \frac{x}{\theta} = \frac{-3500}{730s^2 + 4.1 \times 10^4} \tag{1}$$

For mode 1 ($s^2 = -57.61$),

$$\frac{x}{\theta} = \frac{-3500}{730(-57.61) + 4.1 \times 10^4} = 3.32 \ \text{m}$$

Thus the node is located 3.32 m *behind* the mass center. Because this node is so far from the mass center, the motion in this mode is predominantly a bounce motion, and this node is called the "bounce center"(see Figure 13.4.3b).

For mode 2 ($s^2 = -47.63$),

$$\frac{x}{\theta} = \frac{-3500}{730(-47.63) + 4.1 \times 10^4} = -0.562 \ \text{m}$$

This node is located 0.562 m *ahead* of the mass center (because $x/\theta < 0$). Because this node is close to the mass center, the motion in this mode is predominantly a pitching motion, and this node is called the "pitch center" (see Figure 13.4.3c).

EXAMPLE 13.4.5

Sensitivity of Node Location

■ **Problem**

For the vehicle in Example 13.4.4, investigate the sensitivity of the bounce and pitch node location to uncertainty in the parameter values.

■ **Solution**

Rearrange the mode ratio expression to obtain

$$\frac{A_1}{A_2} = \frac{x}{\theta} = \frac{k_1L_1 - k_2L_2}{m} \frac{1}{\omega_1^2 - \omega^2}$$

where we have defined $\omega^2 = -s^2$, and ω_1 as the natural frequency of a concentrated mass m attached to two springs k_1 and k_2:

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}}$$

The value of $\omega^2 = -s^2$ is obtained from the characteristic equation. If ω^2 is close to ω_1^2, then the denominator of x/θ will be very small, and x/θ will be very large. In addition, any slight change in the values of ω_1^2 or ω^2 due to changes in the parameters k_1, k_2, L_1, L_2, m, or I_G will cause large changes in the value of x/θ and thus large changes in the node location. In Example 13.4.4, $\omega_1^2 = 4.1 \times 10^4/730 = 56.16$. One of the roots gives $s^2 = -57.61$, which is close to -56.16. Thus we can expect large changes in the node location if slightly different parameter

values are used. The other root gives $s^2 = -47.63$, which is not as close to -56.16; thus, the corresponding node location will not be as sensitive.

If it is important to have a reliable prediction for the node location, the parameters must be selected—usually by trial and error—so that ω^2 is not close to ω_1^2.

PLACING THE BOUNCE AND PITCH CENTERS

In the 1930s Maurice Olley discovered that the best road vehicle ride is obtained by placing the bounce center behind the rear axle and the pitch center near the front axle. This is accomplished by designing the front suspension to have a lower natural frequency than the rear suspension, and is true because bounce motion is less annoying than pitch motion. Since road excitation affects the front wheels first, a lower front suspension natural frequency will tend to induce bounce rather than pitch.

Olley developed other rules of thumb, which are still used today [Gillespie, 1992].

1. If the front-rear weight distribution is approximately equal, the front suspension should have a 30% lower ride rate than the rear, or the spring center should be at least 6.5% of the wheelbase behind the center of gravity. (The *spring center* is the point at which an applied vertical force will produce only vertical displacement.) When the spring center is at the center of gravity, there is no coupling between the bounce and pitch motion, and poor ride quality results because the total motion is irregular.

2. The bounce frequency should be within 20% of the pitch frequency. Otherwise the superposition of the two modes will result in irregular, annoying motions. The roll frequency should be close to the bounce and pitch frequencies for the same reason.

3. The bounce and pitch frequencies should be kept as low as possible ($\leq$1.3 Hz if possible), consistent with any upper limit on static deflection. This is because the most severe road acceleration inputs occur at higher frequencies.

EFFECTIVE STIFFNESS AND SUSPENSION DESIGN

The primary purpose of a road vehicle's suspension is to keep the tires in contact with the road. The secondary purpose is to give the passengers a smooth ride. Proper design of the wheel suspension provides the primary source of isolation from the road roughness. The quarter-car model is the basis for the suspension system's design. Referring to Figure 13.4.4, the *ride rate* k_e is the effective stiffness of the suspension spring and the tire stiffness, neglecting the wheel mass m_2. Thus, $k_e = k_1 k_2/(k_1 + k_2)$. If $m_2 \ll m_1$, the suspension system's undamped natural frequency is approximately $\omega_n = \sqrt{k_e/m_1}$ and its damping ratio is $\zeta = c_1/2\sqrt{k_e m_1}$. Most modern cars have a suspension damping ratio ζ between 0.2 and 0.4, and thus their damped natural frequency ω_d is very close to ω_n.

The suspension's *static deflection* due to the weight $m_1 g$ is $\Delta = m_1 g/k_e$, and thus $\Delta = m_1 g/m_1 \omega_n^2 = g/\omega_n^2$. A natural frequency of 1 Hz ($\omega_n = 2\pi$ rad/sec) is considered to be a design optimum for highway vehicles, and this corresponds to a static deflection of $\Delta = 9.8$ in. For a vehicle weighing 3200 lb, this deflection requires an effective stiffness of $k_e = m_1 g/\Delta = 0.25(3200)(9.8) = 81.6$ lb/in. With a tire stiffness of $k_2 = 1200$ lb/in., the suspension stiffness must be $k_1 = k_e k_2/(k_2 - k_e) = 87.6$ lb/in. [Gillespie, 1992].

Figure 13.4.4 Quarter-car model including wheel-tire mass.
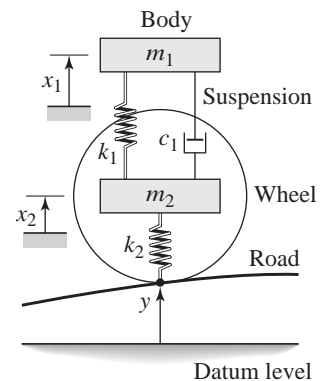
13.5 ACTIVE VIBRATION CONTROL

Sophisticated actuators and sensors, and miniaturized, low-cost computers have greatly increased the number of applications for active control of vibration. The core of an active vibration controller is typically a microprocessor-based system with analog-to-digital converters to process sensor inputs and digital-to-analog converters to convert the microprocessor's output commands into signals to the actuator (Figure 13.5.1). The term *active* originates from the fact that such systems require an active power source.

The actuator applies a force to the mass whose vibration is to be reduced. The sensor measures the motion of the mass (either its displacement, velocity, or acceleration, depending on the application). The control algorithm programmed in the computer uses these measurements to decide how much force should be applied.

The vibration isolators and vibration absorbers treated in Sections 13.1 through 13.3 have the disadvantage of working only at certain frequencies, because their stiffness and damping—and in the case of vibration absorbers, their mass—are fixed and cannot be changed without replacing the appropriate component. Another advantage of active vibration control systems is that they can change the effective stiffness and damping without adding much weight. Because such systems require a power source, many are designed to be *fail-safe*. For example, a fail-safe damper provides the minimum required damping in the event of a power supply or computer system failure.

Several terms are used to describe active vibration control systems. Structures such as buildings, bridges, aircraft cabins, and aircraft wings containing such systems are called *smart* or *intelligent structures*. Structural materials having the controller embedded in them are called *smart materials*. *Smart dampers* and *active shock absorbers* are dampers that can change their damping properties under computer control.

APPLICATIONS OF ACTIVE VIBRATION CONTROL

Examples of current applications of this technology include

- Reduction of building sway caused by wind and seismic waves using *hybrid mass dampers* that apply a controlled force to a large movable weight. Recall that the vibration absorbers we have thus far seen are limited because they must be designed for a known, fixed frequency.
- Active suspensions in motor vehicles.
- Reduction of aircraft cabin noise by reducing the vibration of the large panels of thin metal that form the cabin walls.
- Piezoelectric devices installed on the trailing edge of helicopter blades to dampen vibrations.

Figure 13.5.1 Active suspension system for one wheel of a vehicle.
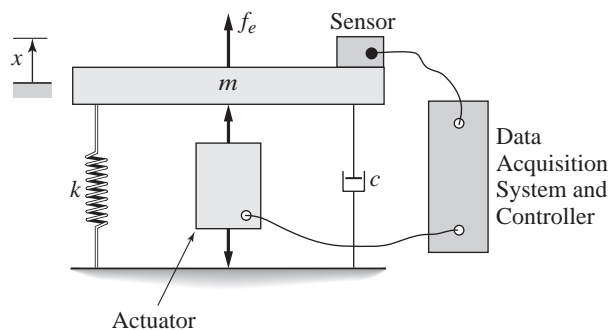
- Reduction of tabletop vibrations to the level necessary for precision manufacturing, such as the manufacture of semiconductors.
- Damping out floor vibrations with an electromagnetic shaker.
- Smart skis that dampen vibrations.
- Active dampers in bicycles.
- Vibration-dampening devices embedded in sporting equipment such as tennis racquets, snowboards, baseball bats, and golf clubs.
- Active stiffness control in running shoes.

ACTUATORS FOR ACTIVE VIBRATION CONTROL

Piezoelectric materials can convert electrical current into motion, and vice versa (see Section 6.6). They change shape when an electric current passes through them, and generate an electric signal when they flex. Thus they can be used as actuators, to create force or motion, and as sensors, to sense motion. Other materials used for high-precision actuation include electrostrictive and magnetostrictive materials, which are similar to piezoelectric materials. These are ferromagnetic materials that expand or contract when subjected to an electric or a magnetic field.

A magneto-rheological fluid changes its viscosity when subjected to a magnetic field. The viscosity of such fluids can be changed from that of water to that of pudding in milliseconds. Some engineering development has been done with using so-called *magneto-rheological fluid dampers* for smart dampers. Such actuators cannot provide enough force for larger applications, such as vehicle suspensions and control of building motions. In building applications, hydraulic cylinders are usually used. Active vehicle suspensions use hydraulic devices, electric motors, and magneto-rheological fluid dampers.

Some active systems using accelerometers and piezoelectric actuators are employed to isolate precision microlithography, metrology, and inspection equipment in advanced semiconductor factories to cancel motion caused by floor motion. In these processes, a silicon wafer must be positioned relative to optical or other components by placing the wafer on a heavy or fast-moving stage that can cause the entire system to vibrate. The resulting small motion of several millimeters may still be larger than the resolution of the instrument, which may be as small as 1 nm.

ELEMENTARY CONTROL THEORY FOR ACTIVE VIBRATION CONTROL

Consider the system shown in Figure 13.5.1. We wish to control the vibration of the mass m with the actuator that applies a force f to the mass. The force $f_e(t)$ is disturbance force such as that caused by rotating unbalance. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = f(t) + f_e(t) \tag{13.5.1}$$

Assume we have a sensor that measures the displacement x and the velocity $\dot{x}$ of the mass. A computer takes these measurements in real time, computes the required force f necessary to control the motion, and commands the actuator to supply that amount of force. The stiffness k and damping c are either inherently part of the system structure or are inserted to provide for fail-safe operation.

If we program the computer to control the force f in a manner that is proportional to x and to $\dot{x}$, then the control algorithm is

$$f(t) = -K_P x - K_D \dot{x} \tag{13.5.2}$$

where K_P and K_D are constants whose value must be determined by the designer and programmed into the computer. In control system terminology, these constants are called the proportional and derivative gains (see Chapter 10). Substituting (13.5.2) into (13.5.1) and collecting terms gives

$$m\ddot{x} + (c + K_D)\dot{x} + (k + K_P)x = f_e(t) \tag{13.5.3}$$

We immediately see that the gain K_P acts like an artificial stiffness and the gain K_D like an artificial damper. Thus we can change the effectiveness stiffness and damping by properly selecting the gains.

For example, the effective damping ratio is

$$\zeta = \frac{c + K_D}{2\sqrt{m(k + K_P)}} \tag{13.5.4}$$

and the effective natural frequency is

$$\omega_n = \sqrt{\frac{k + K_P}{m}} \tag{13.5.5}$$

If $\zeta \leq 1$, then the time constant is

$$\tau = \frac{2m}{c + K_P} \tag{13.5.6}$$

So if we know m, c, and k, we can compute the values of K_P and K_D to achieve desired values of ζ, ω_n, or τ. Note that the gains need not be positive. If the effective stiffness or damping needs to be smaller than k or c, then the gains will be negative.

The success of this scheme depends on several factors:

■ The ability of the sensors to provide accurate measurements of x and $\dot{x}$ in real time. Noise in the measured signal will degrade the system performance. Some systems do not measure the velocity but rather differentiate the x measurement. This will increase the noise in the signal representing $\dot{x}$.

■ The ability of the computer to perform the necessary calculations in real time. Although microprocessor speed is continually increasing, some microprocessors may not be fast enough for certain applications.

■ The ability of the actuator to provide the needed force level. All actuators have a limit on the force they can generate. This limitation frequently gets lost in the mathematics of the design, and must be considered when setting the desired values of ζ, ω_n, or τ. If the magnitudes of the gains K_P and K_D are too large, the actuator might not be able to provide the expected force.

CONTROL OF MULTI-MASS SYSTEMS

Figure 13.5.2 An active suspension.

Figure 13.5.2 illustrates the principle of an active suspension for one wheel. An actuator located between the wheel (mass m_1) and the chassis (mass m_2) supplies force f to both the wheel and the chassis. This force supplements the passive spring k_2 and passive damper c. The tire stiffness is k_1. If we choose the control the force f so that it imitates a spring and damper, like the PD control, then the control algorithm is

$$f = K_P(x_1 - x_2) + K_D(\dot{x}_1 - \dot{x}_2)$$

This requires measurements of the relative displacement $x_1 - x_2$ and the relative velocity $\dot{x}_1 - \dot{x}_2$. If this is substituted into the equations of motion, theoretically we can compute

values for K_P and K_D to achieve a desired set of characteristic roots corresponding to desired damping ratios, natural frequencies, and time constants. The difficulty with this approach is that there will be four roots, because this is a fourth order system, and so we cannot expect to be able to specify all four roots by choosing the values of only two gains.

Another approach is to use measurements of all four variables: x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$, and the control algorithm is called *state variable feedback* (see Chapter 11).

$$f = K_1 x_1 + K_2 x_2 + K_3 \dot{x}_1 + K_4 \dot{x}_2$$

We now have four gains to select in order to place the four roots where desired. It turns out that this is not always possible and that the mathematics required to solve for the gain values is quite difficult. In practice, solution algorithms provided by MATLAB will do this (see the `acker` function in Chapter 11).

EXAMPLES OF CURRENT APPLICATIONS

Active Suspensions   Traditional vehicle suspensions with fixed springs and dampers cannot provide both the smooth ride of a luxury car and the precise handling of a sports car, but, with an active suspension, such a goal is closer to being realized. Versions of active suspensions have been available since the 1990s. As of 2008, however, there is no truly active suspension available commercially. The most sophisticated system currently available, made by Delphi, uses a damper filled with a magneto-rheological fluid. The system uses a controlled magnetic field that can change the viscosity 1000 times a second.

Under development by the Bose Corp. since 1980, another system uses speaker-coil technology to drive linear electrical motors attached to each wheel. Such motors produce translation rather than rotation, and so imitate the forces provided by springs and dampers. One of the challenges is to develop control algorithms that provide effective coordination of all four wheel subsystems. This is necessary for proper vehicle control in sharp turns and sudden stops, for example.

The sports industry has developed a number of applications for active vibration control. These include skis, bike suspensions, and running shoes.

Skis   Skis can vibrate excessively at high speeds, on hard snow, and on rough terrain. This vibration decreases the contact area between the ski edge and the snow surface and results in a reduced ability of the skier to control the motion. Smart skis designed to keep the edge of the ski in contact with the snow have been developed by the K2 Corp and by Head. Because tests indicated that most vibrations are concentrated near the ski binding and propagate outwards to the rest of the ski, a piezoelectric ceramic plate is mounted in front of the ski binding. When current is applied to the element it flexes and applies force to the ski to change its shape. As the ski flexes, it generates electricity to power the system.

Bikes   Piezoelectric materials have been used to develop a smart bicycle shock absorber to assist in keeping the wheel in contact with the ground and to minimize the force transmitted to the rider. Some bikes have suspensions in both the front and the rear, but the control of the front suspension is more critical. The front suspension consists of a pair of telescoping cylinders—called fork tubes—that contain springs and dampers. These can be arranged in parallel, series, or any desired combination. The system monitors the position and velocity of the piston inside the fluid-filled cylinder

using a magnet attached to the piston and a sensor that detects the flux density of the magnetic field and converts it to a voltage.

Compression damping helps to control the rate of shock compression. Too much compression damping will transmit a large force to the rider, but too little compression damping will cause the shock to bottom out quickly. The smart shock absorber automatically adjusts compression damping with varying trail conditions, by commanding a piezoelectric bypass valve to open or close, thus increasing or decreasing the fluid resistance. To do this, some designs use a main hydraulic flow path (for fail-safe operation) and a by-pass flow path that is normally closed by a flexible cantilever beam that acts like a flapper valve. An element consisting of layers of piezoelectric ceramic wafers is bonded to the beam. These layers expand when an electric current is applied. When a current is applied to the piezoelectric element by the controller, the beam flexes and opens an additional hydraulic flow path to reduce the resistance. The beam displacement can be controlled in increments to provide fine control of the damping.

Running Shoes   In 2005 Adidas introduced a running shoe that uses active stiffness control with a battery-powered microprocessor and motor. The motor turns a gear train and shaft whose rotation changes the tension of cables in the base of the shoe. The system uses a magnetic sensor to measure the shoe compression. Based on this measurement, the microprocessor commands the motor and adjusts the shoe stiffness before the shoe strikes the ground again.

The number of applications continues to grow and now includes tennis rackets and other sports equipment.

13.6 NONLINEAR VIBRATION

Nonlinear model response differs significantly from that of linear models in several ways. Here we give a brief overview of these differences. The most direct way of dealing with a nonlinear model is to linearize it about a reference equilibrium. If the motion does not take the system too far from the equilibrium, then the linearized analysis can provide useful results, especially an estimate of the frequency of vibration. As we will see, the natural frequency of a nonlinear model depends on the specific equilibrium.

LINEARIZATION AND SPRING CONSTANTS

Figure 13.6.1 Generic representation of a mass-spring system.

Elastic elements that are extended or compressed beyond their linear range are often described by the following nonlinear model (see Figure 13.6.1).

$$m\ddot{y} = -f(y) + mg = -(k_1 y + k_2 y^3) + mg \qquad (13.6.1)$$

where $f(y) = k_1 y + k_2 y^3$ is the spring force and y is the spring's displacement from its free length. When the mass is in equilibrium at the position $y = y_r$, then $\ddot{y}_r = 0$, and the previous equation becomes

$$k_1 y_r + k_2 y_r^3 - mg = 0 \qquad (13.6.2)$$

This can be solved for y_r.

Evaluating (13.6.1) at the equilibrium point gives

$$m\ddot{y}_r = 0 = -f(y_r) + mg \qquad (13.6.3)$$

To linearize (13.6.1), we expand $f(y)$ in a Taylor series, keeping only the linear term, and subtract (13.6.3) from (13.6.1) to obtain

$$m(\ddot{y} - 0) = f(y_r) - f(y)$$

$$= f(y_r) - \left[f(y_r) + \left(\frac{df}{dy}\right)_r (y - y_r) \right]$$

$$= -\left(\frac{df}{dy}\right)_r (y - y_r)$$

Let $x = y - y_r$, where x denotes the displacement of the mass from its equilibrium position. Substituting $x = y - y_r$ and $\ddot{x} = \ddot{y} - \ddot{y}_r = \ddot{y}$ yields

$$m\ddot{x} = -\left(\frac{df}{dy}\right)_r x \tag{13.6.4}$$

For the *linear* spring relation $f = kx$, the equation of motion is $m\ddot{x} = -kx$. Comparing this with (13.6.4) we see that

$$k = \left(\frac{df}{dy}\right)_r \tag{13.6.5}$$

The important result of this analysis is that *the value of the linearized spring constant k is the slope of the spring force curve $f(y)$ at the equilibrium position y_r.*

Linearizing a Nonlinear Spring Model | **EXAMPLE 13.6.1**

■ **Problem**

Suppose a particular nonlinear spring is described by (13.6.1) with $k_1 = 1400$ N/m and $k_2 = 13000$ N/m^3. Determine its equivalent linear spring constant k for the two equilibrium positions corresponding to the following masses:

a. $m = 4$ kg
b. $m = 14$ kg

In addition, find the linearized equations for motion for each equilibrium.

■ **Solution**

The first step is to determine the equilibrium positions for each mass. With $m = 4$ in (13.6.2), the only real root of this cubic equation is $y_r = 0.0278$ m. With $m = 14$ in (13.6.2), the only real root is $y_r = 0.091$ m. Note that $f(y_r) = mg$, and that

$$k = \left(\frac{df}{dy}\right)_r = k_1 + 3k_2 y_r^2 = 1400 + 39\,000 y_r^2$$

The equivalent linear spring constant k is the slope m of the curve $f(y)$ near the specific equilibrium point. Thus, the linearized spring constants for each equilibrium are:

a. For $y_r = 0.0278$, $k = 1430$ N/m
b. For $y_r = 0.091$, $k = 1723$ N/m

The linearized equations of motion, which are accurate only near their respective equilibrium points, are found from $m\ddot{x} + kx = 0$. They are

a. For $y_r = 0.0278$, $x = y - 0.0278$, $4\ddot{x} + 1430x = 0$, and $\omega_n = 18.91$ rad/s.
b. For $y_r = 0.091$, $x = y - 0.091$, $14\ddot{x} + 1723x = 0$, and $\omega_n = 11.09$ rad/s.

Note that in each case, the mass oscillates about the specific equilibrium position. The oscillation frequency is different for each equilibrium.

In addition to linearized models, some mathematical results are available for approximate solutions to nonlinear equations of motion, but coverage of the methods used to obtain these results is beyond the scope of our study. The following examples serve as a guide to what to look for when using computer solution methods.

A STATIC MODEL

First consider the simple, static model

$$x(t) = f^2(t) = [\sin \omega_1 t + \sin \omega_2 t]^2$$

where $x(t)$ is the response and $f(t)$ is the forcing function. Expanding the right-hand side gives

$$x(t) = \sin^2 \omega_1 t + 2 \sin \omega_1 t \, \sin \omega_2 t + \sin^2 \omega_2 t$$
$$= 1 - \frac{1}{2} \cos 2\omega_1 t + \cos(\omega_1 - \omega_2)t - \cos(\omega_1 + \omega_2)t - \frac{1}{2} \cos 2\omega_2 t$$

Whereas the input contains only two frequencies, the response contains four frequencies, all different than the input frequencies. Also, the response is not a linear combination of the two input terms. In general, the principle of superposition does not apply to nonlinear models.

A DYNAMIC MODEL

One consequence of the superposition property of linear dynamic models is that their solutions consist of the sum of two terms: the free response, which is that part attributable to the initial conditions, and the forced response, which is due to the forcing function or input. The entire solution is called the complete or total response. This property enables us to find the solution for the free response separately from the solution for the forced response. The free response is independent of the forced response, and vice versa. If we need to find the total response, we simply add the free and the forced responses.

Another consequence of the superposition property is that the responses due to more than one input can be separated.

Few nonlinear models have closed-form solutions, but we can illustrate some of their properties by considering the following first-order model. The equation of motion of a mass m subjected to a square-law damping force and a constant force f is $m\dot{v} = f - qv^2$, where v is the velocity of the mass and q is a constant. Let the initial speed $v(0)$ at time $t = 0$ be denoted v_0.

Separation of variables can be used to obtain the following solution where $\alpha = \sqrt{f/q}$.

$$v(t) = \alpha \frac{a + be^{-ct}}{a - be^{-ct}} \tag{13.6.6}$$

where $a = v_0\sqrt{q} + \sqrt{f}$, $b = v_0\sqrt{q} - \sqrt{f}$, and $c = 2\alpha q/m$. This solution is valid if $v_0^2 \neq \alpha^2$ and if $f \neq 0$. Now consider these special cases:

1. If $v_0^2 = \alpha^2 = f/q$, then we can show that $v(t)$ remains constant at v_0.

2. If $v_0 = 0$, the solution reduces to

$$v(t) = \alpha \frac{1 - e^{-ct}}{1 + e^{-ct}} \tag{13.6.7}$$

3. If $f = 0$ the solution can be found using separation of variables as follows.

$$v(t) = \frac{m v_0}{m + q v_0 t} \tag{13.6.8}$$

Note that the form of (13.6.8) is different than that of the general case (13.6.6), where $f \neq 0$. Note also that we *cannot* obtain the total response (13.6.6) for $f \neq 0$, $v_0 \neq 0$ by adding the forced response (13.6.7) (where $f \neq 0$ and $v_0 = 0$) to the free response (13.6.8) (where $f = 0$ and $v_0 \neq 0$). We usually cannot obtain the total response of a nonlinear differential equation by superimposing the free and the forced responses.

LINEARIZED STABILITY ANALYSIS OF DUFFING'S EQUATION

A common model of a nonlinear spring is

$$f_s(y) = k_1 y + k_3 y^3 \tag{13.6.9}$$

which represents a hardening spring if $k_3 > 0$ and a softening spring if $k_3 < 0$. For a mass-spring-damper system with such a spring and an applied force $f(t)$, the equation of motion is

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = f(t) \tag{13.6.10}$$

When $f(t)$ is a cosine function, this equation is called *Duffing's equation*. Its characteristics have been extensively studied, but a closed-form solution is not available (for details, consult [Harris, 2002] or [Greenberg, 1998]).

If f is a constant, f_o, there can possibly be three equilibrium positions for the mass m, which are the solutions of the equation

$$k_1 y_e + k_3 y_e^3 = f_o \tag{13.6.11}$$

where y_e denotes an equilibrium solution of this equation. We can linearize the equation of motion about such an equilibrium by linearizing the cubic term y^3 using a truncated Taylor series of y^3 about the point y_e, as follows.

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 \left[y_e^3 + \frac{\partial y^3}{\partial y}\bigg|_{y=y_e} (y - y_e) \right] = f(t) = f_o + u(t)$$

where $u(t) = f(t) - f_o$ is the deviation in $f(t)$ from its reference value f_o. The equation of motion becomes

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 \left[y_e^3 + 3y_e^2(y - y_e) \right] = f_o + u(t)$$

If we define x to the deviation of y from equilibrium, $x = y - y_e$, then $\dot{x} = \dot{y}$, $\ddot{x} = \ddot{y}$, and

$$m\ddot{x} + c\dot{x} + k_1(x + y_e) + k_3 \left[y_e^3 + 3y_e^2 x \right] = f_o + u(t)$$

In light of (13.6.11), this becomes

$$m\ddot{x} + c\dot{x} + \left(k_1 + 3k_3 y_e^2 \right) x = u(t)$$

The characteristic equation is

$$ms^2 + cs + k_1 + 3k_3 y_e^2 = 0 \qquad (13.6.12)$$

The equilibrium will be stable in a linearized sense (that is, *locally stable*) if both roots have negative real parts. This occurs if m, c, and $k_1 + 3k_3 y_e^2$ all have the same sign. Assuming, of course that $m > 0$ and $c > 0$, local stability therefore requires that $k_1 + k_3 y_e^2 > 0$. Because $y_e^2 > 0$, we see that the equilibrium of a hardening spring is locally stable, while that of a softening spring can be either locally unstable or locally stable, depending of the values of k_1 and k_3.

Since the stability analysis was based on a linearized model, which assumes that y is "close" to y_e, its conclusions are valid only for local stability characteristics. It is possible that the system with either a hardening or a softening spring may be globally stable or globally unstable.

For example, consider the case where $f_o = 0$. Then the equilibria are found from (13.6.11) to be $y_e = 0$ and $y_e = \pm\sqrt{-k_1/k_3}$. Assuming that $k_1 > 0$,

- The only real-valued equilibrium is $y_e = 0$ if $k_3 > 0$.
- If $k_3 < 0$ there are three real-valued equilibria.
- For $y_e = 0$, (13.6.12) becomes $ms^2 + cs + k_1 = 0$, which represents a locally stable equilibrium regardless of the sign of k_3.
- For $y_e^2 = -k_1/k_3$, (13.6.12) becomes $ms^2 + cs - 2k_1 = 0$, which represents a locally unstable equilibrium regardless of the sign of k_3.

Consider the case where $f_o = 0$, $m = 1$, $c = 2$, and $k_1 = 2$. We choose this overdamped case so that the resulting plots will be easier to understand compared to plots showing many oscillations.

1. Suppose that $k_3 = -1$. The equilibria are $y_e = 0$, $y_e = +\sqrt{2}$, and $y_e = -\sqrt{2}$. Figure 13.6.2 shows the numerical solution of the nonlinear model (13.6.10). For all six cases, $\dot{y}(0) = 0$. Curves a and b result from the initial conditions $y(0) = 1$ and $y(0) = -1$, respectively. They both approach the equilibrium at $y_e = 0$, thus

Figure 13.6.2 Free response of Duffing's equation for various initial conditions.
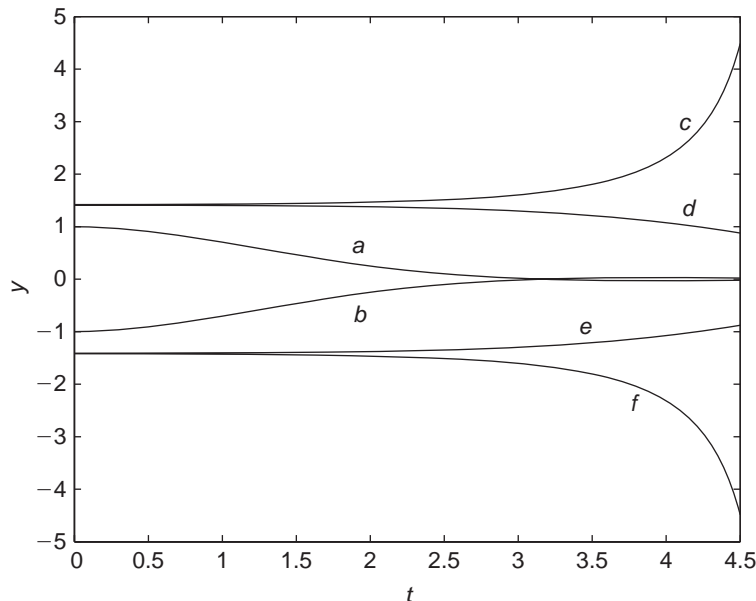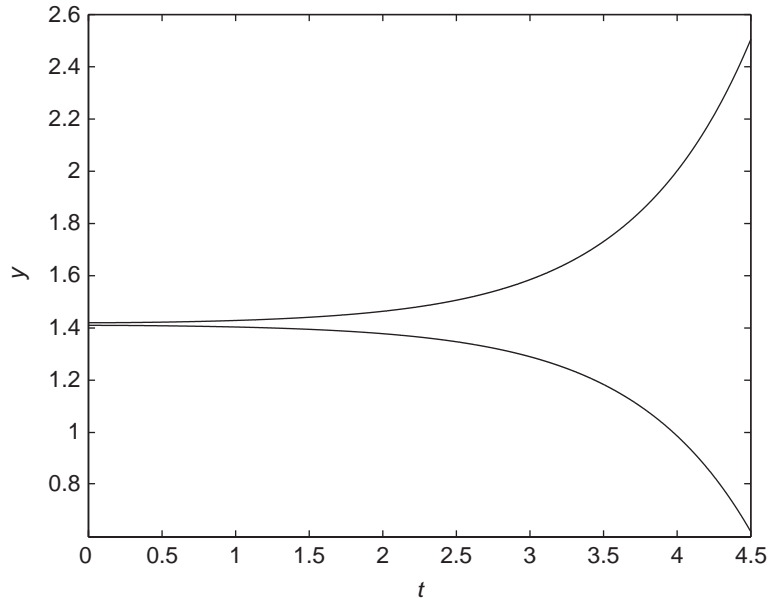
verifying the stability prediction based on the linearized equation. Curves c and d result from the initial conditions $y(0) = 1.42$ and $y(0) = 1.41$, respectively, which are slightly above and slightly below the equilibrium at $\sqrt{2} = 1.414$. Curve d approaches the equilibrium at $y_e = 0$, but curve c approaches ∞ and thus displays unstable behavior. These results show how an equilibrium of a nonlinear model can be stable for some initial conditions, but unstable for others. Similar results are observed with the equilibrium at $-\sqrt{2}$.

2. Figure 13.6.3 shows the responses of the model linearized about $y_e = \sqrt{2}$, for the initial conditions $y(0) = 1.42$ and $y(0) = 1.41$, respectively. Both responses display unstable behavior, as predicted by the linearized model. The nonlinear model, however, is stable for $y(0) = 1.41$, as seen with curve d in Figure 13.6.2. This result shows how the linearized model can give erroneous results if y is not close to y_e.

FREE RESPONSE OF A NONLINEAR SYSTEM

Suppose that $f(t) = 0$ in (13.6.10) and that $m = 1$, $c = 0$, $k_1 = 2$, and $k_3 = 0.1$. Figure 13.6.4 shows the numerical solutions for $\dot{y}(0) = 0$ and two values of $y(0)$: $y(0) = 1$ and $y(0) = 30$. Although it is difficult to tell from the plots, neither response is a pure harmonic. Each response, however, contains a single harmonic that dominates the response. For $y(0) = 1$, the dominant radian frequency is approximately 1.8. For $y(0) = 30$, the dominant radian frequency is approximately 8. We thus conclude that it is possible for the oscillation frequency of the free response of a nonlinear system to be dependent on the specific initial conditions. This result stands in great contrast to the free response of linear systems, whose oscillation frequency depends only on the values of the system parameters and not on the initial conditions.

HARMONIC RESPONSE OF DUFFING'S EQUATION

Suppose that $f(t) = F_o \cos \omega t$ in (13.6.10) and that $m = 1$, $c = 0$, $k_1 = 2$, and $k_3 = 0.1$. Figure 13.6.5 shows the response for $y(0) = \dot{y}(0) = 0$ and $\omega = 2$. Although

Figure 13.6.4 Free response of Duffing's equation for two initial conditions.
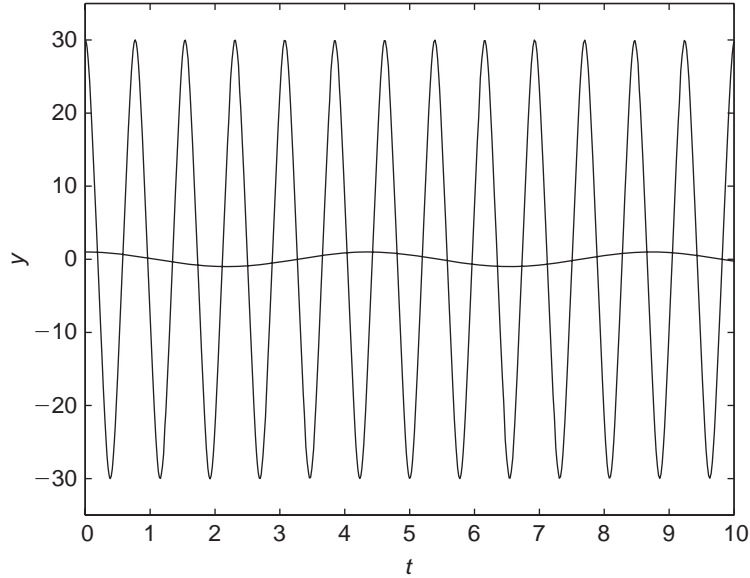
Figure 13.6.5 Harmonic response of Duffing's equation.
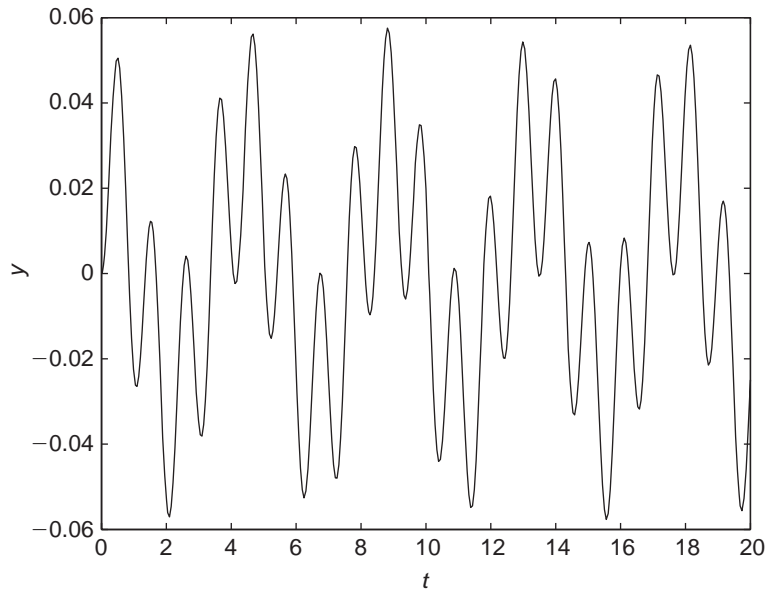
it is difficult to tell from the plot, the response contains two dominant harmonics, with radian frequencies of approximately 1.6 and 6.4. We thus conclude that it is possible for the oscillation frequency of the forced harmonic response of a nonlinear system to contain multiple frequencies that are different than the forcing frequency. We should have expected this after studying the response of the model $x(t) = f^2(t)$. Recall that the harmonic response of a linear system has only one frequency, which is the same as that of the forcing function.

Duffing's equation has other unusual response characteristics, including something called the *jump phenomenon*, in which the response frequency suddenly jumps from one value to another if the forcing frequency gradually changes. This is discussed in the specialized literature; for example, see [Harris, 2002] or [Greenberg, 1998]).

PHASE PLANE PLOTS AND LIMIT CYCLES

The *phase plane plot* is a plot of one state variable, usually velocity, versus another variable, usually displacement. Such a plot provides a concise description of the response. Figure 13.6.6 shows the phase plane plots of the free response of two second-order models of the form $m\ddot{x} + c\dot{x} + kx = 0$, one with medium damping and one with light damping, using the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. The greater the damping, the faster the response spirals into the equilibrium point at $x = \dot{x} = 0$. The phase plane plot of the free response of an undamped system never reaches the point (0, 0), but is a single ellipse passing through the initial condition point. The phase plane plot of the free response of an overdamped system does not spiral but heads directly for the point (0, 0).

The phase plane plot of some nonlinear models displays a *limit cycle*, which closed curve that represents a dynamic equilibrium. Limit cycles can be stable or unstable, and they can be stable for initial conditions within the closed curve and unstable for initial conditions outside the curve, or vice versa. *Van der Pol's equation* is a famous

Figure 13.6.6 Phase plane plots of a second order model with medium damping and light damping.
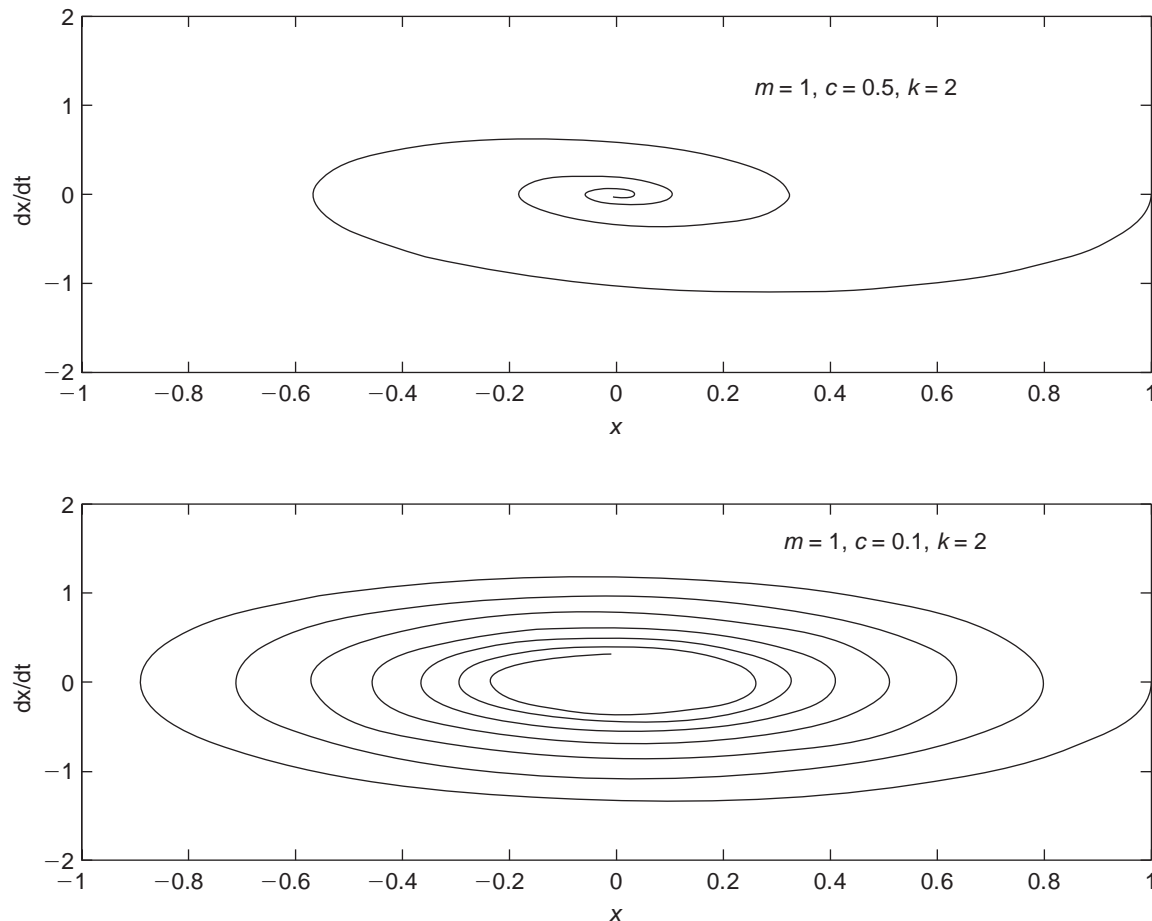
Figure 13.6.7 Limit cycle behavior of Van der Pol's equation for $\mu = 1$ and two initial conditions.
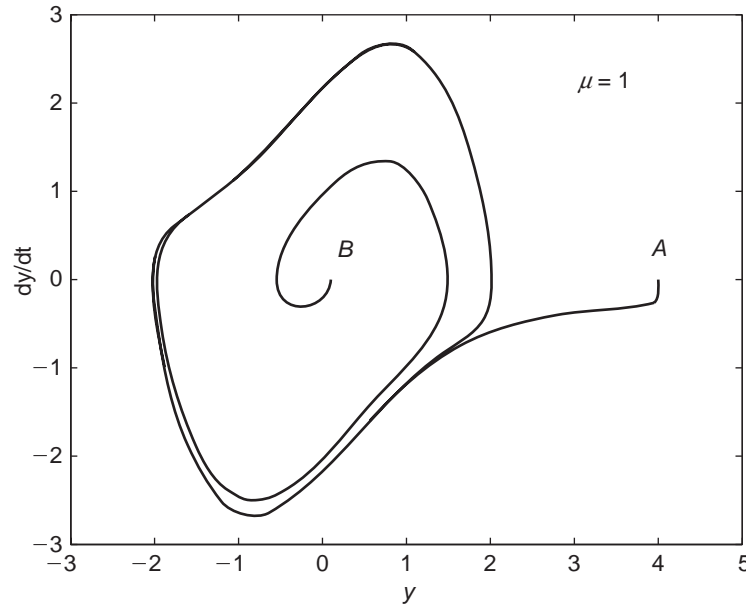
equation in vibration that displays limit cycles.

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0 \qquad \mu > 0 \qquad (13.6.13)$$

Figure 13.6.7 shows a limit cycle for Van der Pol's equation with $\mu = 1$. The response is shown for two sets of initial conditions, one inside and one outside the limit cycle. These are marked *A* and *B* in the figure. In each case the response approaches the limit cycle, which indicates, but does not prove, that the limit cycle is *bilaterally* stable. Analysis of limit cycles is an advanced topic, but you should be aware of their existence. One of the classic works on the subject of nonlinear vibration is [Minorsky, 1962].

SUMMARY OF NONLINEAR RESPONSE CHARACTERISTICS

In general, nonlinear models display the following phenomena.

1. Because the principle of superposition does not apply to nonlinear models, we cannot separate the effects of the initial conditions from the forced response, and we cannot decompose the forced response into a sum of terms, each corresponding to a term in the forcing function.

2. In a linear model, response characteristics such as stability, natural frequency, and logarithmic decrement depend only on the numerical values of the mass, damping, and stiffness parameters. This is not true with nonlinear models, for which the response characteristics depend also on the initial conditions.

3. Nonlinear models can resonate at frequencies that are different than the forcing frequency.

4. Linear models can have only one equilibrium point, and there is no distinction between local and global stability for such a point. Nonlinear models can have multiple equilibria, each of which can be a) globally stable, b) globally unstable, c) locally stable but globally unstable, or d) locally unstable but globally stable.

5. Nonlinear models can have limit cycles, but linear models cannot.

13.7 MATLAB APPLICATIONS

Although the root locus plot has been applied mostly to control system design, it is a very useful but unappreciated method for vibration analysis.

| Root Locus of a Suspension Model | **EXAMPLE 13.7.1** |

■ Problem

Consider the two-mass suspension model developed in Example 4.5.9 in Chapter 4, and shown again in Figure 13.7.1. The equations of motion are

Figure 13.7.1 Two-mass suspension model.

$$m_1\ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) + k_2(y - x_2)$$

We will use the following numerical values: $m_1 = 250$ kg, $m_2 = 40$ kg, $k_1 = 1.5 \times 10^4$ N/m, and $k_2 = 1.5 \times 10^5$ N/m.

a. Use the root locus plot to determine the value of the damping c_1 required to give a dominant root pair having a damping ratio of $\zeta = 0.707$.
b. Using the value of c_1 found in part (a), obtain a plot of the unit-step response.

■ Solution

a. Transforming the equations of motion using zero initial conditions and rearranging gives

$$\left(m_1s^2 + c_1s + k_1\right)X_1(s) - (c_1s + k_1)X_2(s) = 0$$

$$-(c_1s + k_1)X_1(s) + \left(m_2s^2 + c_1s + k_1 + k_2\right)X_2(s) = k_2Y(s)$$

We can find the characteristic polynomial and the transfer functions by using the determinant method.

$$D_3(s) = \begin{vmatrix} \left(m_1s^2 + c_1s + k_1\right) & -(c_1s + k_1) \\ -(c_1s + k_1) & \left(m_2s^2 + c_1s + k_1 + k_2\right) \end{vmatrix}$$

$$= \left(m_1s^2 + c_1s + k_1\right)\left(m_2s^2 + c_1s + k_1 + k_2\right) - \left(c_1s + k_1\right)^2$$

The transfer functions for $X_1(s)$ and $X_2(s)$ with $Y(s)$ as the input can be expressed as

$$\frac{X_1(s)}{Y(s)} = \frac{D_1(s)}{D_3(s)}$$

$$\frac{X_2(s)}{Y(s)} = \frac{D_2(s)}{D_3(s)}$$

where

$$D_1(s) = \begin{vmatrix} 0 & -(c_1s + k_1) \\ k_2 & \left(m_2s^2 + c_1s + k_1 + k_2\right) \end{vmatrix}$$

$$= k_2(c_1s + k_1)$$

$$D_2(s) = \begin{vmatrix} \left(m_1s^2 + c_1s + k_1\right) & 0 \\ -(c_1s + k_1) & k_2 \end{vmatrix}$$

$$= k_2\left(m_1s^2 + c_1s + k_1\right)$$

Now substitute the given values of m_1, m_2, k_1, and k_2 into $D(s)$ to obtain the root locus equation.

$$D_3(s) = \left(250s^2 + c_1 s + 15\,000\right)\left(40s^2 + c_1 s + 165\,000\right) - (c_1 s + 15\,000)^2$$

Factoring out c_1 and rearranging into the standard root-locus form (13.7.1) so that the highest coefficients of $N(s)$ and $D(s)$ are unity, we obtain

$$s^4 + 4185s^2 + 2.25 \times 10^5 + K\left(s^3 + 775.86s\right) = 0$$

where the root locus parameter K is related to c_1 as $K = 0.029c_1$. The MATLAB session for obtaining the root locus plot is as follows:

```
≫sys4 = tf([1, 0, 775.86,0],[1,0,4185,0,2.25*10^5]);
≫rlocus(sys4),axis equal,sgrid(0.707,[])
```

The root locus plot is shown in Figure 13.7.2.

Using the magnifying glass tool to expand the plot, and clicking on the intersection of the suspected dominant root path with the $\zeta = 0.707$ line, we find that the dominant root pair is $s = -5.53 \pm 5.53j$, with a gain of $K = 55.6$. Using the command `r = rlocus(sys4,K)`, we find that the other roots are at $s = -22.2 \pm 56j$, and thus the first root pair is dominant. So to achieve a dominant root pair having a damping ratio of $\zeta = 0.707$, we must set c_1 to be $c_1 = K/0.029 = 1917\,\text{N} \cdot \text{s/m}$.

b.   Using $c = 1917$ and the other parameter values, we find that

$$D_3(s) = 250(40)\left[s^4 + 4185s^2 + 2.25 \times 10^5 + K\left(s^3 + 775.86s\right)\right]$$

$$\frac{X_1(s)}{Y(s)} = \frac{1.5 \times 10^5 \left(1917s + 1.5 \times 10^4\right)}{D_3(s)}$$

$$\frac{X_2(s)}{Y(s)} = \frac{1.5 \times 10^5 \left(250s^2 + 1917s + 1.5 \times 10^4\right)}{D_3(s)}$$

Figure 13.7.2 Root locus plot for Example 13.7.1.
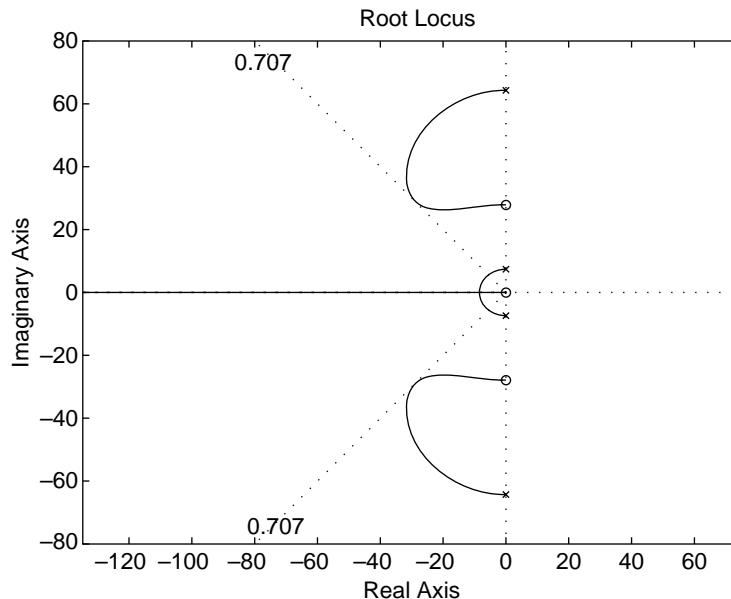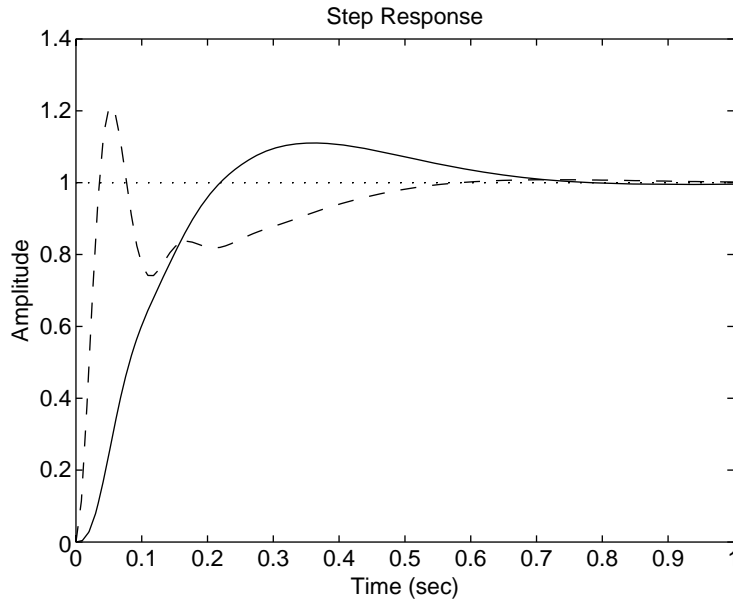
Figure 13.7.3 Step response for Example 13.7.1.

We continue the previous session as follows, using $K = 55.6$.

```
≫[num, den] = tfdata(sys4,'v');
≫D3 = 250*40*(den+55.6*num);
≫sysx1 = tf(150000*[1917,15000],D3);
≫sysx2 = tf(150000*[250,1917,15000],D3);
≫step(sysx1,'-',sysx2,'--')
```

The plot is shown in Figure 13.7.3. By right clicking on the plot and selecting the "Characteristics" menu, we find that for $x_1(t)$, the peak response is 11% at $t = 0.357$ s and the settling time is 0.655 s. For $x_2(t)$, the peak response is 21% at $t = 0.051$ s and the settling time is 0.494 s.

The response characteristics computed from the dominant root pair $s = -5.53 \pm 5.53j$ using the formulas given in Table 9.3.2 for the second-order model without numerator dynamics are a peak response of 4.3% at $t = 0.568$ s and a settling time of 0.723 s. The difference between these results is due to the fact that the present model is fourth order and also has numerator dynamics.

13.8 CHAPTER REVIEW

This chapter treated four important vibration applications of transfer functions and frequency response. A common cause of vibration, base excitation, is due to motion of the supporting structure. Rotating unbalance, which occurs when the center of mass of a rotating machine does not coincide with the center of rotation, is another common vibration source. The chapter showed how to analyze and design vibration isolators for minimizing the transmission of force to a supporting structure from a machine having rotating unbalance. Reduction of vibration amplitude can also be accomplished with a dynamic vibration absorber, which consists of an attached stiffness and mass whose motion "absorbs" the vibration.

The free response of linear multi-degree of freedom systems is the linear combination of the modes. The extent to which a particular mode appears in the free response

depends on the initial conditions. In addition, certain forcing functions can excite a particular mode. The mode concept sometimes enables us to simplify the model so that we can use it for design purposes. This was illustrated for vehicle suspension design.

Passive isolators and absorbers are being replaced by active vibration control systems in many applications. Designing such systems requires knowledge of control theory.

Analysis methods for nonlinear models are limited, and such models often must be analyzed by numerical simulation. Nonlinear response differs quite a bit from that of linear models, and it is important to recognize these differences.

Now that you have finished this chapter, you should be able to

1. Analyze the performance of spring and damping elements for isolating a mass from base excitation.
2. Analyze the performance of spring and damping elements for minimizing the force transmission due to rotating unbalance.
3. Design a vibration absorber.
4. Obtain a description of a multi-degree of freedom system in terms of its modes.
5. Select the spring and damping constants for a basic vehicle suspension.
6. Analyze the effectiveness of active vibration control systems.
7. Describe the differences between linear and nonlinear responses.

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PROBLEMS

Section 13.1 Base Excitation

13.1 The 0.5-kg mass shown in Figure P13.1 is attached to the frame with a spring of stiffness $k = 500$ N/m. Neglect the spring weight and any damping. The frame oscillates vertically with an amplitude of 4 mm at a frequency of 3 Hz. Compute the steady-state amplitude of motion of the mass.

13.2 A quarter-car representation of a certain car has a stiffness $k = 2000$ lb/ft, which is the series combination of the tire stiffness and suspension stiffness, and a damping constant of $c = 360$ lb-sec/ft. The car weighs 2000 lb. Suppose the road profile is given (in feet) by $y(t) = 0.03 \sin \omega t$, where the amplitude

Figure P13.1
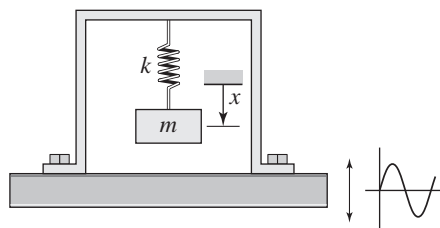
of variation of the road surface is 0.03 ft, and the frequency ω depends on the vehicle's speed and the road profile's period. Suppose the period of the road surface is 20 ft. Compute the steady-state motion amplitude and the force transmitted to the chassis, if the car is traveling at speeds of 20 and 50 mi/hr.

13.3 A certain factory contains a heavy rotating machine that causes the factory floor to vibrate. We want to operate another piece of equipment nearby and we measure the amplitude of the floor's motion at that point to be 0.01 m. The mass of the equipment is 1500 kg and its support has a stiffness of $k = 2 \times 10^4$ N/m and a damping ratio of $\zeta = 0.04$. Calculate the maximum force that will be transmitted to the equipment at resonance.

13.4 An electronics module inside an aircraft must be mounted on an elastic pad to protect it from vibration of the airframe. The largest amplitude vibration produced by the airframe's motion has a frequency of 40 Hz. The module weighs 200 N, and its amplitude of motion is limited to 0.003 m to save space. Neglect damping and calculate the percent of the airframe's motion transmitted to the module.

13.5 An electronics module used to control a large crane must be isolated from the crane's motion. The module weighs 2 lb. (a) Design an isolator so that no more than 10% of the crane's motion amplitude is transmitted to the module. The crane's vibration frequency is 3000 rpm. (b) What percentage of the crane's motion will be transmitted to the module if the crane's frequency can be anywhere between 2500 and 3500 rpm?

13.6 An instrument weighing 5 lb is mounted on the housing of a pump that rotates at 30 rpm. The amplitude of motion of the housing is 0.003 ft. We want no more than 10% of the housing's motion to be transmitted to the instrument. Design a suitable isolator having negligible damping. Compute the force transmitted to the instrument.

13.7 Figure P13.7 shows a system being driven by base excitation through a damping element. Assume that the base displacement is sinusoidal: $y(t) = Y \sin \omega t$.

   a. Derive the expression for X, the steady-state amplitude of motion of the mass m.

   b. Derive the expression for F_t, the steady-state amplitude of the force transmitted to the support.

Figure P13.7

13.8 Consider the vehicle suspension problem in Example 13.1.1. Investigate the choice of the damping constant c so that the displacement transmissibility X/Y is as small as possible for the case where the speed gives a frequency ratio of $r = 2$. Do this by trying different values for the damping ratio ζ. Investigate your design's sensitivity to a $\pm 20\%$ variation in r.

13.9 Consider the vehicle suspension problem in Example 13.1.1. Suppose the amplitude of variation of the road surface is 5/8 in. Determine a set of values for the suspension's stiffness k and damping c so that the force transmitted to the chassis will be as small as possible at 40 mi/hr. For your design, calculate the transmitted force at 40 mi/hr.
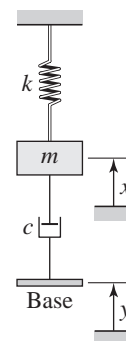
Section 13.2 Rotating Unbalance

13.10 Alternating-current motors are often designed to run at a constant speed, typically either 1750 or 3500 rpm. One such motor for a power tool weighs 20 lb and is to be mounted at the end of a steel cantilever beam. Static-force

calculations and space considerations suggest that a beam 6 in. long, 4 in. wide, and 3/8 in. thick would be suitable. Use $E = 3 \times 10^7$ psi for the modulus of elasticity of steel. Use $\rho = 15.2$ slug/ft^3 for the density of steel. The rotating part of the motor weighs 1 lb and has an eccentricity of 0.01 ft. (This distance can be determined by standard balancing methods or applying the analysis in this example in reverse, using a vibration test.) The damping ratio for such beams is difficult to determine but is usually very small, say, $\zeta \leq 0.1$. Estimate the amplitude of vibration of the beam at steady state. Consider both speeds.

13.11 When a certain motor is started, it is noticed that its supporting frame begins to resonate when the motor speed passes through 900 rpm. At the operating speed of 1750 rpm the support oscillates with an amplitude of 8 mm. Determine the amplitude that would result at 1750 rpm if the support were replaced with one having one-half the stiffness.

13.12 A 500-lb motor is supported by an elastic pad that deflects 0.25 in. when the motor is placed on it. When the motor operates at 1750 rpm, it oscillates with an amplitude of 0.1 in. Suppose a 1500-lb platform is placed between the motor and the pad. Compute the oscillation amplitude that would result at 1750 rpm.

13.13 A certain pump weighs 50 lb and has a rotating unbalance. The unbalanced weight is 0.05 lb and has an eccentricity of 0.1 in. The pump rotates at 1000 rpm. Its vibration isolator has a stiffness of $k = 500$ lb/ft. Compute the force transmitted to the foundation if the isolator's damping ratio is (a) $\zeta = 0.05$ and (b) $\zeta = 0.7$.

13.14 To calculate the effects of rotating unbalance, we need to know the value of the product $m_u \epsilon$, where m_u is the unbalanced mass and ϵ is the eccentricity. These two quantities are sometimes difficult to calculate separately, but sometimes an experiment can be performed to estimate the product $m_u \epsilon$. An experiment was performed on a particular rotating machine whose mass is 75 kg. The machine's support has negligible damping and a stiffness of $k = 2500$ N/m. When the machine operates at 200 rpm, the measured force transmitted to the foundation was 15 N. Estimate the value of $m_u \epsilon$.

13.15 A computer disk drive is mounted to the computer's chassis with an isolator consisting of an elastic pad. The disk drive motor weighs 3 kg and runs at 3000 rpm. Calculate the pad stiffness required to provide a 90% reduction in the force transmitted from the motor to the chassis.

13.16 Figure P13.16 shows a motor mounted on four springs (the second pair of springs is behind the front pair and is not visible). Each spring has a stiffness $k = 2000$ N/m. The distance D is 0.2 m. The inertia of the motor is $I = 0.2$ kg · m^2; its mass is $m = 25$ kg, and its speed is 1750 rpm. Because the motor
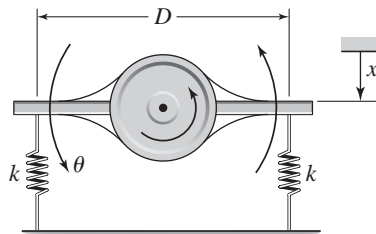
mounts allow the motor to vibrate both vertically and rotationally, the system has a vertical force transmissibility F_t/F and a torque transmissibility T_t/T, where F is the vertical unbalance force and T is the torque due to the unbalance.

Neglect damping in the system and obtain the vertical force transmissibility F_t/F and the torque transmissibility T_t/T.

Section 13.3 Vibration Absorbers

13.17 A motor mounted on a cantilever beam weighs 20 lb and runs at the constant speed of 3500 rpm. The steel beam is 6 in. long, 4 in. wide, and 3/8 in. thick. The unbalanced part of the motor weighs 1 lb and has an eccentricity of 0.01 ft. The damping in the beam is very slight. Design a vibration absorber for this system. The available clearance for the absorber's motion is 0.25 in. Use $E = 3 \times 10^7$ psi for the modulus of elasticity of steel. Use $\rho = 15.2$ slug/ft^3 for the density of steel.

13.18 A motor mounted on a beam vibrates too much when it runs at a speed of 6000 rpm. At that speed the measured force produced on the beam is 60 lb. Design a vibration absorber to attach to the beam. Because of space limitations, the absorber's mass cannot have an amplitude of motion greater than 0.08 in.

13.19 The supporting table of a radial saw weighs 160 lb. When the saw operates at 200 rpm it transmits a force of 4 lb to the table. Design a vibration absorber to be attached underneath the table. The absorber's mass cannot vibrate with an amplitude greater than 1 in.

13.20 A certain machine of mass 8 kg with supports has an experimentally determined natural frequency of 6 Hz. It will be subjected to a rotating unbalance force with an amplitude of 50 N and a frequency of 4 Hz.

    a.   Design a vibration absorber for this machine. The available clearance for the absorber's motion is 0.1 m.

    b.   Let x_1 be the vertical displacement of the machine. The amplitude of the rotating unbalance force is $m_u \epsilon \omega^2$. Plot $X_1/m_u \epsilon$ versus the frequency ω, and use the plot to discuss the sensitivity of the absorber to changes in the frequency ω.

13.21 The operating speed range of a certain motor is from 1500 to 3000 rpm. The motor and its mount vibrate excessively at 2100 rpm. When a vibration absorber weighing 5 lb and tuned to 2100 rpm was attached to the motor, resonance occurred at 1545 and 2850 rpm. Design a more effective absorber that will yield resonant frequencies outside the operating speed range of the motor.

13.22 Figure P13.22 shows another type of vibration absorber that uses only mass and damping, and not stiffness, to reduce vibration. The main mass is m_1 and the absorber's mass is m_2. Suppose the applied force $f(t)$ is sinusoidal.

    a.   Derive the expressions for $X_1(j\omega)/F(j\omega)$ and $X_2(j\omega)/F(j\omega)$.

    b.   Use these expressions to discuss the selection of values for m_2 and c to minimize the motion of mass m_1. To aid in your discussion, plot kX_1/F versus $\omega/\sqrt{k/m_1}$ for several values of $\zeta = c/2\sqrt{km_1}$.

13.23 Figure P13.23 shows another type of vibration absorber that uses mass, stiffness, and damping to reduce vibration. The damping can be used to reduce

Figure P13.22
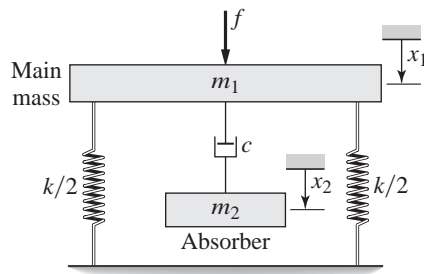
Figure P13.23
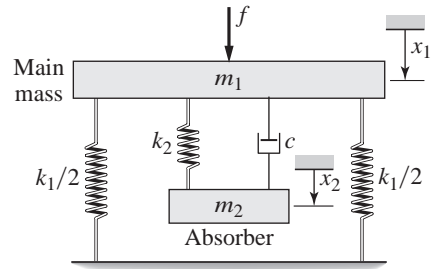
the amplitude of motion near resonance. The main mass is m_1 and the absorber's mass is m_2. Suppose the applied force $f(t)$ is sinusoidal.

a.   Derive the expressions for $X_1(j\omega)/F(j\omega)$ and $X_2(j\omega)/F(j\omega)$.

b.   Use these expressions to discuss the selection of values for m_2, k_2, and c to minimize the motion of mass m_1. To aid in your discussion, plot $k_1 X_1/F$ versus $\omega/\sqrt{k_1/m_1}$ for several values of $\zeta = c/2\sqrt{k_1 m_1}$.

Section 13.4 Modes of Vibrating Systems

13.24 Find and interpret the mode ratios for the system shown in Figure P13.24. The masses are $m_1 = 10$ kg and $m_2 = 30$ kg. The spring constants are $k_1 = 10^4$ N/m and $k_2 = 2 \times 10^4$ N/m.

13.25 Find and interpret the mode ratios for the coupled pendulum system shown in Figure P13.25. Use the values $m_1 = 1$, $m_2 = 4$, $L_1 = 2$, $L_2 = 5$, and $k = 2$.

13.26 Find and interpret the mode ratios for the torsional system shown in Figure P13.26. Use the values $I_1 = 1$, $I_2 = 5$, $k_1 = 1$, and $k_2 = 3$.

13.27 Find and interpret the mode ratios for the system shown in Figure P13.27.

13.28 For the roll-pitch vehicle model described in Example 13.4.2, the suspension stiffnesses are to be changed to $k_1 = 1.95 \times 10^4$ N/m and $k_2 = 2.3 \times 10^4$ N/m. Find the natural frequencies the mode ratios, and the node locations.

13.29 A particular road vehicle weighs 4000 lb. Using the quarter-car model, determine a suitable value for the suspension stiffness, assuming that the tire stiffness is 1300 lb/in.

Figure P13.24
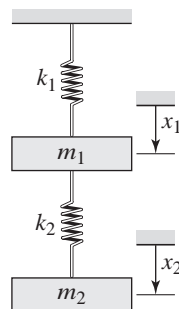
Figure P13.25
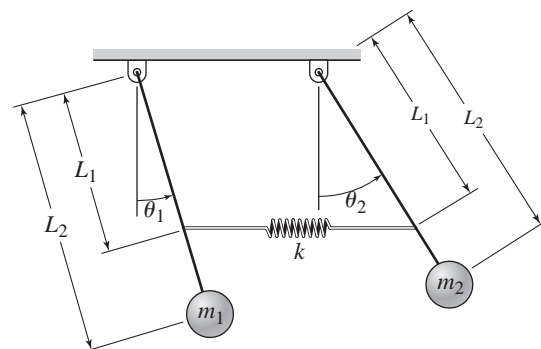
Figure P13.26
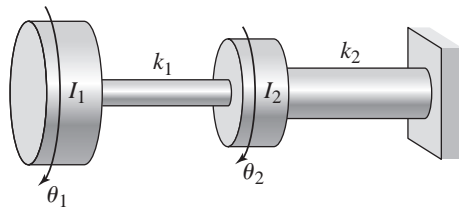
Figure P13.27
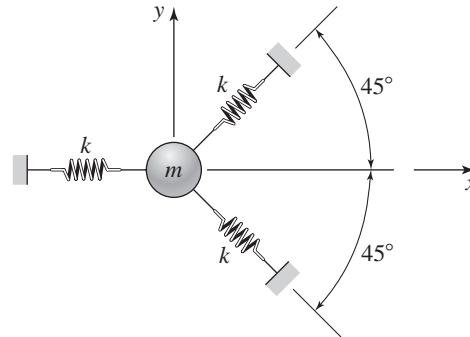
13.30 The vehicle model shown in Figure 13.4.2(a) has the following parameter values: weight $= 4800$ lb, $I_G = 1800$ slug-ft^2, $L_1 = 3.5$ ft, and $L_2 = 2.5$ ft. Design the front and rear suspension stiffnesses to achieve good ride quality.

Section 13.5 Active Vibration Control

13.31 A 125-kg machine has a passive isolation system for which $c = 5000$ N·m/s and $k = 7 \times 10^6$ N/m. The rotating unbalance force has an amplitude of 100 N with a frequency of 2500 rpm. The resonant frequency of this system is 216 rad/s and is close the frequency of the disturbance. In addition, the damping ratio is small ($\zeta = 0.08$). Assuming that c and k cannot be changed, calculate the control gains required to give a damping ratio of $\zeta = 0.5$ and a resonant frequency of 100 rad/s, well below the disturbance frequency.

13.32 A 20-kg machine has a passive isolation system whose damping ratio is 0.28 and whose undamped natural frequency is 13.2 rad/s. Assuming that the passive system remains in place, calculate the control gains required to give a damping ratio of $\zeta = 0.707$ and a resonant frequency of 141 rad/s.

13.33 With the increased availability of powered wheelchairs, improved suspension designs are required for safety and comfort. One chair uses an active suspension like the one shown in Figure P13.33 for each driving wheel.
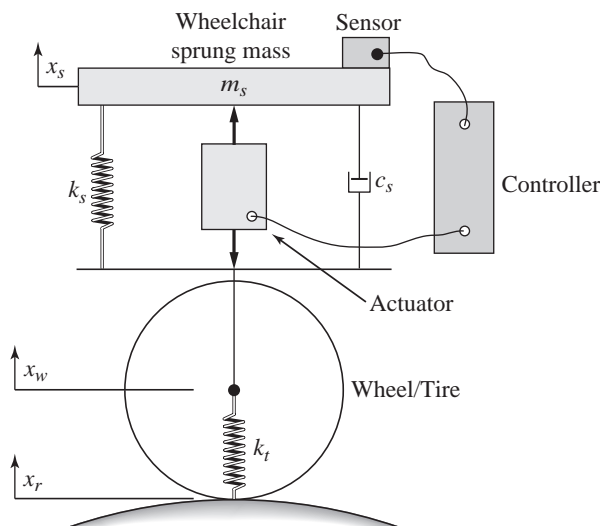
Figure P13.33

An actuator exerts a force f between the tire and the vehicle. Assuming that the control algorithm is such that

$$f = -K_P(x_s - x_w) - K_D(\dot{x}_s - \dot{x}_w)$$

derive the equations of motion of the system, neglecting the tire mass and tire damping.

Section 13.6 Nonlinear Vibration

13.34 The following model describes a mass supported by a nonlinear spring. The units are SI, so $g = 9.81$ m/s^2.

$$5\ddot{y} = 5g - \left(900y + 1700y^3\right)$$

   a. Find the equilibrium position y_r, obtain a linearized model using the equilibrium as the reference operating condition, and compute the oscillation frequency of the linearized model.

   b. Find the free response of the linearized model in terms of $x(t) = y(t) - y_r$, for the initial conditions: $x(0) = 0.002$ m and $\dot{x}(0) = 0.005$ m/s.

13.35 The following is a model of the velocity of an object subjected to cubic damping.

$$m\frac{dv}{dt} = -cv^3$$

Suppose that $m = 1$ and $c = 4$. Obtain the solution in terms of the initial condition $v(0)$.

13.36 Find the equilibria of equation (13.6.10) for $m = 1$, $c = 12$, $k_1 = 16$, and $k_3 = -4$, and use a numerical method to solve and plot the solution for $\dot{y}(0) = 0$ and four values of $y(0)$: ± 1, ± 1.9, and ± 2.1. Compare the results with the stability properties predicted from the linearized model.

13.37 Use a numerical method to compute and plot the free response of equation (13.6.10) for $m = 1$, $c = 0$, $k_1 = 2$, and $k_3 = 0.1$, for $\dot{y}(0) = 0$ and two initial conditions: $y(0) = 10$ and $y(0) = 40$. Compare the results with those shown in Figure 13.6.4. How does the initial condition affect the frequency of the response?

13.38 Plot the phase plane plots for the following equations and the initial conditions: $x(0) = 1$, $\dot{x}(0) = 0$.

   a. $\ddot{x} + 0.1\dot{x} + 2x = 0$

   b. $\ddot{x} + 2\dot{x} + 2x = 0$

   c. $\ddot{x} + 4\dot{x} + 2x = 0$

13.39 Plot the phase plane plot for the following equation with the initial conditions: $y(0) = 1$, $\dot{y}(0) = 0$.

$$\ddot{y} + 2\dot{y} + 2y + 3y^3 = 0$$

13.40 Plot the phase plane plot and identify the limit cycle for Van der Pol's equation (13.6.13) with $\mu = 5$ and the initial conditions: $y(0) = 1$, $\dot{y}(0) = 0$.

A
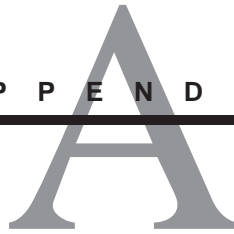
Guide to Selected MATLAB Commands and Functions

This appendix is a guide to those MATLAB commands and functions that are particularly useful for the system dynamics methods covered in this text. For more information, in the Command window type help topic, where topic is the name of the command or function. ■

Operators and special characters

Item	Description
+	Plus. Addition operator.
−	Minus. Subtraction operator.
*	Scalar and matrix multiplication operator.
.*	Array multiplication operator.
^	Scalar and matrix exponentiation operator.
.^	Array exponentiation operator.
\	Left division operator.
/	Right division operator.
.\	Array left division operator.
./	Array right division operator.
:	Colon. Generates regularly spaced elements and represents an entire row or column.
( )	Parentheses. Encloses function arguments and array indices; overrides precedence.
[ ]	Brackets. Encloses array elements.
{ }	Braces. Encloses cell elements.
.	Decimal point.
...	Ellipsis. Line continuation operator.
,	Comma. Separates statements and elements in a row of an array.
;	Semicolon. Separates columns in an array and suppresses line feeds.
%	Percent sign. Designates a comment and specifies formatting.
'	Quote sign and transpose operator.
.'	Nonconjugated transpose operator.
=	Assignment (replacement) operator.

Logical and relational operators.

Item	Description
==	Relational operator: equal to.
~=	Relational operator: not equal to.
<	Relational operator: less than.
<=	Relational operator: less than or equal to.
>	Relational operator: greater than.
>=	Relational operator: greater than or equal to.
&	Logical operator: AND
|	Logical operator: OR
~	Logical operator: NOT

Special variables and constants.

Item	Description
ans	Most recent answer.
eps	Accuracy of floating point precision.
i,j	The imaginary unit $\sqrt{-1}$.
Inf	Infinity.
NaN	Undefined numerical result (<u>N</u>ot <u>a</u> <u>N</u>umber).
pi	The number π.

Commands for managing a session.

Item	Description
clc	Clears Command window.
clear	Removes variables from memory.
doc	Displays documentation.
exist	Checks for existence of file or variable.
global	Declares variables to be global.
help	Searches for a help topic.
helpwin	Displays help text in the Help Browser.
lookfor	Searches help entries for a keyword.
quit	Stops MATLAB.
who	Lists current variables.
whos	Lists detailed information about current variables.

System and file commands.

Item	Description
cd	Changes current directory.
dir	Lists all files in current directory.
load	Loads workspace variables from a file.
path	Displays search path.
pwd	Displays current directory.
save	Saves workspace variables in a file.
type	Displays contents of a file.
what	Lists all MATLAB files.

Input/output commands.

Item	Description
disp	Displays contents of an array or string.
format	Controls screen display format.
input	Displays prompts and waits for input.
menu	Displays a menu of choices.
;	Suppresses screen printing.

Some numeric display formats.

Item	Description
format long	Sixteen decimal digits.
format long e	Scientific notation with sixteen digits plus exponent.
format short	Four decimal digits (the default).
format short e	Scientific notation with five digits plus exponent.

Array functions.

Item	Description
det	Computes determinant of an array.
eig	Computes the eigenvalues of a matrix.
eye	Creates the identity matrix.
find	Finds indices of nonzero elements.
length	Computes number of elements in an array.
linspace	Creates a regularly spaced array.
logspace	Creates a logarithmically spaced array.
max	Returns largest element in an array.
min	Returns smallest element in an array.
ones	Creates an array of ones.
size	Computes array size.
sort	Sorts each array column.
sum	Sums each array column.
zeros	Creates an array of zeros.

Exponential and logarithmic functions.

Item	Description
exp(x)	Exponential, e^x.
log(x)	Natural logarithm, $\ln x$.
log10(x)	Common (base 10) logarithm, $\log x = \log_{10} x$.
sqrt(x)	Square root, $\sqrt{x}$.

Complex functions.

Item	Description
abs(x)	Absolute value, $|x|$.
angle(x)	Angle of a complex number x.
conj(x)	Complex conjugate of x.
imag(x)	Imaginary part of a complex number x.
real(x)	Real part of a complex number x.

Numeric functions.

Item	Description
ceil	Rounds to the nearest integer toward ∞.
fix	Rounds to the nearest integer toward zero.
floor	Rounds to the nearest integer toward $-\infty$.
round	Rounds toward the nearest integer.
sign	Signum function.

Trigonometric functions.

Item	Description
acos(x)	Inverse cosine, $\arccos x = \cos^{-1} x$.
acot(x)	Inverse cotangent, $\text{arccot}\, x = \cot^{-1} x$.
acsc(x)	Inverse cosecant, $\text{arccsc}\, x = \csc^{-1} x$.
asec(x)	Inverse secant, $\text{arcsec}\, x = \sec^{-1} x$.
asin(x)	Inverse sine, $\arcsin x = \sin^{-1} x$.
atan(x)	Inverse tangent, $\arctan x = \tan^{-1} x$.
atan2(y,x)	Four quadrant inverse tangent.
cos(x)	Cosine, $\cos x$.
cot(x)	Cotangent, $\cot x$.
csc(x)	Cosecant, $\csc x$.
sec(x)	Secant, $\sec x$.
sin(x)	Sine, $\sin x$.
tan(x)	Tangent, $\tan x$.

Hyperbolic functions.

Item	Description
acosh(x)	Inverse hyperbolic cosine, $\cosh^{-1} x$.
acoth(x)	Inverse hyperbolic cotangent, $\coth^{-1} x$.
acsch(x)	Inverse hyperbolic cosecant, $\text{csch}^{-1} x$.
asech(x)	Inverse hyperbolic secant, $\text{sech}^{-1} x$.
asinh(x)	Inverse hyperbolic sine, $\sinh^{-1} x$.
atanh(x)	Inverse hyperbolic tangent, $\tanh^{-1} x$.
cosh(x)	Hyperbolic cosine, $\cosh x$.
coth(x)	Hyperbolic cotangent, $\cosh x / \sinh x$.
csch(x)	Hyperbolic cosecant, $1/\sinh x$.
sech(x)	Hyperbolic secant, $1/\cosh x$.
sinh(x)	Hyperbolic sine, $\sinh x$.
tanh(x)	Hyperbolic tangent, $\sinh x / \cosh x$.

Polynomial functions.

Item	Description
conv	Computes product of two polynomials.
deconv	Computes ratio of polynomials.
poly(r)	Computes coefficients of polynomial whose roots are given in the vector r.
poly(A)	Computes coefficients of the characteristic polynomial corresponding to the matrix A.
polyfit	Fits a polynomial to data.
polyval	Evaluates a polynomial at specified values of its independent variable.
residue	Computes residues, poles, and direct term of a partial fraction expansion.
roots	Computes polynomial roots.

Logical functions.

Item	Description
any	True if any elements are nonzero.
all	True if all elements are nonzero.
find	Finds indices of nonzero elements.
finite	True if elements are finite.
isnan	True if elements are undefined.
isinf	True if elements are infinite.
isempty	True if array is empty.
isreal	True if all elements are real.
logical	Converts numeric values to logical values.
xor	Exclusive or.

Miscellaneous mathematical functions.

Item	Description
cross	Cross product.
dot	Dot product.
fminbnd	Finds minimum of single-variable function.
fminsearch	Finds minimum of multivariable function.
function	Creates a user-defined function.
fzero	Finds zero of single-variable function.
mean	Calculates the mean value.
std	Calculates the standard deviation.
trapz	Numerical integration with the trapezoidal rule.

Two-dimensional plotting commands.

Item	Description
axes	Creates axes objects.
axis	Sets axis limits.
fplot	Intelligent plotting of functions.
ginput	Reads coordinates of cursor position.
grid	Displays gridlines.
gtext	Enables label placement with mouse.
hold off	Releases a prior hold on command.
hold on	Holds current graph to enable subsequent plotting.
legend	Enables legend placement with mouse.
loglog	Creates log-log plot.
plot	Generates xy plot.
polar	Creates polar plot.
print	Prints plot or saves plot to a file.
refresh	Redraws current figure window.
semilogx	Creates semilog plot (logarithmic abscissa).
semilogy	Creates semilog plot (logarithmic ordinate).
set	Specifies properties of objects, such as axes.
subplot	Creates plots in subwindows.
text	Places a string in a figure.
title	Puts text at top of plot.
xlabel	Adds text label to abscissa (the x axis).
ylabel	Adds text label to ordinate (the y axis).

Program flow control.

Item	Description
break	Terminates execution of a loop.
case	Provides alternate execution paths within switch structure.
continue	Passes control to the next iteration of a for or while loop.
else	Delineates alternate block of statements.
elseif	Conditionally executes statements.
end	Terminates for, while, and if statements.
for	Repeat statements a specific number of times.
if	Execute statements conditionally.
otherwise	Provides optional control within a switch structure.
pause	Causes the program to stop and wait for a key press before continuing.
switch	Directs program execution by comparing input with case expressions.
while	Repeats statements an indefinite number of times.

LTI model functions.

Item	Description
damp	Computes the characteristic roots, damping ratio, and damped oscillation frequency of complex roots.
ltimodels	Gives help about LTI models.
ltiprops	Gives help about LTI model properties.
ltiview	Interface for analyzing time and frequency response.
ord2	Creates a state-space or transfer function representation of a second-order system from its natural frequency and damping ratio.
ss	Creates an LTI model in state-space form.
ss2tf	Converts from state-space to transfer function form.
ss2zp	Converts from state-space to zero-pole form.
ssdata	Extracts state-space matrices from an LTI model.
tf	Creates an LTI model in transfer function form.
tf2ss	Converts from transfer function form to state-space form.
tf2zp	Converts from transfer function form to zero-pole form.
tfdata	Extracts equation coefficients from an LTI model.
zp2tf	Converts from zero-pole form to transfer function form.
zpk	Creates an LTI model from its poles, zeros, and gain.
zpkdata	Returns the poles, zeros, and gain of an LTI model.

Equation solvers.

Command	Description
impulse	Computes and plots the impulse response of the LTI model sys.
initial	Computes and plots the free response of an LTI model given in state-model form.
lsim	Computes and plots the response of an LTI object to a defined input function of time.
ode23	Solves linear and nonlinear differential equations.
ode45	Solves linear and nonlinear differential equations.
step	Computes and plots the step response of an LTI object.

Predefined input functions.

Command	Description
gensig	Generates a periodic sine, square, or pulse input having a specified period.
stepfun	Generates a step function input.

Frequency response functions.

Command	Description
bode	Computes the magnitude ratio and phase angle of an LTI model and displays the Bode plots.
bodemag	Computes the magnitude ratio of an LTI model and displays the magnitude plot.
evalfr	Evaluates a transfer function model at a specified value of s.
freqresp	Computes the frequency response of an LTI model at multiple specified frequencies.
margin	Computes phase and gain margins of an LTI model and displays the Bode plots.

Root locus functions.

Command	Description
pole	Computes the poles of an LTI model.
pzmap	Computes the poles and zeros of an LTI model.
rlocfind	Enables use of the cursor to select the gain from a specified point on a root locus plot.
rlocus	Computes and displays the root locus plot.
rltool	Starts the root locus GUI interface.
sgrid	Superimposes a grid of constant ζ and constant ω_n lines on the root locus plot.
zero	Computes the zeros of an LTI model.

Control system functions.

Command	Description
acker	Uses Ackermann's method to compute the feedback gain matrix for a single-input system to place the closed-loop poles at specified locations.
c2d	Converts a continuous-time model into a discrete-time model using a zero-order hold on the inputs, with a specified sampling time.
feedback	Creates an LTI model from two subsystems connected with a feedback loop.
ltiview	Starts the LTI viewer.
pade	Pade approximation to the transfer function of the dead time element.
parallel	Creates an LTI model from two subsystems connected in parallel.
series	Creates an LTI model from two subsystems connected in series.
sisotool	Graphical user interface for designing single-input/single-output compensators.

B

Fourier Series

The Fourier series is used to represent a periodic function as a sum of sines, cosines, and a constant. When used with the principle of superposition, which lets you obtain the total response as the sum of the individual responses, the Fourier series enables you to obtain the response of a linear system to any periodic function. All that is needed is the system response to a sine, a cosine, and a constant input. Although the Fourier series is an infinite series, in practice it can be truncated to a small number of terms whose frequencies lie within the bandwidth of the system.

If a function $f(t)$ is periodic with period P, then $f(t + P) = f(t)$. The Fourier series for this function defined on the interval $t_1 \le t \le t_1 + P$, where t_1 and P are constants and $P > 0$, is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{P} + b_n \sin \frac{2n\pi t}{P} \right)$$

where

$$a_n = \frac{2}{P} \int_{t_1}^{t_1+P} f(t) \cos \frac{2n\pi t}{P} \, dt \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{P} \int_{t_1}^{t_1+P} f(t) \sin \frac{2n\pi t}{P} \, dt \quad n = 1, 2, 3, \ldots$$

If $f(t)$ is defined outside the specified interval $[t_1, t_1 + P]$ by a periodic extension of period P, and if $f(t)$ and df/dt are piecewise continuous, then the Fourier series converges to $f(t)$ if t is a point of continuity, and to the average value $[f(t_+) + f(t_-)]/2$ otherwise.

As an example, consider the train of unit pulses of width π and alternating in sign, as shown in Figure B.1. The function is described by

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

The period is $P = 2\pi$, and we can take the constant t_1 to be 0. Using a table of integrals, we find that

$$a_n = 0 \qquad \text{for all } n$$

$$b_n = \frac{4}{n\pi} \quad \text{for } n \text{ odd}$$

$$b_n = 0 \qquad \text{for } n \text{ even}$$
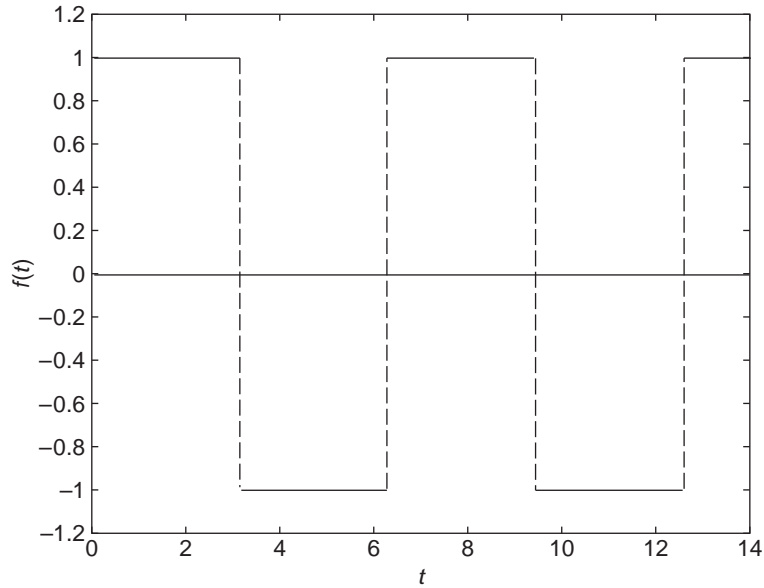
Figure B.1 The function used for the Fourier series example.

The Fourier series is

$$f(t) = \frac{4}{\pi} \left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right)$$

In general, the constant term a_0 and the cosine terms will not appear in the series if the function is odd; that is, if $f(-t) = -f(t)$. If the function is even, then $f(-t) = f(t)$, and no sine terms will appear in the series.

C

Introduction to MATLAB

This appendix provides a "quick start" introduction to MATLAB. You should be familiar with Sections C.1 and C.2 before attempting to cover the MATLAB material in the main body of this text. Sections C.3 and C.4 are useful for more extensive applications such as projects and some of the homework exercises. The extensive MATLAB help system discussed in Section C.5 can be used to obtain more detailed information and more examples. Other sources are [Palm, 2005] and [Palm, 2008].

Appendix A is a guide to the MATLAB commands and functions that are relevant to the system dynamics methods used in this text. To obtain more information about a specific command or function, in the Command window type `help topic`, where `topic` is the name of the command or function. For more information on MATLAB help, type `help help`.

MATLAB has a number of "add-on" software modules, called *toolboxes,* that perform more specialized computations. These can be purchased separately, but all require the "core" MATLAB program to be used. This text uses some features of the core MATLAB program, the Control Systems toolbox, and Simulink, which requires MATLAB to run. ∎

C.1 MATLAB INTERACTIVE SESSIONS

You can use MATLAB in interactive mode to obtain an immediate response to each command as it is typed, or you can run MATLAB programs stored in files. We discuss the interactive mode in this section and program files in Section C.3.

TYPEFACE CONVENTIONS

We use `typewriter font` to represent MATLAB commands, any text that you type in the computer, and any MATLAB responses that appear on the screen; for example, `y = 6*x`. Variables in normal mathematics text appear in italics; for example, $y = 6x$. We use boldface type for three purposes: to represent vectors and matrices in normal mathematics text (for example, $\mathbf{Ax} = \mathbf{b}$), to represent a key on the keyboard (for example, **Enter**), and to represent the name of a screen menu or an item that appears in such a menu (for example, **File**). It is assumed that you press the **Enter** key after you type a command. We do not show this action with a separate symbol.

USER INTERFACE

To start MATLAB on a Windows system, double-click on the MATLAB icon. You will then see the MATLAB Desktop. The Desktop manages the Command window and a Help Browser, as well as other tools. The default appearance of the Desktop consists of three windows. These are the Current Directory window, the Command History window, and the Documents window. The Command History window displays your previous commands for quick reference. The Current Directory window provides easy access to program files and to the Array Editor. Underneath the Documents window is the Workspace Browser, which enables you to examine your variables. Across the top of the Desktop are a row of menu names and a row of icons called the "toolbar."

You use the Command window to communicate with the MATLAB program, by typing instructions of various types called *commands, functions,* and *statements*. To simplify the discussion, we will just call the instructions by the generic name *commands*. In the Windows operating system the professional version of MATLAB displays a prompt ($\gg$) to indicate that it is ready to receive instructions. Other MATLAB versions, such as the student version, might use a different prompt. You can quit MATLAB by typing `quit`. On Windows systems you can also click on the **File** menu, and then click on **Exit Matlab.**

The Workspace Browser can be accessed by clicking on its tab at the bottom of the Current Directory window. You can use this window to determine the type, size, and values of all the variables you create. The Command History window shows all the previous keystrokes you entered in the Command window. You can alter the appearance of the Desktop if you wish. To eliminate a window, click on its Close Window button ($\times$) in its upper right-hand corner.

ENTERING COMMANDS AND EXPRESSIONS

After you type a command, press **Enter** to execute it. For example, to divide 8 by 10, type `8/10` and press **Enter** (the symbol/is the MATLAB symbol for division). Your entry and the MATLAB response looks like the following on the screen (we call this interaction between you and MATLAB an interactive *session,* or simply a *session*).

```
>>8/10
ans =
   0.8000
```

MATLAB has assigned the answer to a variable called `ans`, which is an abbreviation for *answer*. You can use the variable `ans` for further calculations; for example, using the MATLAB symbol for multiplication (`*`):

```
>>5*ans
ans =
   4.0000
```

Note that the variable `ans` now has the value 4.

MATLAB uses high precision for its computations, but by default it usually displays its results using four decimal places. This is called the *short* format. This default can be changed by using the `format` command. MATLAB uses the notation `e` to represent exponentiation to a power of ten; for example, MATLAB displays the number 5.316×10^2 as `5.316e+02`.

Because MATLAB retains your previous keystrokes in a "command file," you can use the up arrow key ↑ to scroll back through the commands. Press the key once to see the previous entry, twice to see the entry before that, and so on. Use the down arrow key ↓ to scroll forward through the commands. When you find the line you want, you can edit it using the left and right arrow keys ← and →, and the **Back Space** and **Delete** keys. Press the **Enter** key to execute the command.

VARIABLES

A *variable* in MATLAB is a symbol used to contain a value. You can use variables to write mathematical expressions and assign the result to a variable of your own choosing, say `r`, as follows:

```
>>r=8/10
r =
   0.8000
```

You can put spaces in the line to improve its readability; for example, you can put a space before and after the = sign if you want. MATLAB ignores these spaces when making its calculations.

A semicolon at the end of a line suppresses printing the results to the screen. Even if you suppress the display with the semicolon, MATLAB still retains the variable's value. You can put several commands on the same line if you separate them with a comma—if you want to see the results of the previous command—or semicolon if you want to suppress the display. If you need to type a long line, you can use an *ellipsis,* by typing three periods, to delay execution.

The term *workspace* refers to the names and values of any variables in use in the current work session. Variable names must begin with a letter and must contain less than 32 characters; the rest of the name can contain letters, digits, and underscore characters. MATLAB is case-sensitive. Thus the following names represent five different variables: `speed`, `Speed`, `SPEED`, `Speed_1`, and `Speed_2`.

MATLAB retains the last value of a variable until you clear its value or quit MATLAB. You can use the `clear` command to remove the values of *all* variables from memory, or you can use the form `clear var1 var2` to clear the variables named `var1` and `var2`. The effect of the `clc` command is different; it clears the Command window of everything in the window display, but the variables retain their values.

You can type the name of a variable and press **Enter** to see its current value. If the variable does not have a value (i.e., if it does not exist), you will see an error message. You can also use the `exist` command. Type `exist('x')` to see if the variable `x` is in use. If a 1 is returned, the variable exists; a 0 indicates that it does not exist. The `who` command lists the names of all the variables in memory, but does not give their values. The form `who var1 var2` restricts the display to the variables specified. The wildcard character `*` can be used to display variables that match a pattern. For instance, `who A*` finds all variables in the current workspace that start with A. The `whos` command lists the variable names and their sizes, and indicates whether or not they have nonzero imaginary parts.

SCALAR ARITHMETIC

A *scalar* is a single number and a *scalar variable* is a variable that is used to contain a single number. MATLAB uses the symbols + – * / ^ for addition, subtraction, multiplication, division, and exponentiation (power) of scalars. For example, typing `x = 8 + 3*5` returns the answer `x = 23`. Typing `2^3-10` returns the answer `ans = -2`. The *forward slash* / represents *right division,* which is the normal division operator familiar to you. Typing `15/3` returns the result `ans = 5`. MATLAB has another division operator, called *left division,* which is denoted by the *backslash* \. The left division operator is useful for solving sets of linear algebraic equations.

The mathematical operations represented by the symbols + – * / \ and ^ follow a set of rules called *precedence*. Mathematical expressions are evaluated starting from the left, with the exponentiation operation having the highest order of precedence, followed by multiplication and division with equal precedence, followed by addition and subtraction with equal precedence. Parentheses can be used to alter this order. Evaluation begins with the innermost pair of parentheses, and proceeds outward. Note the effect of precedence in the following session.

```
>>3*4^2 - 12 - 8/4*2 + 8 + 3*5
ans =
   55
>>3*(4^2) - 12 - 8/(4*2) + (8 + 3)*5
ans =
   90
>>3*(4^2) + 5 + 27^1/3 + 32^0.2
ans =
   64
>>(3*4)^2 + 5 + 27^(1/3) + 32^(0.2)
ans =
   154
```

To avoid mistakes, you should feel free to insert parentheses wherever you are unsure of the effect precedence will have on the calculation.

THE ASSIGNMENT OPERATOR

The = sign in MATLAB is the called the *assignment* operator or *replacement* operator. It works differently than the equals sign you are familiar with from mathematics. When you type `x = 3`, you tell MATLAB to assign the value 3 to the variable `x`. This usage is no different than in mathematics. However, in MATLAB we can also type something like this: `x = x + 2`. This tells MATLAB to add 2 to the current value of `x`, and to

replace the current value of x with this new value. If x originally had the value 3, its new value would be 5. This usage of the = operator is different than its use in mathematics. For example, the mathematics equation $x = x + 2$ is invalid because it implies that $0 = 2$ (subtract x from both sides of the equation to see this).

The variable on the *left-hand* side of the = operator is replaced by the value generated by the *right-hand* side. Thus, one variable, and only one variable, must be on the left-hand side of the = operator. Thus, in MATLAB you cannot type 6 = x. Another consequence of this restriction is that you cannot write in MATLAB expressions like x + 2 = 20. The corresponding equation $x + 2 = 20$ is acceptable in algebra, and has the solution $x = 18$, but MATLAB cannot solve such an equation without additional commands that are available in the Symbolic Math toolbox.

Another restriction is that the right-hand side of the = operator must have a computable value. For example, if the variable y has not been assigned a value, then typing x = 5 + y will generate an error message in MATLAB.

C.2 COMPUTING WITH MATLAB

MATLAB has several predefined special constants. The symbol Inf stands for ∞, which in practice means a number so large that MATLAB cannot represent it. For example, typing 5/0 will generate the answer Inf. The symbol NaN stands for "<u>N</u>ot <u>a</u> <u>N</u>umber." It indicates an undefined numerical result such as that obtained by typing 0/0. The symbol eps is the smallest number which, when added to 1 by the computer, creates a number greater than 1. It is used as an indicator of the accuracy of computations. The symbol pi represents the number $\pi = 3.14159\ldots$. The symbols i and j denote the imaginary unit, where $i = j = \sqrt{-1}$. They are used to create and represent complex numbers, such as x = 5 + 8i.

COMPLEX NUMBER ALGEBRA

MATLAB handles complex number algebra automatically. It accepts both i and j to designate the imaginary part, but it displays imaginary parts using i only. For example, the number $c_1 = 1 - 2i$ is entered as follows: c1 = 1-2i. Note that an asterisk is not needed between i or j and a number, although it is required with a variable, such as c2 = 5 - i*c1. This can cause errors if you are not careful. For example, the expressions y = 7/2*i and x = 7/2i give two different results: $y = (7/2)i = 3.5i$ and $x = 7/(2i) = -3.5i$.

Addition, subtraction, multiplication, and division of complex numbers are easily done. For example,

```
>>s = 3+7i;w = 5-9i;
>>w+s
ans =
   8.0000 - 2.0000i
>>w*s
ans =
  78.0000 + 8.0000i
>>w/s
ans =
  -0.8276 - 1.0690i
```

BUILT-IN FUNCTIONS

MATLAB has hundreds of built-in functions. One of these is the *square root* function, `sqrt`. A pair of parentheses is used after the function's name to enclose the value—called the function's *argument*—that is operated on by the function. For example, to compute the square root of 9, you type `sqrt(9)`. To compute sin x, where x has a value in radians, you type `sin(x)`. To compute cos x, type `cos(x)`. The exponential function e^x is computed from `exp(x)`. The natural logarithm, ln x, is computed by typing `log(x)`. You compute the base 10 logarithm by typing `log10(x)`. The inverse sine, or arcsine, is obtained by typing `asin(x)`. It returns an answer in radians, not degrees. Appendix A lists the available elementary functions. Type `help elfun` to obtain more information.

The difference between a function and a command or a statement is that functions have their arguments enclosed in parentheses. For example, to compute sin 2 using the `sin` function, you type `sin(2)`. Commands, such as `clear`, need not have arguments, but if they do, they are not enclosed in parentheses; for example, `clear x`. Statements cannot have arguments; for example, `clc` and `quit` are statements.

ARRAYS

One of the strengths of MATLAB is its ability to handle collections of numbers, called *arrays,* as if they were a single variable. A numerical array is an ordered collection of numbers (a set of numbers arranged in a specific order). An example of an array is one that contains the numbers 0, 1, 3, and 6 in that order. We can use square brackets to define the variable x to contain this collection by typing `x = [0, 1, 3, 6]`. The elements of the array must be separated by commas or spaces. Note that the variable y defined as `y = [6, 3, 1, 0]` is not the same as x because the order is different.

We can add the two arrays x and y to produce another array z by typing the single line `z = x + y`. To compute z, MATLAB adds all the corresponding numbers in x and y to produce z. The resulting array z is `[6, 4, 4, 6]`.

Array addition and subtraction require that both arrays have the same size. The only exception to this in MATLAB occurs when we add or subtract a *scalar* to or from an array. In this case, the scalar is added or subtracted from each element in the array. For example, if `x = [0, 1, 3, 6]`, typing `v = x + 2` gives `v = [2, 3, 5, 8]`.

Arrays that display on the screen as a single row of numbers with more than one column are called *row arrays*. You can create *column* arrays, which have more than one row, by using a semicolon to separate the rows. So, for example, `[3, 7, 2]` is not the same array as `[3; 7; 2]`.

CREATING AND ACCESSING ARRAYS

To create an array having many entries, you need not type all the numbers if they are regularly spaced. Instead, you type the first number, the spacing, and the last number, separated by colons. For example, the numbers $0, 0.1, 0.2, \ldots, 10$ can be assigned to the variable u by typing `u = [0:0.1:10];` to create a row array having 101 elements. Use the transpose operator `'` to convert a row array into a column array, and vice versa.

You can see all the values in u by typing u after the prompt or, for example, you can see the seventh value by typing `u(7)`. The number 7 is called an *array index,* because it points to a particular element in the array. For example, typing `u(7)` gives

ans = 0.6000. You can use the length function to determine how many values are in an array. For example, typing m = length(u) gives m = 101.

One advantage of MATLAB is that it can operate on arrays with a single command. For example, to compute $w = 5 \sin u$ for $u = 0, 0.1, 0.2, \ldots, 10$, the session is

```
>>u = [0:0.1:10]; w = 5*sin(u);
```

or more compactly,

```
>>w = sin([0:0.1:10]);
```

This single line computes the formula $w = 5 \sin u$ 101 times, once for each value in the array u, to produce an array w that has 101 values. To achieve this result with conventional programming languages requires that a loop be written.

MATRICES

A *matrix* is a two-dimensional array; that is, one that has multiple rows and multiple columns. One way to create a matrix is to type each row as a row array (with its elements separated by a comma or a space) followed by a semicolon except for the last row. The semicolon serves to separate the rows. For example, to create the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix}$$

the session is

```
>>A = [4, 7; 9, 2];
```

To access the element of a matrix in row r and column c, type A(r,c). For example, typing A(2,1) returns the value 9. Note that the row number comes first.

Matrix addition, subtraction, and multiplication by a scalar follow the same rules as for row and column arrays.

ARRAY MULTIPLICATION

Multiplication of arrays by a scalar is easily understood. For example, to double each element of the array r = [3, 5, 2], you type v = 2*r to obtain v = [6, 10, 4]. Similarly, multiplying a matrix **A** by a scalar w produces a matrix whose elements are the elements of **A** multiplied by w. However, multiplication of two arrays is not so straightforward. In fact, there are two definitions of multiplication used by MATLAB: (1) array multiplication and (2) matrix multiplication.

To compute the matrix product **AB** you type A*B. The resulting matrix is computed using the standard rules of matrix multiplication. However, typing A.*B produces a different matrix, in which each element is the product of the two corresponding elements in A and B. This form is called *array or element-by-element multiplication*. Note that .* is *one* symbol.

To illustrate the difference between the two types of multiplication, consider this table, which gives the speed of an aircraft on each leg of a certain trip, and the time spent on each leg.

	Leg			
	1	**2**	**3**	**4**
Speed (mi/hr)	200	250	400	300
Time (hr)	2	5	3	4

We can define a row array `s` containing the speeds and a row array `t` containing the times for each leg. Thus `s = [200, 250, 400, 300]` and `t = [2, 5, 3, 4]`. To find the miles traveled on each leg, we multiply the speed by the time. To do this we use the multiplication `s.*t` to produce the row array whose elements are the products of the corresponding elements in `s` and `t`. This process is illustrated as follows.

$$s.*t = [200(2), 250(5), 400(3), 300(4)] = [400, 1250, 1200, 1200]$$

The resulting array contains the miles traveled by the aircraft on each leg of the trip.

To find only the total miles traveled, we can use matrix multiplication `s*t'`, where the `'` denotes the transpose operation (note that the operator `.*` is not used here). The transpose converts the row array `t` into a column array. This product gives the *sum* of the individual element products; that is

$$s*t' = [200(2) + 250(5) + 400(3) + 300(4)] = 4050$$

The complete session to perform these calculations is

```
>>s = [200, 250, 400, 300];
>>t = [2, 5, 3, 4];
>>miles_per_leg = s.*t
miles_per_leg =
   400    1250    1200    1200
>>total_miles = s*t'
total_miles =
   4050
```

ARRAY DIVISION AND EXPONENTIATION

Array division, also called element-by-element division, is defined similarly to array multiplication, except of course that the elements of one array are divided by the elements of the other array. Both arrays must have the same size. The symbol for array right division is `./`. For example, if `x = [8, 12, 15]` and `y = [-2, 6, 5]` then `z = x./y` gives `z = [-4, 2, 3]`.

In MATLAB not only can we can raise arrays to powers, but we can also raise scalars and arrays to *array* powers. To perform exponentiation on an element-by-element basis, we must use the `.^` symbol. For example, if `x = [3, 5, 8]`, then typing `x.^3` produces the array $[3^3, 5^3, 8^3] = [27, 125, 512]$.

We can raise a scalar to an array power. For example, if `p = [2, 4, 5]`, then typing `3.^p` produces the array $[3^2, 3^4, 3^5] = [9, 81, 243]$. This illustrates a common situation where it helps to remember that `.^` is a *single* symbol; the dot in `3.^p` is not a decimal point associated with the number 3. Thus `3.^p` gives the same result as `3..^p` or as `(3).^p`.

POLYNOMIAL ROOTS

We can describe a polynomial in MATLAB with an array whose elements are the polynomial's coefficients, *starting with the coefficient of the highest power*. For example, the polynomial $4x^3 - 8x^2 + 7x - 5$ would be represented by the array `[4, -8, 7, -5]`. Polynomial roots can be found with the `roots(a)` function, where `a` is the polynomial's coefficient array. The result is a column array that contains the polynomial's

roots. For example, to find the roots of $x^3 - 7x^2 + 40x - 34 = 0$, the session is

```
>>roots([1,-7,40,-34])
ans =
   3.0000 + 5.000i
   3.0000 - 5.000i
   1.0000
```

The roots are $x = 3 \pm 5i$ and $x = 1$.

The `poly(r)` function computes the coefficients of the polynomial whose roots are specified by the array `r`. The result is a row array that contains the polynomial's coefficients. To find the polynomial whose roots are 1 and $3 \pm 5i$, the session is

```
>>poly([1,3+5i,3-5i])
ans =
   1.0000      -7.0000      40.0000      -34.0000
```

Thus, the polynomial is $x^3 - 7x^2 + 40x - 34$.

POLYNOMIAL ALGEBRA

You can do polynomial algebra with MATLAB. To add two polynomials, add the arrays that describe their coefficients. If the polynomials are of different degrees, add zeroes to the coefficient array of the lower-degree polynomial. For example, consider $f(x) = 9x^3 - 5x^2 + 3x + 7$, whose coefficient array is `f = [9, -5, 3, 7]`, and $g(x) = 6x^2 - x + 2$, whose coefficient array is `g = [6, -1, 2]`. The degree of $g(x)$ is one less that of $f(x)$. Therefore, to add $f(x)$ and $g(x)$, we append one 0 to `g` to "fool" MATLAB into thinking $g(x)$ is a third-degree polynomial. That is, we type `g = [0 g]` to obtain `[0, 6, -1, 2]` for `g`. This represents $g(x) = 0x^3 + 6x^2 - x + 2$. To add the polynomials, type `h = f + g`. The result is `h = [9, 1, 2, 9]`, which corresponds to $h(x) = 9x^3 + x^2 + 2x + 9$. Subtraction is done in a similar way.

To multiply a polynomial by a scalar, simply multiply the coefficient array by that scalar. For example, $5h(x)$ is obtained from `5*h`, where `h = [45, 5, 10, 45]`.

Multiplication of polynomials by hand can be tedious, and polynomial division is even more so, but these operations are easily done with MATLAB. Use the `conv` command to multiply polynomials (it stands for "convolve"), and use the `deconv` command to perform synthetic division (`deconv` stands for "deconvolve").

Consider the product

$$f(x)g(x) = (9x^3 - 5x^2 + 3x + 7)(6x^2 - x + 2) = 54x^5 - 39x^4 + 41x^3 + 29x^2 - x + 14$$

Dividing $f(x)$ by $g(x)$ using synthetic division gives a quotient of

$$\frac{f(x)}{g(x)} = \frac{9x^3 - 5x^2 + 3x + 7}{6x^2 - x + 2} = 1.5x - 0.5833$$

with a remainder of $-0.5833x + 8.1667$. The MATLAB session to perform these operations is

```
>>f = [9, -5, 3, 7]; g = [6, -1, 2];
>>product = conv(f,g)
product =
   54    -39    41    29    -1    14
```

```
>>[quotient, remainder] = deconv(f,g)
quotient =
   1.5    -0.5833
remainder =
   0    0    -0.5833    8.1667
```

The `conv` and `deconv` commands do not require that the polynomials have the same degree.

The `polyval` function is useful for plotting polynomials, and is discussed later. Roots of functions other than polynomials can be obtained with the `fzero` function. Type `help fzero` for more information.

PLOTTING WITH MATLAB

MATLAB contains many powerful functions and commands for easily creating plots of several different types, such as rectilinear, logarithmic, surface, and contour plots, for example. To plot the function $y = \sin 2x$ for $0 \leq x \leq 10$, we choose an increment of 0.01 to generate a large number of `x` values in order to produce a smooth curve. The function `plot(x,y)` generates a plot with the `x` values on the horizontal axis (the abscissa) and the `y` values on the vertical axis (the ordinate). The session is

```
>>x = [0:0.01:10]; y = sin(2*x);
>>plot(x,y),xlabel('x'),ylabel('sin(2x)')
```

The `xlabel` function places the text in single quotes as a label on the horizontal axis. The `ylabel` function performs a similar function for the vertical axis. The plot appears on the screen in a graphics window. If a hard copy of the plot is desired, the plot can be printed by selecting **Print** from the **File** menu on the graphics window. The window can be closed by selecting **Close** on the **File** menu in the graphics window. You will then be returned to the prompt in the Command window.

Other useful plotting functions are `title` and `gtext`. These functions place text on the plot. Both accept text within parentheses and single quotes, as with the `xlabel` function. The `title` function places the text at the top of the plot; the `gtext` function places the text at the point on the plot where the cursor is located when you click the left mouse button.

Sometimes it is useful or necessary to obtain the coordinates of a point on a plotted curve. The function `ginput` can be used for this purpose. Place it at the end of all the plot and plot formatting statements, so that the plot will be in its final form. The command `[x,y] = ginput(n)` gets *n* points and returns the *x* and *y* coordinates in the vectors `x` and `y`, which have a length *n*. Position the cursor using a mouse, and press the mouse button. The returned coordinates have the same scale as the coordinates on the plot.

In cases where you are plotting *data,* as opposed to functions, you should use *data markers* to plot each data point (unless there are very many data points). To mark each point with a plus sign +, the required syntax for the `plot` function is `plot(x,y,'+')`. You can connect the data points with lines if you wish. In that case, you must plot the data twice, once with a data marker, and once without a marker. For example, suppose the data for the independent variable is `x = [15:2:23]` and the dependent variable values are `y = [20, 50, 60, 90, 70]`. To plot the *y* data with plus signs connected by straight lines, type `plot(x,y,'+',x,y)`. Other data markers are available. Type `help plot` for information about other markers.

MULTIPLE PLOTS

You can create multiple plots—called *overlay* plots—by including another set or sets of values in the `plot` function. For example, to plot the functions $y = 2\sqrt{x}$ and $z = 4\sin 3x$ for $0 \le x \le 5$ on the same plot, the session is

```
>>x = [0:0.01:5]; y = 2*sqrt(x); z = 4*sin(3*x);
>>plot(x,y,x,z),xlabel('x'),gtext('y'),gtext('z')
```

After the plot appears on the screen, the program waits for you to position the cursor and click the mouse button, once for each `gtext` function used. Although MATLAB displays different colors for each curve, if you are going to print the plot on a black-and-white printer, you should label each curve so that you know which curve represents y and which curve represents z. One way of doing this is to use the `gtext` function to place the labels y and z next to the appropriate curves, as shown in the preceding session. Another way is to use the `legend` function. The plotting functions `xlabel`, `ylabel`, `title`, and `gtext` must be placed after the `plot` function and separated by commas.

You can also distinguish curves from one another by using different line types for each curve. For example, to plot the z curve using a dashed line, replace the `plot(x,y,x,z)` function in the preceding session with `plot(x,y,x,z,'--')`. Other line types can be used. Type `help plot` for information about these line types.

You can use the `subplot` function to create multiple but separate plots in the same figure window (and on the same printed page). The `subplot` function creates a rectangular tiling having r rows and c columns of "panes," each containing a plot. The syntax is `subplot(r,c,n)`, where n denotes the pane number currently being plotted. For example, to plot `y` and `z` created in the previous session as two separate plots, with the plot of `y` above the plot of `z`, the session is

```
>>subplot(2,1,1),plot(x,y),xlabel('x'),ylabel('y')
>>subplot(2,1,2),plot(x,z),xlabel('x'),ylabel('z')
```

PLOTTING POLYNOMIALS

The `polyval(a,x)` function evaluates a polynomial at specified values of its independent variable `x`. The polynomial's coefficient array is `a`. The result is the same size as `x`. The `polyval` function is very useful for plotting polynomials. To do this you should define an array that contains many values of the independent variable x in order to obtain a smooth plot. For example, to plot the polynomial $f(x) = 9x^3 - 5x^2 + 3x + 7$ for $-2 \le x \le 5$, the session is

```
>>a = [9,-5,3,7]; x = [-2:0.01:5];
>>f = polyval(a,x);
>>plot(x,f),xlabel('x'),ylabel('f(x)'),grid
```

The `grid` command puts grid lines on the plot.

THE PLOT EDITOR

The aforementioned plot formatting commands are useful when you want to generate plots automatically with a program file. This is most convenient when you need to make many plots of similar type. If you need a small number of plots, you can use the Plot Editor to place text, lines, and arrows on the plot. Once the plot is displayed, click on the northwest-facing arrow to start the Plot Editor mode. This mode enables you to

edit the plot elements by double-clicking on an element, such as an axis or a plotted curve. To insert an axis label or to change the limits or scale, double-click on any axis, then select the tab of the desired axis, enter its label, and change its scale and limits, if desired.

Click on the Data Cursor button to use the mouse to extract coordinates from a plotted curve. Click on the Show Plot Tools button to display the Figure Palette. This palette enables you to add elements such as arrows, text, rectangles, and ellipses to the plot.

C.3 WORKING WITH FILES

MATLAB uses several types of files that enable you to save programs, data, and user-defined functions. MATLAB program files and function files are saved with the extension '`.m`', and thus are called M-files. Files are stored in *directories,* which are called *folders* on some computer systems. Directories can have subdirectories below them. The *path* tells us and MATLAB how to find a particular file.

THE SEARCH PATH

Suppose you have saved the program file `problem6.m` in the directory `\homework` on a disk that you insert in drive a:. The path for this file is `a:\homework`. As MATLAB is normally installed, when you type `problem6` in the Command window,

1. MATLAB first checks to see if `problem6` is a variable and if so, displays its value.
2. If not, MATLAB then checks to see if `problem6` is one of its own built-in commands, and executes it if it is.
3. If not, MATLAB then looks in the current directory for a program file named `problem6.m` and executes `problem6` if it finds it.
4. If not, MATLAB then searches the directories in its search path, in order, for `problem6.m` and then executes it if found.

You can display the MATLAB search path by typing `path`. If `problem6.m` is on the disk only and if directory `a:\homework` is not in the search path, MATLAB will not find the file and will generate an error message, unless you tell it where to look. You can do this by typing `cd a:\homework`, which stands for "change directory to a:\homework." This will change the current directory to `a:\homework` and force MATLAB to look in that directory to find your file. The general syntax of this command is `cd dirname`, where `dirname` is the full path to the directory. The *main* directory on the disk is `a:`, so if your file is in the main directory, be sure to include the colon, and type `cd a:`.

You can determine the current directory (the one where MATLAB looks for your file) by typing `pwd`. To see a list of all the files in the current directory, type `dir`. To see the files in the directory `dirname`, type `dir dirname`. The `what` command displays a list of the MATLAB-specific files in the current directory. The `what dirname` command does the same for the directory `dirname`.

SCRIPT FILES AND THE EDITOR/DEBUGGER

In addition to using the interactive mode, in which all commands are entered directly in the Command window, you can also perform operations by running a MATLAB

program stored in *script* file. This type of file has MATLAB commands in it, so that when you run such a file it is equivalent to typing all the commands—one at a time—at the Command window prompt. You can run the file by typing its name at the Command window prompt. When the problem requires many commands, or has arrays with many elements, or you need to repeat a set of commands several times, it is more convenient to use a script file. If you need to access the same set of data frequently, you can store the data in an array within a script file.

You write and save MATLAB programs in M-files, which have the extension '.m'; for example, `program1.m`. You execute a script file at the Command window prompt by typing its name without the extension '.m'. Another type of M-file is a *function file,* which is useful when you need to repeat the operation of a set of commands. You can create your own function files; at the end of this section we discuss how to do this.

The symbol `%` designates a comment, which is not executed by MATLAB. Comments are put in script files for the purpose of documenting the file. The comment symbol may be put anywhere in the line. MATLAB ignores everything to the right of the `%` symbol. A very simple example of a script file is shown next. It computes the sine of the square root of several numbers and displays the results on the screen.

```
% Program example1.m
% This program computes the sine of the square root,
% and displays the result.
x = sqrt([3:2:11]);
y = sin(x)
```

To create this new M-file in the MS Windows environment, in the Command window select **New** from the **File** menu, then select **M-file.** You will then see a new edit window. This is the Editor/Debugger window. Type in the file as just shown. You can use the keyboard and the **Edit** menu in the Editor/Debugger as you would in most word processors to create and edit the file. When finished, select **Save** from the **File** menu in the Editor/Debugger. In the dialog box that appears, replace the default name provided (usually named `Untitled`) with the name `example1`, and click on **Save.** The Editor/Debugger will automatically provide the extension '.m' and save the file in the MATLAB current directory, which for now we will assume to be on the hard drive.

Once the file has been saved, in the MATLAB Command window type the script file's name `example1` to execute the program. You should see displayed in the Command window the computed values of the array `y = 0.9870 0.7867 0.4758 0.1411 -0.1741`.

Keep in mind the following when using script files:

1. The name of a script file must follow the MATLAB convention for naming variables; that is, the name must begin with a letter, and may include digits and the underscore character, up to 31 characters.
2. Recall that typing a variable's name at the Command window prompt causes MATLAB to display the value of that variable. Thus, do not give a script file the same name as a variable it computes, because MATLAB will not be able to execute that script file more than once, unless you clear the variable.
3. Do not give a script file the same name as a MATLAB command or function. You can check to see if a command, function, or file name already exists by using the `exist` command.

4. You can use the `type` command to view an M-file without opening it with a text editor. For example, to view the file `example1`, the command is `type example1`.

INPUT AND OUTPUT FOR SCRIPT FILES

You can use the `disp` function to display the value of a variable from within a script file. For example, type `disp(x)` to display the value of x. You can also display messages with the `disp` function. Type `disp('text')` to display the message enclosed within single quotes. The `input` function displays a prompt and assigns an entered value to a variable. The syntax is `x = input('Enter a value for x:')`. The function displays the message within the single quotes and assigns the value typed at the keyboard to the variable x. Of course, any variable name and any text message can be used.

DEBUGGING PROGRAMS

Debugging a program is the process of finding and removing the "bugs," or errors, in a program. Such errors usually fall into one of two categories: (1) Syntax errors such as omitting a parenthesis or comma, or spelling a command name incorrectly. MATLAB usually detects the more obvious errors and displays a message describing the error and its location. (2) Errors due to an incorrect mathematical procedure. These are called *runtime errors*. They do not necessarily occur every time the program is executed; their occurrence often depends on the particular input data. A common example is division by zero.

The MATLAB error messages usually enable you to find syntax errors. However, runtime errors are more difficult to locate. To locate such an error, try the following: (1) Test your program with a simple version of the problem, whose answers can be checked by hand calculations. (2) Display any intermediate calculations by removing semicolons at the end of statements. (3) Use the debugging features of the Editor/Debugger. However, one of advantages of MATLAB is that it requires relatively simple programs to accomplish many types of tasks. Thus, you probably will not need to use the Debugger for most of the problems encountered in this text.

USER-DEFINED FUNCTIONS

Another type of M-file is a *function file*. Unlike a script file, all the variables in a function file are *local,* which means their values are available only within the function. Function files are useful when you need to repeat a set of commands several times. The first line in a function file must begin with a *function definition line* that has a list of inputs and outputs. This line distinguishes a function M-file from a script M-file. Its syntax is as follows:

```
function [output variables] = function_name(input variables)
```

Note that the output variables are enclosed in *square brackets* while the input variables must be enclosed with *parentheses*. The `function_name` must be the same as the filename in which it is saved (with the .m extension). That is, if we name a function `drop`, it must be saved in the file `drop.m`. The function is "called" by typing its name (e.g., `drop`) at the command line. The word `function` in the function definition line must be *lower case*.

The following examples show permissible variations in the format of the function line. The differences depend on whether there is no output, a single output, or multiple outputs.

Function definition line	File name
1. `function [area_square] = square(side);`	square.m
2. `function area_square = square(side);`	square.m
3. `function [volume_box] = box(height,width,length);`	box.m
4. `function [area_circle,circumf] = circle(radius);`	circle.m
5. `function sqplot(side);`	sqplot.m

Example 1 is a function with one input and one output. The square brackets are optional when there is only one output (see Example 2). Example 3 has one output and three inputs. Example 4 has two outputs and one input. Example 5 has no output variable (an example of this would be a function that generates a plot). In such cases, the equals sign should be omitted.

Comment lines starting with the % sign may be put anywhere. However, it is important to note that all comment lines immediately following the function definition line are displayed by MATLAB if `help` is used to obtain information about the function. The first comment line can be accessed by the `lookfor` command, which is discussed in Section C.5.

Both built-in and user-defined functions can be called either with the output variables explicitly specified, as with the previous Examples 1 through 4, or without any output variables specified. For example, we can call the function `square` as follows: `square(side)` if we are not interested in accessing its output variable `area_square` (the function might perform some other operation that we want to occur, such as displaying a result). Note that if the semicolon is omitted at the end of the function call statement, the first variable in the output variable list will be displayed using the default variable name `ans`.

For example, the following function, called `drop`, computes a falling object's velocity and distance dropped. The input variables are the acceleration g, the initial velocity v_0, and the elapsed time t. Note that we must use the element-by-element operations for any operations involving function inputs that are arrays. Here we anticipate that t will be an array, so we use the element-by-element operator (`.^`).

```
function [dist,vel] = drop(g,vO,t);
% Computes velocity and distance of a dropped object.
% The distance and velocity are computed as functions
% of g, the initial velocity vO, and the time t.
vel = g*t + vO;
dist = 0.5*g*t.^2 + vO*t;
```

The following examples show various ways the function `drop` can be called.

1. The variable names used in the function definition may, but need not, be used when the function is called. For example, consider the session:

```
>>a = 32.2;
>>initial_speed = 10;
>>time = 5;
>>[feet_dropped,speed] = drop(a,initial_speed,time)
```

2. The input variables need not be assigned values outside the function prior to the function call. For example,

```
>>[feet_dropped,speed] = drop(32.2,10,5)
```

3. The inputs and outputs may be arrays. For example,

```
>>[feet_dropped,speed] = drop(32.2,10,[0:1:5])
```

produces the arrays `feet_dropped` and `speed` each with six values, corresponding to the six values of time in the array `time`.

Some of the MATLAB commands act on functions, so if the function of interest is not a simple function, you must define the function in an M-file to use one of these commands. For example, we can use the `fzero` function to find the zero of a function of a single variable, which we denote by x. One form of its syntax is `fzero('function', x0)`, where `function` is a string containing the name of the function. The `fzero` function returns a value of x that is near x0. It identifies only points where the function crosses the x axis, not points where the function just touches the axis. If $y = x + 2e^{-x} - 3$, define the following function file:

```
function y = f1(x)
y = x + 2*exp(-x) - 3;
```

To find the zero near $x = 3$, type `x = fzero('f1',3)`. The returned answer is $x = 2.8887$.

The `fminbnd` function can be used to find the minimum of a function of a single variable, which we denote by x. One form of its syntax is `fminbnd('function', x1, x2)`, where `function` is a string containing the name of the function. The `fminbnd` function returns a value of x that minimizes the function in the interval x1 ≤ x ≤ x2. For example, if $y = 1 - xe^{-x}$, define the following function file:

```
function y = f2(x)
y = 1-x.*exp(-x);
```

To find the value of x that gives a minimum for $0 \leq x \leq 5$, type `x = fmin('f2',0,5)`. The returned answer is $x = 1$. To find the minimum value of y, type `y = f2(x)`. The result is `y = 0.6321`.

C.4 LOGICAL OPERATORS AND LOOPS

MATLAB has relational operators, logical operators, and conditional statements that enable you to write decision-making programs whose operations depend on the results of calculations made by the program. MATLAB also has two types of loop structures for performing calculations repeatedly a specified number of times or until some condition is satisfied.

RELATIONAL OPERATORS

MATLAB has six *relational* operators to make comparisons between arrays. These operators are shown in Table C.4.1. Note that the "equal to" operator consists of two = signs, not a single = sign as you might expect. The single = sign is the *assignment* operator in MATLAB. The result of a comparison using the relational operators is either a 0 (if the comparison is *false*), or a 1 (if the comparison is *true*), and the result can be used as a variable.

Table C.4.1 Relational operators.

Relational operator	Meaning
<	less than
<=	less than or equal to
>	greater than
>=	greater than or equal to
==	equal to
~=	not equal to

Table C.4.2 Logical operators.

Operator	Name	Definition
~	NOT	~ A returns an array the same dimension as A, which has 1's where A is 0, and 0's where A is nonzero.
&	AND	A & B returns an array the same dimension as A and B, which has 1's where both A and B have nonzero elements, and 0's where either A or B is zero.
|	OR	A | B returns an array the same dimension as A and B, which has 1's where at least one element in A or B is nonzero, and 0's where A and B are both zero.

For example, if $x = 2$ and $y = 5$, typing $z = x < y$ returns the value $z = 1$, because x is less than y. Typing $u = x==y$ returns the value $u = 0$ because x does not equal y. To make the statements more readable, we can group the operations using parentheses. For example, $z = (x < y)$ and $u = (x==y)$. Note, however, that the variables z and u are *logical* variables, as opposed to *numeric* variables. A logical variable may take on only the logical values, true or false. These values are represented by 1 and 0, but they should not be used for numerical calculations. For example, typing $b = \sin(z)$ generates an error message. This distinction is discussed in more detail in the subsection "Logical Arrays."

When used to compare arrays, the relational operators compare the arrays on an element-by-element basis. The arrays being compared must have the same size. The only exception occurs when we compare an array to a scalar. In that case, all the elements of the array are compared to the scalar. For example, suppose that $x = [6, 3, 9]$ and $y = [14, 2, 9]$. Then typing $z = (x < y)$ gives the result $z = [1, 0, 0]$. Typing $z = (x == y)$ gives the result $z = [0, 0, 1]$. Typing $z = (x > 8)$ gives $z = [0, 0, 1]$.

The relational operators can be used for array addressing. For example, with $x = [6, 3, 9, 11]$ and $y = [14, 2, 9, 13]$, typing $z = x(x < y)$ finds all the elements in x that are less than the corresponding elements in y. The result is the array $z = [6, 11]$.

LOGICAL OPERATORS

MATLAB has three *logical* operators, which are sometimes called *Boolean* operators (see Table C.4.2). These operators perform element-by-element operations. With the exception of the NOT operator ($\sim$), they have a lower precedence than the arithmetic and relational operators (see Table C.4.3). The NOT symbol is called the *tilde*.

The NOT operation $\sim$ A returns an array of the same dimension as A, having 1's where A is 0, and 0's where A is nonzero. For example, if $x = [6, 3, 9]$ and $y = [14, 2, 9]$, then the statement $z = \sim x$ returns the array $z = [0, 0, 0]$, and

Table C.4.3 Order of precedence for operator types.

Precedence	Operator type
First	Parentheses, evaluated starting with the innermost pair.
Second	Arithmetic operators and NOT (~), evaluated from left to right.
Third	Relational operators, evaluated from left to right.
Fourth	Logical AND.
Fifth	Logical OR.

the statement `z = ~x > y` returns the result `z = [0, 0, 0]`. The later statement is equivalent to `z = (~x) > y`, whereas `z = ~(x > y)` gives the result `z = [1, 0, 1]`. This is equivalent to `z = (x <= y)`.

The `&` and `|` operators compare two arrays of the same dimension. The only exception, as with the relational operators, is that an array can be compared to a scalar. The AND operator `&` returns 1's where both `A` and `B` have nonzero elements, and 0's where any element of `A` or `B` is zero. The expression `z = 0&3` returns `z = 0`, `z = 2&3` returns `z = 1`, `z = 0&0` returns `z = 0`, and `z = [5, -3, 0, 0]&[2, 4, 0, 5]` returns `z = [1, 1, 0, 0]`. Because of operator precedence, `z = 1&2 + 3` is equivalent to `z = 1&(2 + 3)`, which returns `z = 1`. Similarly, `z = 5 < 6&1` is equivalent to `z = (5 < 6)&1`, which returns `z = 1`.

Let `x = [6, 3, 9]` and `y = [14, 2, 9]`, and let `a = [4, 3, 12]`. The expression

```
(x > y) & a
```

gives `z = [0, 1, 0]`, and the expression

```
z = (x > y) & (x > a)
```

returns the result `z = [0, 0, 0]`.

Be careful when using the logical operators with inequalities. For example, note that `~(x > y)` is equivalent to `y >= x`. It is *not* equivalent to `y > x`. As another example, the relation $5 < x < 10$ must be written as

```
(5 < x) & (x < 10)
```

in MATLAB.

The OR operator `|` returns 1's where at least one of `A` and `B` has nonzero elements, and 0's where both `A` and `B` are zero. The expression `z = 0|3` returns `z = 1`, the expression `z = 0|0` returns `z = 0`, and the expression

```
z = [5, -3, 0, 0]|[2, 4, 0, 5]
```

returns `z = [1, 1, 0, 1]`. Because of operator precedence, the expression `z = 3 < 5|4 == 7` is equivalent to `z = (3 < 5)|(4 == 7)`, which returns `z = 1`. Similarly, the expression `z = 1|0 & 1` is equivalent to `z = (1|0)&1`, which returns `z = 1`, while the expression `z = 1|0 & 0` returns `z = 0`, and the expression `z = 0 & 0|1` returns `z = 1`.

Because of the precedence of the NOT operator the statement `z = ~3 == 7|4 == 6` returns the result `z = 0`, which is equivalent to `z = ((~3) == 7)|(4 == 6)`.

Table C.4.4 is a so-called *truth table* that defines the operations of the logical operators, where "true" is equivalent to 1, and "false" is equivalent to 0. Let `x` and `y`

Table C.4.4 Truth table for logical operators.

x	y	~x	x|y	x&y
true	true	false	true	true
true	false	false	true	false
false	true	true	true	false
false	false	true	false	false

represent the first two columns of the truth table in terms of 1's and 0's. The following MATLAB session generates the truth table in terms of 1's and 0's.

```
>> x = [1, 1, 0, 0]'; y = [1, 0, 1, 0]';
>> Truth_Table = [x,y,~x,x|y,x&y]
Truth_Table =
   1  1  0  1  1
   1  0  0  1  0
   0  1  1  1  0
   0  0  1  0  0
```

THE `find` FUNCTION

The `find` function is very useful for creating decision-making programs, especially when combined with the relational and logical operators. The function `find(x)` computes an array containing the indices of the nonzero elements of the array x. For example, consider the session

```
>>x = [-2, 0, 4]; y = find(x)
y =
   1     3
```

The resulting array `y = [1, 3]` indicates that the first and third elements of x are nonzero. Note that the `find` function returns the *indices,* and not the *values*. In the following session, note the difference between the result obtained by `x(x<y)` and the result obtained by `find(x < y)`.

```
>>x = [6, 3, 9, 11];y = [14, 2, 9, 13];
>>values = x(x < y)
values =
   6    11
>>indices = find(x < y)
indices =
   1     4
```

Thus, there are two values in the array x that are less than the *corresponding* values in the array y. They are the first and fourth values, which are 6 and 11.

The logical operators can be used with the `find` function. For example, consider the session

```
>>x = [5, -3, 0, 0, 8]; y = [2, 4, 0, 5, 7];
>>z = find(x & y)
z =
   1    2    5
>>values = y(x & y)
values =
   2    4    7
```

The array obtained with the `find` function, `z = [1, 2, 5]`, indicates that the first, second, and fifth elements of `x` and `y` are both nonzero. The array `values = [2, 4, 7]`, which is obtained from `y(x & y)`, shows that there are three nonzero values in `y` that correspond to nonzero values in `x`. These are 2, 4, and 7, the first, second, and fifth values.

LOGICAL ARRAYS

Logical variables can have only the values 1 (true) and 0 (false), and a logical array contains logical values. However, just because an array contains only 0's and 1's, it is not necessarily a logical array. For example, in the following session `k` and `w` appear the same, but `k` is a logical array and `w` is a numeric array, and thus an error message is issued.

```
>> x = [-2:2]
x =
   -2   -1    0    1    2
>> k = (abs(x)>1)
k =
     1    0    0    0    1
>> z = x(k)
z =
   -2    2
>> w = [1,0,0,0,1];
>> v = x(w)
??? Subscript indices must either be real positive...
   integers or logicals.
```

Logical arrays can be created with the relational and logical operators and with the `logical` function. The `logical` function returns an array that can be used for logical indexing and logical tests. Typing `B = logical(A)`, where `A` is a numeric array, returns the logical array `B`. So to correct the error in the previous session, you can type instead `w = logical([1,0,0,0,1])`.

You can use the `double` function to convert a logical array to an array of numerical 0's and 1's of class `double`. For example, `x = (5>3); y = double(x);`. In addition, most arithmetic operations convert a logical array to a double array.

ACCESSING ARRAYS

Typing `A(B)`, where `B` is a logical array of the same size as `A`, returns the values of `A` at the indices where `B` is nonzero. Given `A = [5,6,7;8,9,10;11,12,13]` and `B = logical(eye(3))`, we can extract the diagonal elements of `A` by typing `C = A(B)` to obtain `C = [5;9;13]`. Specifying array subscripts with logical arrays extracts the elements that correspond to the true (1) elements in the logical array.

Note, however, that using the *numeric* array `eye(3)`, as `C = A(eye(3))` results in an error message because the elements of `eye(3)` do not correspond to locations in `A`. If the numeric array values correspond to valid locations, you may use a numeric array to extract the elements. For example, to extract the diagonal elements of `A` with a numeric array, type `C = A([1,5,9])`.

CONDITIONAL STATEMENTS

The MATLAB *conditional* statements enable us to write programs that make decisions. Conditional statements contain one or more of the `if`, `else`, and `elseif` statements.

The end statement is used to denote the end of a conditional statement. These conditional statements read somewhat like their English language equivalents. For example, suppose that x is a scalar, and that we want to compute $y = \sqrt{x}$ only if $x \geq 0$. In English, we could specify this as: "If x is greater than or equal to zero, compute y from $y = \sqrt{x}$, otherwise, do nothing." The if statement in the next script file accomplishes this in MATLAB, assuming that the variable x already has a scalar value.

```
if x >= 0
   y = sqrt(x)
end
```

If x is negative, the program takes no action.

When more than one action must occur as a result of a decision, we can use the else and elseif statements. The statements after the else are executed if all the preceding if and elseif expressions are false. The general form of the if statement is

```
if logical expression
   statement group 1
elseif logical expression
   statement group 2
else
   statement group 3
end
```

There can be more than one elseif statement. The else and elseif statements can be omitted if not required. However, if both are used, the else statement must come after the elseif statements to take care of all conditions that might be unaccounted for.

Suppose that we want to compute y from $y = \sqrt{x}$ for $x \geq 0$, and $y = -\sqrt{-x}$ for $x < 0$. The next script file will calculate y, assuming that the variable x already has a scalar value.

```
if x >= 0
   y = sqrt(x)
else
   y = -sqrt(-x)
end
```

As another example, suppose that we want to compute y such that $y = 0$ if $x < 0$, $y = 10x$ if $0 \leq x < 25$, and $y = 50\sqrt{x}$ if $x \geq 25$. The next script file will compute y, assuming that the variable x already has a scalar value.

```
if x >= 25
   y = 50*sqrt(x)
elseif x >= 0
   y = 10*x
else
   y = 0
end
```

Note that the elseif statement does not require a separate end statement.

When the test, if *logical expression,* is performed, where the logical expression may be an array, the test returns a value of true only if *all* the elements of the logical

expression are true! For example, if we fail to grasp this, the following statements do not perform the way we might expect.

```
x = [4, -9, 25];
if x < 0
   disp('Some of the elements of x are negative.')
else
   y = sqrt(x)
end
```

When this program is run it gives the result $y = 2, 0 + 3.000i, 5$. The program does not compute the square root of each element in x in sequence. Instead it tests the truth of the relation $x < 0$. The test if $x < 0$ returns a false value because it generates the vector $[0, 1, 0]$. Compare the last program with the following statements:

```
x=[4, -9, 25];
if x >= 0
   y = sqrt(x)
else
   disp('Some of the elements of x are negative.')
end
```

When executed, it produces the result: Some of the elements of x are negative. The difference is due to the different conditional statements used. Here the test, if $x < 0$, is true, whereas the test, if $x >= 0$, returns a false value because $x >= 0$ returns the vector $[1, 0, 1]$.

Conditional structures can be nested; that is, one structure can contain another structure.

THE `for` LOOP

A *loop* is a structure for repeating a calculation a number of times. Each repetition of the loop is a *pass*. There are two types of explicit loops in MATLAB: the `for` loop, which is used when the number of passes is known or can be computed ahead of time, and the `while` loop, which is used when the looping process must terminate when a specified condition is satisfied, and thus the number of passes is not known or computable in advance.

A simple example of a `for` loop is

```
m = 0;
x(1) = 10;
for k = 2:3:11
   m = m+1;
   x(m+1) = x(m) + k^2;
end
```

The *loop variable* k is initially assigned the value 2. During each successive pass through the loop k is incremented by 3, and x is calculated until k exceeds 11. Thus, k takes on the values 2, 5, 8, and 11. The variable m indicates the index of the array x. The program then continues to execute any statements following the `end` statement. When the loop is finished the array x will have the values $x(1)=14, x(2)=39, x(3)=103, x(4)=224$. The name of the loop variable need not be k.

Note the following rules when using `for` loops with the loop variable expression `k = m:s:n:`

1. The step value `s` can be negative. For example, `k = 10:-2:4` produces k = 10, 8, 6, 4.
2. If `s` is omitted, the step value defaults to 1.
3. If `s` is positive, the loop will not be executed if `m` is greater than `n`.
4. If `s` is negative, the loop will not be executed if `m` is less than `n`.
5. If `m` equals `n`, the loop will be executed only once.
6. If the step value `s` is not an integer, roundoff errors can cause the loop to execute a different number of passes than intended.

You should not alter the value of the loop variable `k` within the *statements*. Doing so can cause unpredictable results.

THE `while` LOOP

The `while` loop is used in cases where the looping process must terminate when a specified condition is satisfied, and thus the number of passes is not known or computable in advance. The basic structure of a `while` loop is the following.

```
while logical expression
    statement group
end
```

MATLAB first tests the truth of the *logical expression,* which must contain a loop variable. If the *logical expression* is true, the statements in the *statement group* will be executed. For the `while` loop to function properly, the following two conditions are required:

1. The loop variable must have a value before the `while` statement is executed.
2. The loop variable must be changed somehow by the statements in the *statement group*.

The statements in the *statement group* are executed once during each pass, using the current value of the loop variable. The looping continues until the *logical expression* is false.

A simple example of a `while` loop is

```
x = 5; k = 0;
while x < 25
   k = k + 1;
   y(k) = 3*x;
   x = 2*x - 1;
end
```

The loop variable `x` is initially assigned the value 5, and it has this value until the statement `x = 2*x - 1` is encountered the first time. Its value then changes to 9. Before each pass through the loop, `x` is checked to see if its value is less than 25. If so, the pass is made. If not, the loop is skipped and the program continues to execute any statements following the `end` statement. The variable `x` takes on the values 9, 17, and 33 within the loop. The resulting array `y` contains the values `y(1) = 15`, `y(2) = 27`, and `y(3) = 51`.

Every `for` and `while` statement must be matched by an accompanying `end` statement. As with conditional statements, loops can be nested.

C.5 THE MATLAB HELP SYSTEM

Two principal ways to obtain more information about MATLAB commands and features are to use the help commands in the Command window and to access the Help Browser. The help commands can be used to display syntax information for a specified topic in the Command window. The Help Browser is a graphical user interface for searching and viewing documentation. For additional help, you can run demos, contact technical support, search documentation for other MathWorks products, view a list of other books, and participate in a newsgroup.

HELP COMMANDS

Four MATLAB commands are useful for obtaining online information about MATLAB topics.

- ■   Typing `help topic` displays in the Command window a description of the specified `topic`.
- ■   Typing `helpwin topic` displays the help text for the specified `topic` inside the Help Browser window.
- ■   Typing `lookfor topic` displays in the Command window a brief description for all items whose description includes the specified keyword `topic`.
- ■   Typing `doc topic` opens the Help Browser to the reference page for the specified topic, providing a description, additional remarks, and examples.

The `help` command is the basic way to determine the syntax of a particular command, function, or statement. For example, typing `help log10` produces the following display:

```
LOG10 Common (base 10) logarithm.
   LOG10(X) is the base 10 logarithm of the elements of X.
   Complex results are produced if X is not positive.

   See also LOG, LOG2, EXP, LOGM.
```

Note that the display describes what the function does, warns about any unexpected results if nonstandard argument values are used, and directs the user to other related functions.

All the MATLAB functions are organized into groups, and the MATLAB directory structure is based on this grouping. For instance, all elementary mathematical functions such as `log10` reside in the `elfun` directory, and the polynomial functions reside in the `polyfun` directory. To list the names of all the functions in that directory, with a brief description of each, type `help polyfun`. If you are unsure of what directory to search, type `help` to obtain a list all the directories, with a description of the function category each represents.

Typing `helpwin topic` displays the help text for the specified `topic` inside the Help Browser window. Links are created to functions referenced in the "See also" line of the help text. You can also access the Help Window by selecting the **Help** option under the **Help** menu, or by clicking the question mark button on the toolbar.

The `lookfor` command enables you to search for topics based on a keyword. It searches through the first line of help text, which is known as the H1 line, for each MATLAB function, and returns all the H1 lines containing a specified keyword. For example, MATLAB does not have a function named `sine`. So the response from `help sine` is `sine.m not found`. However, typing `lookfor sine` produces over a dozen matches, depending on which toolboxes you have installed. For example, some of the matches you will see are

```
ACOS      Inverse cosine.
ACOSH     Inverse hyperbolic cosine.
ASIN      Inverse sine.
ASINH     Inverse hyperbolic sine.
COS       Cosine.
COSH      Hyperbolic cosine.
SIN       Sine.
SINH      Hyperbolic sine.
```

From this list you can find the correct name for the sine function. Note that all words containing sine are returned, such as cosine. Adding `-all` to the lookfor command searches the entire help entry, not just the H1 line.

Typing `doc topic` displays the documentation for the topic in the Help Browser. For example, typing `doc step` displays information about the function `step`. Typing `doc toolbox` displays the documentation road map page for the specified toolbox. For example, typing `doc control` displays information about the Control Systems toolbox.

THE HELP BROWSER

To open the Help Browser, select **Matlab Help** from the **Help** menu on the Desktop, or click the question mark button in the toolbar. The Help Browser contains two window "panes": the Help Navigator pane and the Display pane. The Help Navigator contains four tabs:

- **Contents:** a contents listing tab,
- **Index:** a global index tab,
- **Search:** a search tab having a find function and full text search features, and
- **Demos:** a tab for running built-in demonstrations of various MATLAB features.

Use the tabs in the Help Navigator to find documentation, which can then be viewed in the Display pane.

While viewing a page of documentation, you can scroll to see contents not currently visible in the window and navigate page by page forward or backward by clicking on the appropriate arrows.

Click the **Contents** tab in the Help Navigator to list the titles and table of contents for all product documentation. Click on an item to select it. The first page of that document appears in the display pane. The **Contents** pane is synchronized with the Display pane so that the item selected in the **Contents** pane always matches the documentation appearing in the Display pane.

Click the **Index** tab in the Help Navigator pane to find specific index entries (keywords) from all of your MathWorks documentation. Type a word or words in the "Search index for" box. As you type, the index displays matching entries. Click on an entry to display the corresponding documentation page.

Click the **Search** tab in the Help Navigator pane to find all MATLAB documents containing a specified phrase. Type the phrase in the "Search for" box. Then click the **Go** button. The list of documents and the heading under which the phrase is found in that document then appear in the Help Navigator pane. Select an entry from the list of results to view that document in the Display pane.

MATHWORKS WEBSITE

The MathWorks maintains a very useful website that you can use to ask questions, make suggestions, and report possible bugs. You can also search for solutions by querying an up-to-date database of technical support information. You can access the website by clicking on **Web Resources** under the **Help** menu in the Desktop.

REFERENCES

[Palm, 2005] W. J. Palm III *Introduction to MATLAB 7 for Engineers*, McGraw-Hill, New York, 2005.

[Palm, 2008] W. J. Palm III *A Concise Introduction to MATLAB*, McGraw-Hill, New York, 2008.

PROBLEMS

C.1   Assuming that the variables a, b, c, d, and f are scalars, write MATLAB statements to compute and display the following expressions. Test your statements for the values $a = 1.12$, $b = 2.34$, $c = 0.72$, $d = 0.81$, and $f = 19.83$.

$$x = 1 + \frac{a}{b} + \frac{c}{f^2} \qquad\qquad s = \frac{b-a}{d-c}$$

$$r = \frac{1}{1/a + 1/b + 1/c + 1/d} \qquad y = ab\frac{1}{c}\frac{f^2}{2}$$

C.2   Suppose that $x = -7 - 5j$ and $y = 4 + 3j$. Use MATLAB to compute
a. $x + y$ b. xy c. x/y d. $\frac{3}{2}j$ e. $\frac{3}{2j}$

C.3   Suppose x takes on the values $x = 1, 1.2, 1.4, \ldots, 5$. Use MATLAB to compute the array y that results from the function $y = 7 \sin(4x)$. Use MATLAB to determine how many elements are in the array y, and the value of the third element in the array y.

C.4   The following table shows the hourly wages, hours worked, and output (number of widgets produced) in 1 week for five widget makers.

	Worker				
	1	**2**	**3**	**4**	**5**
Hourly wage ($)	5	5.50	6.50	6	6.25
Hours worked	40	43	37	50	45
Output (widgets)	1000	1100	1000	1200	1100

Use MATLAB to answer these questions:
a.   How much did each worker earn in the week?
b.   What is the total salary amount paid out?
c.   How many widgets were made?

d. What is the average cost to produce one widget?

e. How many hours does it take to produce one widget, on average?

f. Assuming that the output of each worker has the same quality, which worker is the most efficient? Which is the least efficient?

C.5 Write a MATLAB assignment statement for each of the following functions, assuming that w, x, y, and z are vector quantities of equal length, and that c and d are scalars.

$$f = \frac{1}{\sqrt{(2\pi c)/x}} \qquad E = \frac{x + w/(y+z)}{x + w/(y-z)}$$

$$A = \frac{e^{-c/(2x)}}{(\ln y) \sqrt{dz}} \qquad S = \frac{x(2.15 + 0.35y)^{1.8}}{z(1-x)^y}$$

C.6 The following tables show the costs associated with a certain product, and the production volume for the four quarters of the business year. Use MATLAB to find (a) the quarterly costs for materials, labor, and transportation; (b) the total material, labor, and transportation costs for the year; and (c) the total quarterly costs.

	Unit product costs $\left(\$ \times 10^3\right)$		
Product	**Materials**	**Labor**	**Transportation**
1	7	3	2
2	3	1	3
3	9	4	5
4	2	5	4
5	6	2	1

	Quarterly production volume			
Product	**First quarter**	**Second quarter**	**Third quarter**	**Fourth quarter**
1	16	14	10	12
2	12	15	11	13
3	8	9	7	11
4	14	13	15	17
5	13	16	12	18

C.7 Use MATLAB to find the roots of $13s^3 + 182s^2 - 184s + 2503 = 0$, and use `poly` to confirm your answer.

C.8 Use MATLAB to find the roots of the polynomial $36s^3 + 12s^2 - 5s + 10$. Use the `polyval` function to verify the solution.

C.9 Use MATLAB to find the polynomial whose roots are $3 \pm 6j$, 8, 8, and 20. Use MATLAB to confirm your answer.

C.10 Use MATLAB to find the following product:

$$\left(10s^3 - 9s^2 - 6s + 12\right)\left(5s^3 - 4s^2 - 12s + 8\right)$$

C.11 Use MATLAB to find the quotient and remainder of

$$\frac{14s^3 - 6s^2 + 3s + 9}{5s^2 + 7s - 4}$$

C.12   Use MATLAB to plot the functions $u = 2 \log_{10} (60x + 1)$ and $v = 3 \cos(6x)$ over the interval $0 \leq x \leq 2$. Properly label the plot and each curve. The variables u and v represent speed in miles per hour; the variable x represents distance in miles.

C.13   Use MATLAB to plot the polynomials $y = 3x^4 - 6x^3 + 8x^2 + 4x + 90$ and $z = 3x^3 + 5x^2 - 8x + 70$ over the interval $-3 \leq x \leq 3$. Properly label the plot and each curve. The variables y and z represent current in milliamps; the variable x represents voltage in volts.

C.14   Write a script file using conditional statements to evaluate the following function, assuming that the scalar variable x has a value. The function is $y = e^{x+1}$ for $x < -1$, $y = 2 + \cos(\pi x)$ for $-1 \leq x < 5$, and $y = 10(x - 5) + 1$ for $x \geq 5$. Use your file to evaluate y for $x = -5$, $x = 3$, and $x = 15$, and check the results by hand.

C.15   Suppose that `x = [-15,-8,9,8,5]` and `y = [-20,12,-4,8,9]`. What is the result of the following operations? Determine the answers by hand, and then use MATLAB to check your answers.

   a.   `z = (x < y)`

   b.   `z = (x > y)`

   c.   `z = (x ~= y)`

   d.   `z = (x == y)`

   e.   `z = (y > -4)`

C.16   The given arrays `price_A`, `price_B` and `price_C` contain the price in dollars of three stocks over ten days.

   a.   Use MATLAB to determine how many days the price of stock A was above both the price of stock B and the price of stock C.

   b.   Use MATLAB to determine how many days the price of stock A was above either the price of stock B or the price of stock C.

   c.   Use MATLAB to determine how many days the price of stock A was above either the price of stock B or the price of stock C, but not both.

```
price_A = [19, 18, 22, 21, 25, 19, 17, 21, 27, 29]
price_B = [22, 17, 20, 19, 24, 18, 16, 25, 28, 27]
price_C = [17, 13, 22, 23, 19, 17, 20, 21, 24, 28]
```

C.17   The price, in dollars, of a certain stock over a ten day period is given in the following array.

```
price = [19, 18, 22, 21, 25, 19, 17, 21, 27, 29]
```

Suppose you owned 1000 shares at the start of the ten day period, and you bought 100 shares every day the price was below $20, and sold 100 shares every day the price was above $25. Use MATLAB to compute (a) the amount you spent in buying shares, (b) the amount you received from the sale of shares, (c) the total number of shares you own after the tenth day, and (d) the net increase in the worth of your portfolio of stock.

C.18   Use a `for` loop to determine the sum of the first 10 terms in the series $5k^3$, $k = 1, 2, 3, \ldots, 10$.

C.19   Use a `while` loop to determine how many terms in the series 2^k, $k = 1, 2, 3, \ldots$, are required for the sum of the terms to exceed 2000. What is the sum for this number of terms?

C.20   One bank pays 5.5% annual interest, while a second bank pays 4.5% annual interest. Determine how much longer it will take to accumulate at least $50,000 in the second bank account if you deposit $1000 initially, and $1000 at the end of each year.

C.21   Given a number x and the quadrant q ($q = 1, 2, 3, 4$), write a program to compute $\sin^{-1}(x)$ in degrees, taking into account the quadrant. The program should display an error message if $|x| > 1$.

C.22   The function $y = 1 + e^{-0.2x} \sin(x + 2)$ has two minimum points in the interval $0 < x < 10$. Create a user-defined function and use the `fminbnd` function to find the values of x and y at each minimum.

C.23   An object thrown vertically with a speed v_0 will reach a height h at time t, where

$$h = v_0 t - \frac{1}{2} g t^2$$

Write and test a function that computes the time t required to reach a specified height h, for a given value of v_0. The function's inputs should be h, v_0, and g. Test your function for the case where $h = 100$ m, $v_0 = 50$ m/s, and $g = 9.81$ m/s^2. Interpret both answers.

D

Numerical Methods

D.1 INTRODUCTION TO NUMERICAL ALGORITHMS

Numerical methods for solving differential equations can be applied to both linear and nonlinear models, although their primary application is to nonlinear models. A nonlinear ordinary differential equation can be recognized by the fact that the dependent variable or its derivatives appears raised to a power or in a transcendental function. For example, the following equations are nonlinear.

$$y\ddot{y} + 5\dot{y} + y = 0 \qquad \dot{y} + \sin y = 0 \qquad \dot{y} + \sqrt{y} = 0$$

The algorithms presented here are simplified versions of the ones used by MATLAB and Simulink, and so an understanding of these methods will improve your understanding of these two programs. The numerical solution algorithms used by the MATLAB ODE solvers are very complicated. Therefore, we will limit our coverage to simple algorithms so that we can highlight the important issues to be considered when using numerical methods.

Numerical methods require that the derivatives in the model be described by finite-difference expressions, and that the resulting difference equations be solved in a step-by-step procedure. The issues related to these methods are

■   What finite-difference expressions provide the best approximations for derivatives?
■   What are the effects of step size used to obtain the approximations?
■   What are the effects of roundoff error when solving the finite-difference equations?

We will explore these issues and make you aware of some pitfalls that can be encountered. Such difficulties are more likely to occur when the solution is rapidly changing with time, and can happen if the step size is not small compared to the smallest time constant of the system or the smallest oscillation period.

TEST CASES

We now develop three test cases, whose solution can be found analytically, to use for checking the results of our numerical methods.

1. The following equation is used to illustrate the effect of step size relative to the system time constant, which is $\tau = 1/10$.

    $$\frac{dy}{dt} + 10y = 0 \qquad y(0) = 2$$

    The solution is $y(t) = 2e^{-10t}$.

2. The following equation is used to illustrate the effect of step size relative to the solution's oscillation period.

    $$\frac{dy}{dt} = \sin \omega t \qquad y(0) = y_0$$

    The solution is

    $$y(t) = y_0 + \frac{1 - \cos \omega t}{\omega}$$

    and the period is $2\pi/\omega$.

3. A *stiff* differential equation is one whose response changes rapidly over a time scale that is short compared to the time scale over which we are interested in the solution. For this reason, stiff equations present a challenge to solve numerically. The following equation has such characteristics.

    $$\dot{y} + y = 0.001e^{10t} \qquad y(0) = 10$$

    The solution is

    $$y(t) = 10e^{-t} + \frac{0.001}{11}\left(e^{10t} - e^{-t}\right)$$

    That part of the response due to the term e^{-t} is approximately 10 at $t = 0$ and decays quickly (it is approximately 0.2 at $t = 4$). However, the term due to e^{10t} is 9.09×10^{-5} at $t = 0$ but grows at a fast rate (it is 2.14×10^{13} at $t = 4$!). Thus it would be difficult for a plot to show the solution accurately over the range $0 \le t \le 4$. More importantly, a numerical solver would need a very small step size to compute the rapid changes due to the e^{10t} term, much smaller than the step size required to compute the slower response due to the e^{-t} term. The result can be a large accumulated error because of the small step size combined with the large number of steps required to obtain the full solution.

THE EULER METHOD

The essence of a numerical method is to convert the differential equation into a difference equation that can be programmed on a calculator or digital computer. Numerical algorithms differ partly as a result of the specific procedure used to obtain the difference

equations. In general, as the accuracy of the approximation is increased, so is the complexity of the programming involved. It is important to understand the concept of "step size" and its effects on solution accuracy. To provide a simple introduction to these issues, we begin with the simplest numerical method, the *Euler method*.

Consider the equation

$$\frac{dy}{dt} = f(t, y) \qquad\qquad\qquad (D.1.1)$$

where $f(t, y)$ is a known function. From the definition of the derivative,

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

If the time increment Δt is chosen small enough, the derivative can be replaced by the approximate expression

$$\frac{dy}{dt} \approx \frac{y(t + \Delta t) - y(t)}{\Delta t} \qquad\qquad\qquad (D.1.2)$$

Assume that the right-hand side of (D.1.1) remains constant over the time interval $(t, t + \Delta t)$, and replace (D.1.1) by the following approximation:

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} = f(t, y)$$

or

$$y(t + \Delta t) = y(t) + f(t, y)\Delta t \qquad\qquad\qquad (D.1.3)$$

The smaller Δt is, the more accurate are our two assumptions leading to (D.1.3). This technique for replacing a differential equation with a difference equation is the *Euler method*. The increment Δt is called the *step size*.

A more concise representation is obtained by using the following notation:

$$y_k = y(t_k) \qquad y_{k+1} = y(t_{k+1}) = y(t_k + \Delta t)$$

where $t_{k+1} = t_k + \Delta t$. In this notation, the Euler algorithm (D.1.3) is expressed as

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t \qquad\qquad\qquad (D.1.4)$$

The Euler Method and a Decaying Solution | **EXAMPLE D.1.1**

■ Problem
Use the Euler method to solve our first test case,

$$\frac{dy}{dt} + 10y = 0 \qquad y(0) = 2$$

which has the exact solution $y(t) = 2e^{-10t}$. Use a step size of $\Delta t = 0.02$, which is one-fifth of the time constant.

■ Solution
Here $f(t, y) = -10y$. Thus the Euler algorithm (D.1.4) in this case becomes

$$y_{k+1} = y_k - (10y_k)\Delta t$$

or

$$y_{k+1} = y_k - (10y_k)0.02 = 0.8y_k$$

We show the computations for the first few steps, using four significant figures. The exact value, obtained from $y(t) = 2e^{-10t}$, and the percent error are shown in the following table.

Step	Numerical solution	Exact solution	Percent error
$k = 0$	$y_1 = 0.8y_0 = 1.6$	1.637	2.3%
$k = 1$	$y_2 = 0.8y_1 = 1.28$	1.341	4.5%
$k = 2$	$y_3 = 0.8y_2 = 1.024$	1.098	6.7%
$k = 3$	$y_4 = 0.8y_3 = 0.8192$	0.8987	8.8%
$k = 4$	$y_5 = 0.8y_4 = 0.6554$	0.7358	10.9%
$k = 5$	$y_6 = 0.8y_5 = 0.5243$	0.6024	13%

Notice how the percent error grows with each step. This is because the calculated result from the previous step is not exact. The numerical and exact solutions are shown in Figure D.1.1, where the numerical solution is shown by the small circles and the exact solution is shown by the solid curve.

Another observation here is that the step size should be much smaller than the time constant τ. A commonly used rule of thumb is that $\Delta t \leq \tau/20$.

Figure D.1.1 Euler and exact solutions of $\dot{y} + 10y = 0$, $y(0) = 2$.
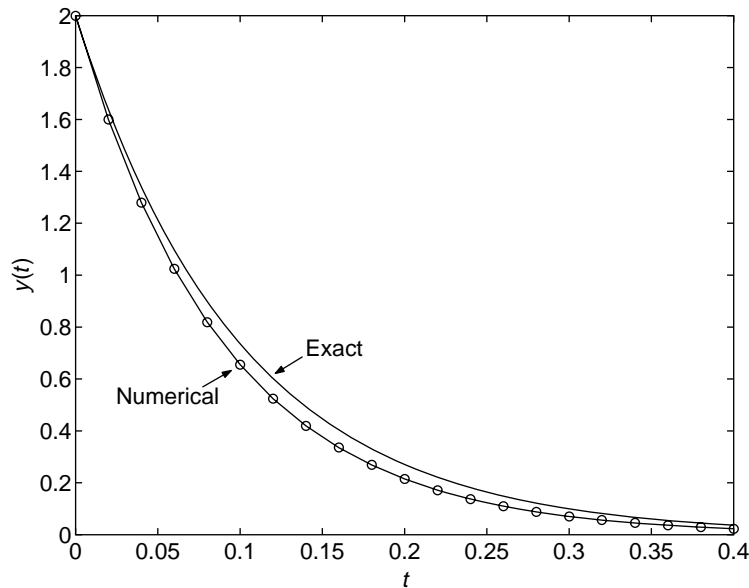
Roundoff Error   There is another reason why the error increases with the number of steps. If we had retained six significant figures instead of four, we would have obtained $y_5 = 0.65536$ and $y_6 = 0.524288$. Even though computers can retain many more than four significant figures, nevertheless, they cannot represent numbers with infinite accuracy. Thus, the calculated solution obtained by computer at each step is in effect rounded off to a finite number of significant figures. This rounded number is then used in the calculations for the next step, and so on, just as we rounded the value of y_5 to 0.6554 before using it to compute y_6. Therefore, the error in the numerical solution will increase with the number of steps required to obtain the solution.

Thus, because roundoff error increases with each step, there is a trade-off between using a step size small enough to obtain an accurate solution yet not so small that many steps are required to obtain the solution.

Numerical methods have their greatest errors when trying to obtain solutions that are rapidly changing. The difficulties caused by an oscillating solution are illustrated in the following example.

The Euler Method and an Oscillating Solution │ **EXAMPLE D.1.2**

■ **Problem**

Consider the following equation, which is our second test case.

$$\dot{y} = \sin t$$

for $y(0) = 0$ and $0 \leq t \leq 4\pi$. The exact solution is $y(t) = 1 - \cos t$ and its period is 2π. Solve this equation with Euler's method.

■ **Solution**

We choose a step size equal to one-thirteenth of the period, or $\Delta t = 2\pi/13$, so that we can compare the answer with that obtained by a method to be introduced later. The Euler algorithm (D.1.4) becomes

$$y_{k+1} = y_k + (\sin t_k)\frac{2\pi}{13}$$

For successive values of $k = 0, 1, 2, \ldots$, we have $t_k = 0, 2\pi/13, 4\pi/13, \ldots$. Retaining four significant figures, we have

$$y_1 = y_0 + (\sin t_0)\frac{2\pi}{13} = 0 + (\sin 0)\frac{2\pi}{13} = 0$$

$$y_2 = y_1 + (\sin t_1)\frac{2\pi}{13} = 0 + \left(\sin\frac{2\pi}{13}\right)\frac{2\pi}{13} = 0.2246$$

$$y_3 = y_2 + (\sin t_2)\frac{2\pi}{13} = 0.2246 + \left(\sin\frac{4\pi}{13}\right)\frac{2\pi}{13} = 0.6224$$

and so on. The numerical and exact solutions are shown in Figure D.1.2, where the numerical solution is shown by the small circles and the exact solution is shown by the solid curve. There is noticeable error, especially near the peaks and valleys, where the solution is rapidly changing.
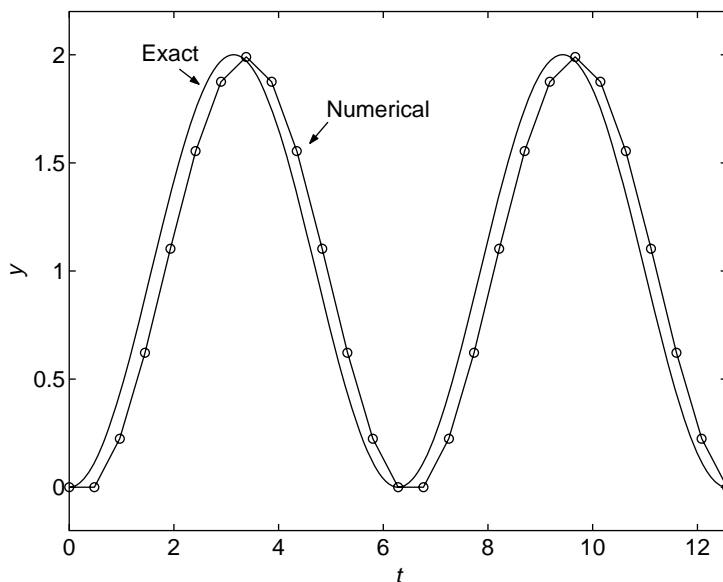
Figure D.1.2 Euler and exact solutions of $\dot{y} = \sin t$, $y(0) = 0$.

The accuracy of the Euler method can be improved by using a smaller step size. However, very small step sizes require longer run times and can result in a large accumulated error due to roundoff effects. So we seek better algorithms to use for more challenging applications.

PREDICTOR-CORRECTOR METHODS

We now consider *predictor-corrector* methods, which serve as the basis for many powerful algorithms. The Euler method can have a serious deficiency in problems where the variables are rapidly changing, because the method assumes the variables are constant over the time interval Δt. One way of improving the method is to use a better approximation to the right-hand side of the equation

$$\frac{dy}{dt} = f(t, y) \tag{D.1.5}$$

The Euler approximation is

$$y(t_{k+1}) = y(t_k) + \Delta t\, f[t_k, y(t_k)] \tag{D.1.6}$$

Suppose instead we use the average of the right-hand side of (D.1.5) on the interval (t_k, t_{k+1}). This gives

$$y(t_{k+1}) = y(t_k) + \frac{\Delta t}{2}\,(f_k + f_{k+1}) \tag{D.1.7}$$

where

$$f_k = f[t_k, y(t_k)] \tag{D.1.8}$$

with a similar definition for f_{k+1}. Equation (D.1.7) is equivalent to integrating (D.1.5) with the trapezoidal rule, whereas the Euler method is equivalent to integrating with the rectangular rule (see Figure D.1.3).

The difficulty with (D.1.7) is that f_{k+1} cannot be evaluated until $y(t_{k+1})$ is known, but this is precisely the quantity being sought. A way out of this difficulty is to use the Euler formula (D.1.6) to obtain a preliminary estimate of $y(t_{k+1})$. This estimate is then used to compute f_{k+1} for the use in (D.1.7) to obtain the required value of $y(t_{k+1})$.

Figure D.1.3
Illustration of numerical integration by (a) the rectangular rule and (b) the trapezoidal rule.
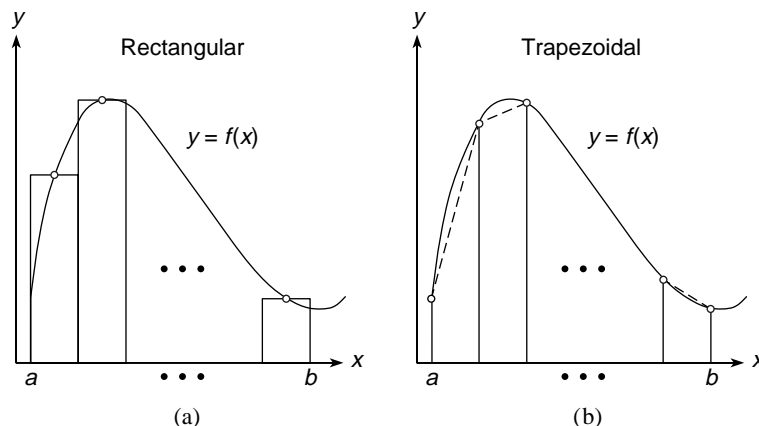
The notation can be changed to clarify the method. Let $h = \Delta t$ and $y_k = y(t_k)$, and let x_{k+1} be the estimate of $y(t_{k+1})$ obtained from the Euler formula (D.1.6). Then, by omitting the t_k notation from the other equations, we obtain the following description of the predictor-corrector process:

$$\text{Euler predictor:} \quad x_{k+1} = y_k + h\, f(t_k, y_k) \tag{D.1.9}$$

$$\text{Trapezoidal corrector:} \quad y_{k+1} = y_k + \frac{h}{2}\,[f(t_k, y_k) + f(t_{k+1}, x_{k+1})] \tag{D.1.10}$$

This version of a predictor-corrector algorithm is sometimes called the *modified-Euler method*. However, note that any algorithm can be tried as a predictor or a corrector. Thus many other methods can also be classified as predictor-corrector.

The Modified-Euler Method and a Decaying Solution | **EXAMPLE D.1.3**

■ Problem
Use the modified-Euler method to solve our first test case:

$$\dot{y} = -10y \qquad y(0) = 2$$

for $0 \le t \le 0.5$. The exact solution is $y(t) = 2e^{-10t}$.

■ Solution
To illustrate the effect of the step size on the solution's accuracy, we use a step size $h = 0.02$, the same size used with the Euler method. The modified-Euler algorithm for this case has the following form.

$$x_{k+1} = y_k + h(-10y_k) = (1 - 10h)y_k = 0.8y_k$$

$$y_{k+1} = y_k + \frac{h}{2}(-10y_k - 10x_{k+1}) = 0.9y_k - 0.1x_{k+1}$$
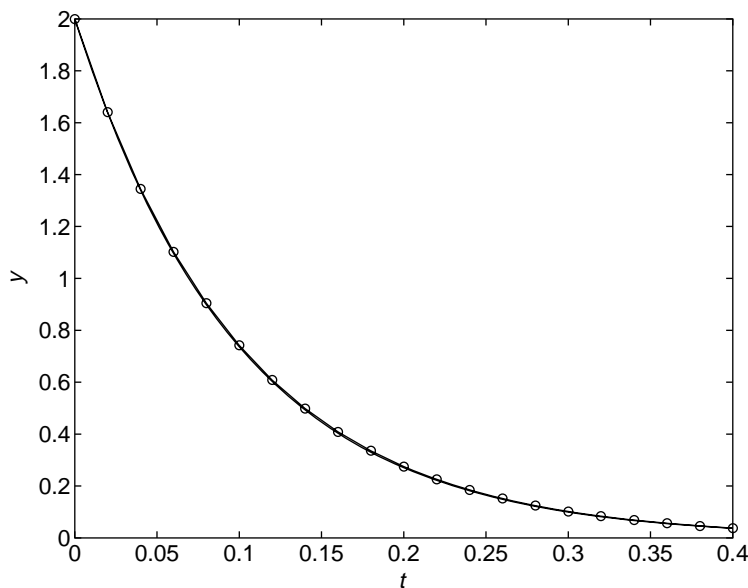
Figure D.1.4 Modified-Euler and exact solutions of $\dot{y} + 10y = 0$, $y(0) = 2$.

The following table shows the numerical and exact solutions, rounded to four significant figures, and the percent error, for a few steps.

Step	Numerical solution	Exact solution	Percent error
$k = 0$	$x_1 = 0.8y_0 = 1.6$		
	$y_1 = 0.9y_0 - 0.1x_1 = 1.64$	1.637	0.2%
$k = 1$	$x_2 = 0.8y_1 = 1.312$		
	$y_2 = 0.9y_1 - 0.1x_2 = 1.3448$	1.341	0.3%
$k = 2$	$x_3 = 0.8y_2 = 1.07584$		
	$y_3 = 0.9y_2 - 0.1x_3 = 1.102736$	1.098	0.4%

There is less error than with the Euler method using the same step size. Figure D.1.4 shows the results, with the numerical solution shown by the small circles and the exact solution by the solid line.

The modified-Euler method is a special case of the Runge-Kutta family of algorithms, to be discussed next. For purposes of comparison with the Runge-Kutta methods, we can express the modified-Euler method as follows:

$$y_{k+1} = y_k + \frac{1}{2}(g_1 + g_2) \qquad (D.1.11)$$

$$g_1 = hf(t_k, y_k) \qquad (D.1.12)$$

$$g_2 = hf(t_k + h, y_k + g_1) \qquad (D.1.13)$$

RUNGE-KUTTA METHODS

The Taylor series representation forms the basis of several methods for solving differential equations, including the Runge-Kutta methods. The Taylor series may be used to represent the solution $y(t + h)$ in terms of $y(t)$ and its derivatives, as follows:

$$y(t + h) = y(t) + h\,\dot{y}(t) + \frac{1}{2}h^2\ddot{y}(t) + \cdots \qquad (D.1.14)$$

The number of terms kept in the series determines its accuracy. The required derivatives are calculated from the differential equation. If these derivatives can be found, (D.1.14) can be used to march forward in time. In practice, the high-order derivatives can be difficult to calculate, and the series (D.1.14) is truncated at some term. The Runge-Kutta methods were developed because of the difficulty in computing the derivatives. These methods use several evaluations of the function $f(t, y)$ in a way that approximates the Taylor series. The number of terms in the series that is duplicated determines the order of the Runge-Kutta method. Thus, a fourth-order Runge-Kutta algorithm duplicates the Taylor series through the term involving h^4.

SECOND-ORDER RUNGE-KUTTA METHODS

The second-order Runge-Kutta methods express y_{k+1} as

$$y_{k+1} = y_k + w_1 g_1 + w_2 g_2 \qquad (D.1.15)$$

where w_1 and w_2 are constant weighting factors, and

$$g_1 = hf(t_k, y_k) \qquad (D.1.16)$$

$$g_2 = hf(t_k + \alpha h, y_k + \beta h f_k) \qquad (D.1.17)$$

The family of second-order Runge-Kutta algorithms is categorized by the parameters α, β, w_1, and w_2. To duplicate the Taylor series through the h^2 term, these coefficients must satisfy the following:

$$w_1 + w_2 = 1 \qquad w_1\alpha = \frac{1}{2} \qquad w_2\beta = \frac{1}{2} \qquad \text{(D.1.18)}$$

Thus one of the parameters can be chosen independently.

The modified-Euler algorithm, (D.1.11) through (D.1.13), is thus seen to be a second-order Runge-Kutta algorithm with $\alpha = \beta = 1$ and $w_1 = w_2 = 1/2$.

FOURTH-ORDER RUNGE-KUTTA METHODS

The family of fourth-order Runge-Kutta algorithms expresses y_{k+1} as

$$y_{k+1} = y_k + w_1g_1 + w_2g_2 + w_3g_3 + w_4g_4 \qquad \text{(D.1.19)}$$

where
$$
\begin{aligned}
g_1 &= hf(t_k, y_k) \\
g_2 &= hf(t_k + \alpha_1 h, y_k + \alpha_1 g_1) \\
g_3 &= hf[t_k + \alpha_2 h, y_k + \beta_2 g_2 + (\alpha_2 - \beta_2)g_1] \\
g_4 &= hf[t_k + \alpha_3 h, y_k + \beta_3 g_2 + \gamma_3 g_3 + (\alpha_3 - \beta_3 - \gamma_3)g_1]
\end{aligned}
\qquad \text{(D.1.20)}
$$

Comparison with the Taylor series yields eight equations for the 10 parameters. Thus, two parameters can be chosen in light of other considerations. A number of different choices have been used. For example, the *classical* Runge-Kutta method, which reduces to Simpson's rule for integration if $f(t, y)$ is a function of only t, uses the following set of parameters.

$$
\begin{aligned}
w_1 = w_4 &= \tfrac{1}{6} & w_2 = w_3 &= \tfrac{1}{3} \\
\alpha_1 = \alpha_2 &= \tfrac{1}{2} & \beta_2 &= \tfrac{1}{2} \\
\gamma_3 = \alpha_3 &= 1 & \beta_3 &= 0
\end{aligned}
\qquad \text{(D.1.21)}
$$

The Runge-Kutta algorithms are very tedious to compute by hand, so we not show the steps involved in Examples D.1.4 and D.1.5. The algorithms are easily programmed, however. The MATLAB programs for these examples, using the parameter values for the classical fourth-order Runge-Kutta algorithm, are given in Section D.2.

Runge-Kutta Method for an Oscillating Solution   **EXAMPLE D.1.4**

■ Problem
Illustrate how the fourth-order Runge-Kutta method works with an oscillating solution by using the method to solve our second test case:

$$\dot{y} = \sin t \qquad y(0) = 0$$

for $0 \le t \le 4\pi$. Use the parameter values given by (D.1.21).

■ Solution
To compare the results with those obtained with the Euler method, we will use the same step size $\Delta t = 2\pi/13$. The results are shown in Figure D.1.5, with the numerical solution shown by the small circles and the exact solution by the solid line. There is much less error than with the Euler method using the same step size.

Figure D.1.5 Fourth-order Runge-Kutta and exact solutions of $\dot{y} = \sin t$, $y(0) = 0$.
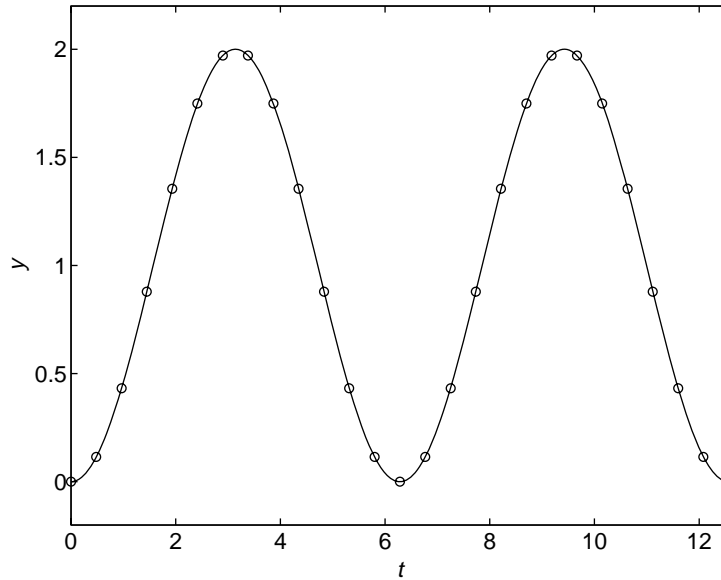
EXAMPLE D.1.5

Runge-Kutta Method for a Stiff Equation

■ Problem

Our third test case is an example of an equation that requires a step size small enough to solve for the rapid changes in the solution, but for which many steps are needed to obtain the solution over the longer time interval. Thus an accurate numerical algorithm is needed to prevent large errors from accumulating. The problem is

$$\dot{y} + y = 0.001e^{10t} \qquad y(0) = 10$$

The closed form solution is

$$y(t) = 10e^{-t} + \frac{0.001}{11}(e^{10t} - e^{-t})$$

Compare this solution with that obtained with the fourth-order Runge-Kutta using the parameter values given by (D.1.21).

■ Solution

The following table gives the results for every twentieth step, along with the exact solution, to 11 decimal places. The numerical solution is correct to seven or more significant figures, which is quite good considering the wide range of values of y.

t	**Exact solution**	**Numerical solution**
0.2	8.18790483308	8.18790483325
0.4	6.70810299036	6.70810299083
0.6	5.52474181384	5.52474181600
0.8	44.76424497372	44.76424498796
1.0	5.68116694947	5.68116705275
1.2	17.80780486683	17.80780562797
1.4	111.79360941831	111.79361504050
1.6	809.84717596280	809.84721750362
1.8	5970.74107724873	5970.74138419445

    Programming these algorithms is a very good way to improve your understanding of them. Examples of MATLAB programs that generated the data and the plots in this section are given in Section D.2.

D.2 PROGRAMMING NUMERICAL METHODS IN MATLAB

In this section, we first show how to program the Euler, modified-Euler, and Runge-Kutta methods in MATLAB. Although MATLAB has built-in solvers, learning to program such algorithms will improve your understanding of these methods.

PROGRAMMING THE EULER METHOD

The Euler algorithm for the equation $\dot{y} = f(t, y)$ is

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t$$

where $t_{k+1} = t_k + \Delta t$.

    This equation can be applied successively at the times t_k, for example, by putting it in a `for` loop in a MATLAB program.

The Euler Method for $\dot{y} = -10y$ | **EXAMPLE D.2.1**

■ **Problem**

Use the Euler method to solve our first test case:

$$\dot{y} = -10y \qquad y(0) = 2$$

■ **Solution**

The following script file solves the problem and plots the solution over the range $0 \le t \le 0.5$. The exact solution is $y(t) = 2e^{-10t}$. For comparison purposes, we use the same step size as in Example D.1.1, which is $\Delta t = 0.02$.

```
delta = 0.02; y(1) = 2;
k = 0;
for time = [delta:delta:0.5]
   k = k + 1;
   y(k+1) = y(k) - 10*y(k)*delta;
end
t = [0:delta: 0.5];
y_exact = 2*exp(-10*t);
plot(t,y,'o',t,y_exact),xlabel('t'),ylabel('y')
```

    Figure D.1.1 in Section D.1 shows the results. The numerical solution is shown by the small circles. The exact solution is shown by the solid line. There is some noticeable error. If we had used a step size equal to 0.005, for example, the error would not be noticeable on the plot.

PROGRAMMING THE MODIFIED-EULER METHOD

The modified-Euler algorithm for the equation $\dot{y} = f(t, y)$ is

$$\text{Euler predictor:} \quad x_{k+1} = y_k + hf(t_k, y_k)$$

$$\text{Trapezoidal corrector:} \quad y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, x_{k+1})]$$

The Modified-Euler Method for $\dot{y} = -10y$

■ Problem

Use the modified-Euler method to solve our first test case:

$$\dot{y} = -10y \qquad y(0) = 2$$

■ Solution

The following script file solves the problem and plots the solution over the range $0 \leq t \leq 0.5$. The exact solution is $y(t) = 2e^{-10t}$. We use a step size $\Delta t = 0.02$ to compare the results with those of the Euler method.

```
delta = 0.02; y(1) = 2;
k = 0;
for time = [delta:delta: 0.5]
   k = k + 1;
   x(k+1) = y(k) - 10*delta*y(k);
   y(k+1) = y(k) - 10*(delta/2)*(y(k) + x(k+1));
end
t = [0:delta:0.5];
y_exact = 2*exp(-10*t);
plot(t,y,'o',t,y_exact),xlabel('t'),ylabel('y')
```

Figure D.1.4 in Section D.1 shows the results, with the numerical solution shown by the small circles and the exact solution by the solid line. There is less error than with the Euler method using the same step size.

PROGRAMMING THE RUNGE-KUTTA METHOD

The fourth-order Runge-Kutta algorithm is

$$y_{k+1} = y_k + w_1 g_1 + w_2 g_2 + w_3 g_3 + w_4 g_4$$

where

$$g_1 = hf(t_k, y_k)$$
$$g_2 = hf(t_k + \alpha_1 h, y_k + \alpha_1 g_1)$$
$$g_3 = hf[t_k + \alpha_2 h, y_k + \beta_2 g_2 + (\alpha_2 - \beta_2)g_1]$$
$$g_4 = hf[t_k + \alpha_3 h, y_k + \beta_3 g_2 + \gamma_3 g_3 + (\alpha_3 - \beta_3 - \gamma_3)g_1]$$

We will use the parameters for the classical Runge-Kutta method, which are

$$w_1 = w_4 = \tfrac{1}{6} \qquad w_2 = w_3 = \tfrac{1}{3}$$
$$\alpha_1 = \alpha_2 = \tfrac{1}{2} \qquad \beta_2 = \tfrac{1}{2}$$
$$\gamma_3 = \alpha_3 = 1 \qquad \beta_3 = 0$$

| The Runge-Kutta Method for an Oscillating Solution | **EXAMPLE D.2.3** |

■ Problem
Use the fourth-order Runge-Kutta method to solve our second test case:

$$\dot{y} = \sin t \qquad y(0) = 0$$

$0 \leq t \leq 4\pi$. The exact solution is $y(t) = 1 - \cos t$.

■ Solution
Here $f(t, y) = \sin t$ and thus is not a function of y. Therefore, the β and γ parameters are not needed for this problem and the g_i functions are

$$g_1 = h \sin(t_k)$$

$$g_2 = h \sin(t_k + \alpha_1 h)$$

$$g_3 = h \sin(t_k + \alpha_2 h)$$

$$g_4 = h \sin(t_k + \alpha_3 h)$$

We use a step size equal to one-thirteenth of the period, or $\Delta t = 2\pi/13$, so that we can compare the results with those obtained from the modified-Euler method. The following script file implements the method:

```
h = 2*pi/13;
w1 = 1/6; w2 = 1/3; w3 = 1/3; w4 = 1/6;
a1 = 1/2; a2 = 1/2; a3 = 1;
tk = 0;
% Set the initial condition.
y(1) = 0;
% Set the upper limit on the number of steps.
tf = 4*pi;
klimit = round(tf/h)-1;
for k = 0:klimit
   tk = k*h;
   g1 = h*sin(tk);
   g2 = h*sin(tk+a1*h);
   g3 = h*sin(tk+a2*h);
   g4 = h*sin(tk+a3*h);
   m = k+1;
   y(m+1) = y(m)+w1*g1+w2*g2+w3*g3+w4*g4;
end
t = [0:h:(klimit+1)*h];
% Compute the exact solution.
te = [0:0.01:tf];
ye = 1-cos(te);
plot(te,ye,t,y,'o'),xlabel('t'),ylabel('y'),axis([0 tf 0 2])
```

Figure D.1.5 in Section D.1 shows the results, with the numerical solution shown by the small circles and the exact solution by the solid line. There is less error than with the modified-Euler method using the same step size.

EXAMPLE D.2.4

The Runge-Kutta Method for a Stiff Equation

■ **Problem**

Use the fourth-order Runge-Kutta method to solve our third test case:

$$\dot{y} + y = 0.001e^{10t} \qquad y(0) = 10$$

for $0 \leq t \leq 1.8$. The exact solution is

$$y(t) = 10e^{-t} + \frac{0.001}{11}\left(e^{10t} - e^{-t}\right)$$

■ **Solution**

Here $f(t, y) = -y + 0.001e^{10t}$ and it is helpful to write a user-defined function file for $f(t, y)$. This file is

```
function ydot = stiff(t,y)
ydot = 0.001*exp(10*t)-y;
```

We try a step size of $h = 0.01$. The script file is

```
h = 0.01; tk = 0;
w1 = 1/6; w2 = 1/3; w3 = 1/3; w4 = 1/6;
a1 = 1/2; a2 = 1/2; a3 = 1;
b2 = 1/2; b3 = 0; gam3 = 1;
% Set the initial condition.
y(1) = 10;
% Set the upper limit on the number of steps.
tf = 1.8;
klimit = round(tf/h)-1;
for k = 0:klimit
   tk = k*h;
   yk = y(k+1);
   g1 = h*stiff(tk,yk);
   g2 = h*stiff(tk+a1*h,yk+a1*g1);
   g3 = h*stiff(tk+a2*h,yk+b2*g2+(a2-b2)*g1);
   g4 = h*stiff(tk+a3*h,yk+b3*g2+gam3*g3+(a3-b3-gam3)*g1);
   m = k+1;
   y(m+1) = y(m)+w1*g1+w2*g2+w3*g3+w4*g4;
end
% Compute the exact solution.
te = [0:0.01:tf];
ye = 10*exp(-te)+(0.001/11)*(exp(10*te)-exp(-te));
% Plot every 20th point of the numerical solution.
kp = 0;
for k = 1:20:klimit+1
   kp = kp+1;
   yp(kp) = y(k);
end
% Compute times for numerical solution, every 20th point.
tp = [0:20*h:tf];
plot(te,ye,tp,yp,'o'),xlabel('t'),ylabel('y')
```
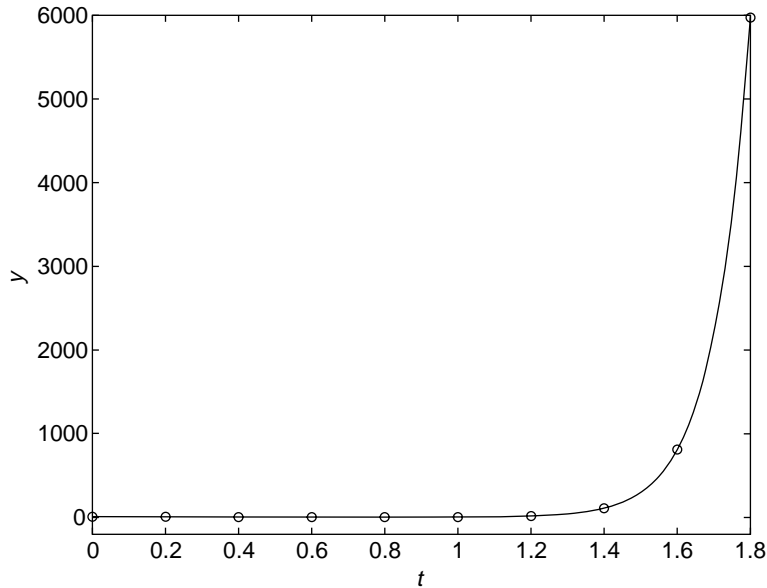
Figure D.2.1 shows the results, with the numerical solution shown for every 20 steps by the small circles and the exact solution by the solid line. Notice the wide range of values for y, and yet the results are very accurate.

PROBLEMS

Section D.1 Introduction to Numerical Algorithms

D.1 a. Use the Euler method with a step size of $\Delta t = 0.02$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} + 5y = 0 \quad y(0) = 6$$

  b. Use the closed-form solution to check the accuracy of the numerical method.

D.2 a. Use the Euler method with a step size of $\Delta t = 0.3$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = \cos t \quad y(0) = 6$$

  b. Use the closed-form solution to check the accuracy of the numerical method.

D.3 a. Use the Euler method with a step size of $\Delta t = 0.1$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = 6 \sin 3t \quad y(0) = 10$$

  b. Use the closed-form solution to check the accuracy of the numerical method.

D.4 a. Use the Euler method with a step size of $\Delta t = 0.025$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = 5e^{-4t} \quad y(0) = 2$$

  b. Use the closed-form solution to check the accuracy of the numerical method.

D.5 a. Use the Euler method with a step size of $\Delta t = 0.01$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} + 3y = 5e^{4t} \quad y(0) = 10$$

b. Use the closed-form solution to check the accuracy of the numerical method.

D.6 a. Use the modified-Euler method with a step size of $\Delta t = 0.02$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} + 5y = 0 \quad y(0) = 6$$

b. Use the closed-form solution to check the accuracy of the numerical method.

D.7 a. Use the modified-Euler method with a step size of $\Delta t = 0.03$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = \cos t \quad y(0) = 6$$

b. Use the closed-form solution to check the accuracy of the numerical method.

D.8 a. Use the modified-Euler method with a step size of $\Delta t = 0.1$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = 6 \sin 3t \quad y(0) = 10$$

b. Use the closed-form solution to check the accuracy of the numerical method.

D.9 a. Use the modified-Euler method with a step size of $\Delta t = 0.025$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} = 5e^{-4t} \quad y(0) = 2$$

b. Use the closed-form solution to check the accuracy of the numerical method.

D.10 a. Use the modified-Euler method with a step size of $\Delta t = 0.01$ to solve the following equation for five steps, using four significant figures.

$$\dot{y} + 3y = 5e^{4t} \quad y(0) = 10$$

b. Use the closed-form solution to check the accuracy of the numerical method.

Section D.2 Programming Numerical Methods in MATLAB

D.11 Consider the system for lifting a mast, discussed in Example 2.2.4 in Chapter 2. The 70-ft-long mast weighs 500 lb. The winch applies a force $f = 380$ lb to the cable. The mast is supported initially at an angle of 30°, and the cable at A is initially horizontal. The equation of motion of the mast is

$$25{,}400\,\ddot{\theta} = -17{,}500 \cos \theta + \frac{626{,}000}{Q} \sin(1.33 + \theta)$$

where

$$Q = \sqrt{2020 + 1650 \cos(1.33 + \theta)}$$

Use a numerical method to solve for and plot $\theta(t)$ for $\theta(t) \le \pi/2$ rad.

D.12 a. Program the Euler method and use the program to solve the following equation for $0 \leq t \leq 1$. Use a step size of $\Delta t = 0.02$.

$$\dot{y} + 5y = 0 \quad y(0) = 6$$

   b. Use the closed-form solution to check the accuracy of the numerical method.

D.13 a. Program the Euler method and use the program to solve the following equation for $0 \leq t \leq 12$. Use a step size of $\Delta t = 0.3$.

$$\dot{y} = \cos t \quad y(0) = 6$$

   b. Use the closed-form solution to check the accuracy of the numerical method.

D.14 a. Program the Euler method and use the program to solve the following equation for $0 \leq t \leq 1$. Use a step size of $\Delta t = 0.025$.

$$\dot{y} = 5e^{-4t} \quad y(0) = 2$$

   b. Use the closed-form solution check the accuracy of the numerical method.

D.15 a. Program the modified-Euler method and use the program to solve the following equation for $0 \leq t \leq 1$. Use a step size of $\Delta t = 0.02$.

$$\dot{y} + 5y = 0 \quad y(0) = 6$$

   b. Use the closed-form solution to check the accuracy of the numerical method.

D.16 a. Program the modified-Euler method and use the program to solve the following equation for $0 \leq t \leq 12$. Use a step size of $\Delta t = 0.3$.

$$\dot{y} = \cos t \quad y(0) = 6$$

   b. Use the closed-form solution to check the accuracy of the numerical method.

D.17 a. Program the modified-Euler method and use the program to solve the following equation for $0 \leq t \leq 1$. Use a step size of $\Delta t = 0.025$.

$$\dot{y} = 5e^{-4t} \quad y(0) = 2$$

   b. Use the closed-form solution check the accuracy of the numerical method.

D.18 a. Program the fourth-order Runge-Kutta method and use the program to solve the following equation for $0 \leq t \leq 12$. Use a step size of $\Delta t = 0.3$.

$$\dot{y} = \cos t \quad y(0) = 6$$

   b. Use the closed-form solution check the accuracy of the numerical method.

D.19 a. Program the fourth-order Runge-Kutta method and use the program to solve the following equation for $0 \leq t \leq 1$. Use a step size of $\Delta t = 0.01$.

$$\dot{y} + 3y = 5e^{4t} \quad y(0) = 10$$

   b. Use the closed-form solution to check the accuracy of the numerical method.

GLOSSARY

A

active network   An electrical network that has its own energy source.

actuator   The "muscle" in a control system; the device that provides the power, force, heat, etc.

analog signal   A signal that has an infinite number of possible values at an infinite number of possible times.

analog-to-digital converter (ADC or A/D)   A device that converts an analog signal to a digital signal.

angle of arrival   The angle at which the root locus approaches a complex zero.

angle of departure   The angle at which the root locus leaves a complex pole.

asymptote   (a) On root locus plots, the line approached by a root locus path as the root locus parameter approaches infinity. (b) On Bode plots, the lines describing the low-frequency and high-frequency approximations of the magnitude and phase.

B

back emf   A voltage that is generated by a motor armature's speed and that opposes the supply voltage applied to the armature.

bandwidth   The frequency range over which the logarithmic magnitude ratio is within 3 dB of its peak value.

break frequency   See corner frequency.

breakaway point   A point where a root locus path leaves the real axis.

breakin point   A point where a root locus path enters the real axis.

Bode plots   The plots of the logarithmic magnitude ratio and phase angle of a frequency transfer function versus the logarithm of frequency.

C

characteristic equation   The equation obtained by equating to zero the denominator of the transfer function.

closed-loop system   A system that uses feedback.

compensation   The use of an additional device or algorithm to supplement the main controller.

corner frequency   The frequency at which the asymptotic approximation of the frequency response changes slope.

critical damping   The value of the damping constant that forms the boundary between oscillatory and nonoscillatory response.

current source   A power supply that provides the required current, regardless of the load.

D

damping constant   The constant, equal to the applied force (or torque) divided by the resulting velocity, that expresses the resistance of a damping element.

damping ratio   The ratio of the actual damping value to the critical damping value.

dead time   The time between an action and the response, during which time no response occurs.

decade   A range of frequencies separated by a factor of 10.

decibel (dB)   The logarithmic unit for gain.

delay time   The time required for the response to first reach 50% of its steady-state value.

digital-to-analog converter (DAC or D/A)   A device that converts a digital signal to an analog signal.

digital signal   A signal that is quantized in time and amplitude.

disturbance   An unwanted or unpredictable input.

dominant root   The root or root pair that has the largest time constant.

E

equilibrium   A state of no change.

equivalent inertia   The inertia of a fictitious single-inertia system that has the same kinetic energy as the real system.

equivalent mass   The mass of a fictitious single-mass system that has the same kinetic energy as the real system.

error signal   The difference between the desired output value and the actual output value.

F

feedback compensation   The use of an additional device or algorithm in a feedback loop around the main controller to supplement the main controller.

feedback signal   The measured system output.

feedforward compensation   The use of an additional device or algorithm to supplement the main controller by acting on the command input and feeding the result forward to modify the output of the main controller.

final value theorem   A mathematical method, based on the Laplace transform, for computing the steady-state value of a time function.

fluid capacitance   A constant that relates the fluid mass stored in a container to the resulting pressure.

fluid resistance   A constant that relates the pressure drop to the flow rate.

forced response   That part of the response due to the input.

Fourier series   A representation of a periodic function in terms of a constant plus the sum of sines and cosines of different frequencies.

free response   That part of the response due to the initial conditions.

frequency response   The steady-state response of a stable system to a sinusoidal input.

frequency transfer function   The transfer function with the Laplace variable s replaced with $j\omega$.

G

gage pressure   Pressure measured relative to atmospheric pressure.

gain margin   The difference in decibels between 0 dB and the open-loop gain at the frequency where the phase equals $-180°$.

H

hydraulic capacitance   The fluid capacitance of a hydraulic element.

hydraulic resistance   The fluid resistance of a hydraulic element.

hydraulic system   A system that operates with an incompressible fluid.

I

impedance   For an electrical device, the ratio of the Laplace transform of the voltage across the device to the Laplace transform of the current through the device.

impulse   In mathematics, the Dirac delta function. A model of an input that has a large amplitude but a very short duration.

initial value theorem   A mathematical method, based on the Laplace transform, for computing the initial value of a time function.

K

Kirchhoff's current law   The sum of currents at a node equals zero.

Kirchhoff's voltage law   The sum of voltages around a closed loop equals zero.

L

lag compensation   A device or algorithm, used in series with the main controller, that provides a negative phase shift and attenuated gain over the desired frequency range.

Laplace transform   An integral transformation that converts a time-domain function into an algebraic function of the Laplace variable s.

lead compensation   A device or algorithm, used in series with the main controller, that provides a positive phase shift over the desired frequency range.

lead-lag compensation   A device or algorithm, used in series with the main controller, that is equivalent to lead compensation and lag compensation in series.

linear system   A system that satisfies the superposition property.

linearization   The process of replacing a nonlinear expression with a linear one that is approximately correct near a reference state of the system.

logarithmic plots   See Bode plots.

M

maximum overshoot   The difference between the peak response and the steady-state response.

mode   A fundamental behavior pattern of a dynamic system.

N

natural frequency   See undamped natural frequency.

negative feedback   A feedback signal that is subtracted from the input.

Newton's second law   States, in its simplest form, that the mass times the acceleration equals the sum of the forces.

numerator dynamics   The presence of an s term in the numerator of a transfer function. Indicates that the system responds to the derivative of the input.

O

Ohm's law   The electrical resistance equals the ratio of voltage to current.

open-loop control system   A control system that does not use feedback.

operational amplifier (op amp)   An electrical amplifier having a very high gain, a very high input impedance, and a very low output impedance.

P

partial-fraction expansion   A representation of the ratio of two polynomials as a sum of simpler terms involving the factors of the denominator.

passive network   An electrical network that only stores or dissipates energy.

peak time   The time at which the maximum overshoot occurs.

phase margin   The difference in degrees between $-180°$ and the open-loop phase at the frequency where the logarithmic gain equals 0 dB.

PID controller   A device or algorithm used as the main controller to give an output that is a sum of terms proportional to the error, the integral of the error, and the derivative of the error.

plant   The device or process to be controlled.

pneumatic capacitance   The fluid capacitance of a pneumatic element.

pneumatic resistance   The fluid resistance of a pneumatic element.

pneumatic system   A system that operates with a compressible fluid.

positive feedback   A feedback signal that is added to the input.

pulse width modulation (PWM)   A control technique that varies the time duration of controller output pulses as a function of the error.

R

ramp function   A time function whose slope is constant.

resonant frequency   The frequency at which the frequency response magnitude is a maximum.

reset windup   In a control system using integral action, the tendency of that action to continue increasing its output even while the error is decreasing.

rise time   The time required for the response to first reach 100% of its steady-state value.

root locus   The paths traced by the characteristic roots as a parameter is varied.

Routh-Hurwitz criterion   A test applied to the characteristic polynomial to determine whether or not the system is stable.

S

saturation   Occurs when the actuator output reaches its maximum or minimum possible value.

settling time   The time required for the response to settle to within a certain percent (usually 2% or 5%) of its steady-state value.

specifications   A set of statements that describe the performance requirements of a system.

spring constant   The constant, equal to the applied force (or torque) divided by the resulting deflection, that expresses the stiffness of an elastic element.

stable equilibrium   An equilibrium state to which the system state returns and remains when disturbed.

stable system   A system that possesses a stable equilibrium.

stability test   A linear system is stable if its characteristic roots all have negative real parts.

state variables   A set of variables that completely describe a system.

state variable feedback   A control scheme that uses measurements of all the state variables.

steady-state error   The error remaining after the transient response has disappeared.

step function   In mathematics, a time function whose value instantaneously changes from one constant value to another. Used as a model of an input that changes rapidly from one constant value (usually zero) to another constant value.

system   A set of elements connected to achieve a common purpose.

T

tachometer   A generator that produces a voltage proportional to its rotational speed.

thermal capacitance   A constant that relates heat stored in a mass to the temperature of the mass.

thermal resistance   A constant that relates heat flow rate to temperature difference.

thermal system   A system whose dynamics are governed by temperature difference and the flow of heat energy.

time constant   A measure of exponential decay. After an elapsed time equal to one time constant, the response is approximately 37% of its initial value.

time delay   See dead time.

transducer   A sensing device that converts a signal from one form to another, which is usually a voltage or a current.

transfer function   The ratio of the Laplace transform of the forced response to the Laplace transform of the input.

transient response   That part of the response that disappears with time.

tuning   The process of setting or adjusting control-system gains.

type number   The number of poles at the origin in the forward-path transfer function.

U

undamped natural frequency   The oscillation frequency of the free response of an undamped second-order linear system.

V

vibration absorber   A device that uses a mass and an elastic element to reduce the displacement of an object.

vibration isolator   A device that uses elastic and damping elements to reduce the force or displacement transmitted to the object being isolated.

voltage source   A power supply that provides the required voltage, regardless of the load.

Z

zero-order hold (ZOH)   The staircase approximation method used by some digital-to-analog converters to produce an analog output.

Ziegler-Nichols guidelines   Two sets of guidelines, the process reaction method and the ultimate cycle method, used for control-system tuning.

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Table 3.1.3 Solution forms.

Equation	Solution form
First order: $\dot{x} + ax = b \quad a \neq 0$	$x(t) = \dfrac{b}{a} + Ce^{-at}$
Second order: $\ddot{x} + a\dot{x} + bx = c \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \dfrac{c}{b}$
2. $(a^2 = 4b)$ repeated, real roots: s_1, s_1	$x(t) = (C_1 + tC_2)e^{s_1 t} + \dfrac{c}{b}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$, $\omega = \sqrt{b}$	$x(t) = C_1 \sin \omega t + C_2 \cos \omega t + \dfrac{c}{b}$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = \sigma \pm j\omega$, $\sigma = -a/2, \omega = \sqrt{4b - a^2}/2$	$x(t) = e^{\sigma t}(C_1 \sin \omega t + C_2 \cos \omega t) + \dfrac{c}{b}$

Table 3.3.1 Table of Laplace transform pairs.

	$X(s)$	$x(t), t \geq 0$
1.	1	$\delta(t)$, unit impulse
2.	$\dfrac{1}{s}$	$u_s(t)$, unit step
3.	$\dfrac{c}{s}$	constant, c
4.	$\dfrac{e^{-sD}}{s}$	$u_s(t - D)$, shifted unit step
5.	$\dfrac{n!}{s^{n+1}}$	t^n
6.	$\dfrac{1}{s + a}$	e^{-at}
7.	$\dfrac{1}{(s + a)^n}$	$\dfrac{1}{(n - 1)!}t^{n-1}e^{-at}$
8.	$\dfrac{b}{s^2 + b^2}$	$\sin bt$
9.	$\dfrac{s}{s^2 + b^2}$	$\cos bt$
10.	$\dfrac{b}{(s + a)^2 + b^2}$	$e^{-at} \sin bt$
11.	$\dfrac{s + a}{(s + a)^2 + b^2}$	$e^{-at} \cos bt$
12.	$\dfrac{a}{s(s + a)}$	$1 - e^{-at}$
13.	$\dfrac{1}{(s + a)(s + b)}$	$\dfrac{1}{b - a}\left(e^{-at} - e^{-bt}\right)$
14.	$\dfrac{s + p}{(s + a)(s + b)}$	$\dfrac{1}{b - a}\left[(p - a)e^{-at} - (p - b)e^{-bt}\right]$
15.	$\dfrac{1}{(s + a)(s + b)(s + c)}$	$\dfrac{e^{-at}}{(b - a)(c - a)} + \dfrac{e^{-bt}}{(c - b)(a - b)} + \dfrac{e^{-ct}}{(a - c)(b - c)}$
16.	$\dfrac{s + p}{(s + a)(s + b)(s + c)}$	$\dfrac{(p - a)e^{-at}}{(b - a)(c - a)} + \dfrac{(p - b)e^{-bt}}{(c - b)(a - b)} + \dfrac{(p - c)e^{-ct}}{(a - c)(b - c)}$

Table 9.1.1 Free, step, and ramp response of $\tau \dot{y} + y = r(t)$.

Free response $[r(t) = 0]$
$$y(t) = y(0)e^{-t/\tau}$$
$$y(\tau) \approx 0.37y(0)$$
$$y(4\tau) \approx 0.02y(0)$$

Step response $[r(t) = Ru_s(t), y(0) = 0]$
$$y(t) = R(1 - e^{-t/\tau})$$
$$y(\infty) = y_{ss} = R$$
$$y(\tau) \approx 0.63y_{ss}$$
$$y(4\tau) \approx 0.98y_{ss}$$

Ramp response $[r(t) = mt, y(0) = 0]$
$$y(t) = m(t - \tau + \tau e^{-t/\tau})$$

Table 3.1.2 The exponential function.

Taylor series
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

Euler's identities
$$e^{j\theta} = \cos \theta + j \sin \theta$$
$$e^{-j\theta} = \cos \theta - j \sin \theta$$

Limits
$$\lim_{x \to \infty} xe^{-x} = 0 \quad \text{if } x \text{ is real.}$$
$$\lim_{t \to \infty} e^{-st} = 0 \quad \text{if the real part of } s \text{ is positive.}$$

If a is real and positive,
$$e^{-at} < 0.02 \text{ if } t > 4/a.$$
$$e^{-at} < 0.01 \text{ if } t > 5/a.$$

The *time constant* is $\tau = 1/a$.

Table 9.3.1 Unit step response of a stable second-order model.

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. Overdamped case ($\zeta > 1$): distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2}e^{-r_1 t} - \frac{r_1}{r_1 - r_2}e^{-r_2 t} + 1\right)$$

2. Critically damped case ($\zeta = 1$): repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t)e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1)e^{-r_1 t} + 1]$$

3. Underdamped case ($0 \leq \zeta < 1$): complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$x(t) = Be^{-t/\tau}\sin\left(\omega_n\sqrt{1 - \zeta^2}\,t + \phi\right) + \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1 - \zeta^2}\,t + \phi\right) + 1\right]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right) + \pi \quad \text{(third quadrant)}$$

Time constant: $\tau = 1/\zeta\omega_n$

Table 10.5.3 Routh-Hurwitz stability conditions.

1. **Second-Order:** $a_2 s^2 + a_1 s + a_0 = 0$
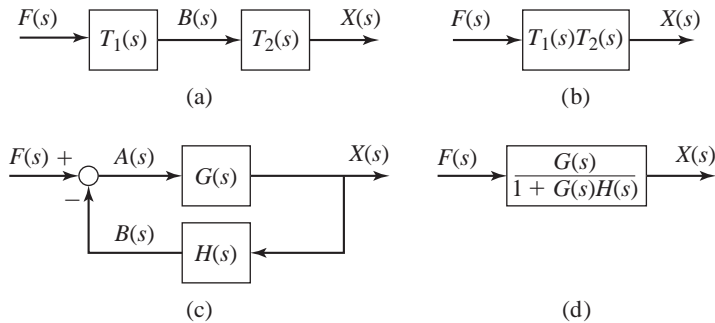   Stable if and only if a_2, a_1, and a_0 all have the same sign.
2. **Third-Order:** $a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$
   Assuming $a_3 > 0$, stable if and only if a_2, a_1, and a_0 are all positive and $a_2 a_1 > a_3 a_0$.
3. **Fourth-Order:** $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$
   Assuming $a_4 > 0$, stable if and only if a_3, a_2, a_1, and a_0 are all positive, $a_2 a_3 > a_1 a_4$, and

$$a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^2 > 0$$

Figure 9.5.4 (a) and (b) Simplification of series blocks. (c) and (d) Simplification of a feedback loop.

Table 10.5.2 Useful results for second-order systems.

1. Model: $m\ddot{x} + c\dot{x} + kx = f(t)$
2. Transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

3. Characteristic equation: $ms^2 + cs + k = 0$
4. Characteristic roots:

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

5. Damping ratio and undamped natural frequency:

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

6. Time constant: If $\zeta \le 1$,

$$\tau = \frac{2m}{c}$$

If $\zeta > 1$, the dominant (larger) time constant is

$$\tau_1 = \frac{2m}{c - \sqrt{c^2 - 4mk}}$$

and the secondary (smaller) time constant is

$$\tau_2 = \frac{2m}{c + \sqrt{c^2 - 4mk}}$$

7. Maximum percent overshoot and peak time:

$$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$
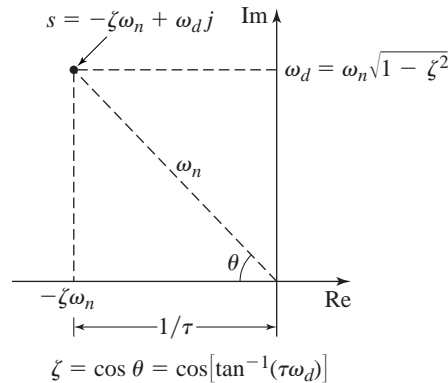
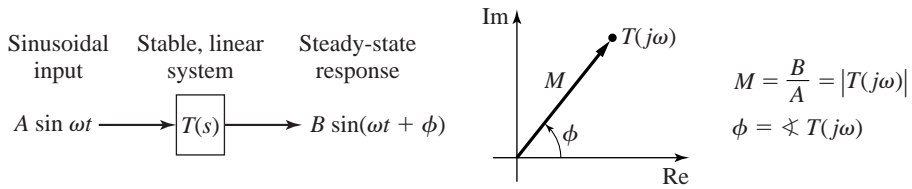8. The complex root pair $s = -a \pm bj$ corresponds to the characteristic equation

$$(s + a)^2 + b^2 = 0$$

9. The value $\zeta = 0.707$ corresponds to a root pair having equal real and imaginary parts: $s = -a \pm aj$.

Figure 9.2.5 Graphical interpretation of the parameters ζ, τ, ω_n, and ω_d.

$$\zeta = \cos\theta = \cos\left[\tan^{-1}(\tau\omega_d)\right]$$

Sinusoidal input → $A \sin \omega t$ → Stable, linear system $T(s)$ → Steady-state response → $B \sin(\omega t + \phi)$

$M = \dfrac{B}{A} = |T(j\omega)|$

$\phi = \measuredangle\, T(j\omega)$

Figure 8.1.2 Frequency response of a stable linear system.

Table 8.1.2 Frequency response of the model $\tau \dot{y} + y = f(t)$.

$$M = \frac{|Y|}{|F|} = \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \qquad (1)$$

$$\phi = -\tan^{-1}(\omega \tau) \qquad (2)$$

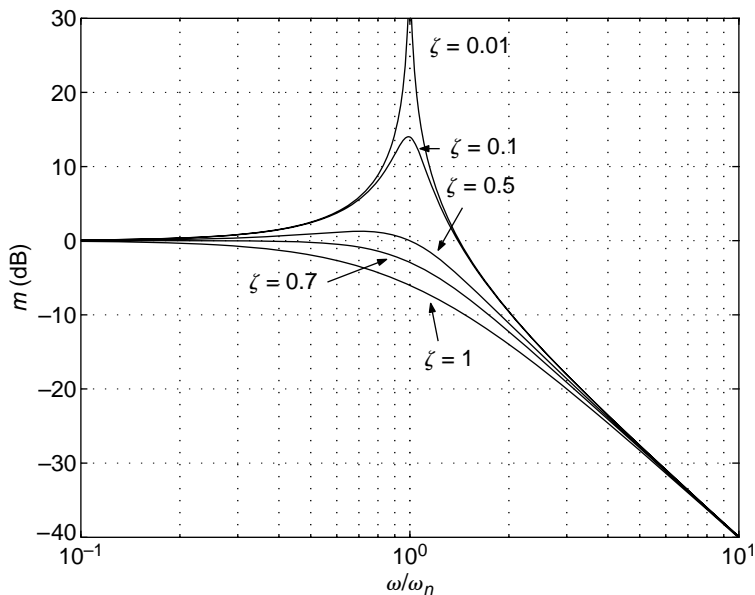**Table 8.2.2** Frequency response of a second-order system.

Model:
$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Resonant frequency:
$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \quad 0 \le \zeta \le 0.707$$

Resonant response:
$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \quad 0 \le \zeta \le 0.707$$

$$m_r = -20\log\left(2\zeta\sqrt{1 - \zeta^2}\right) \quad 0 \le \zeta \le 0.707$$

$$\phi_r = -\tan^{-1}\frac{\sqrt{1 - 2\zeta^2}}{\zeta} \quad 0 \le \zeta \le 0.707$$

Figure 8.2.4 Semilog plot of log magnitude ratio of the model $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

With its strong emphasis on practical applications that help students understand the relevance of what they are learning, the second edition of *System Dynamics* builds on the strengths of the first edition with a careful and focused reorganization to further improve student accessibility of the material.

System Dynamics includes the strongest treatment of computational software and system simulation of any available text, with its early introduction of MATLAB and Simulink. The text's extensive coverage also includes discussion of the root locus and frequency response plots, among other methods, for assessing system behavior in the time and frequency domains as well as topics such as function discovery, parameter estimation, and system identification techniques, motor performance evaluation, and system dynamics in everyday life.

New features and their benefits:

- Block diagrams are now presented in Chapter 9 to be closer to their applications in control system analysis. The material in Chapter 5 dealing with transfer functions and state variable methods has been reorganized to better delineate the advantages of each method.

- Introduction to MATLAB, offered on the text website, provides readers with a practical, concise guide to the program.

- The dynamics review in Chapter 2 and the introduction to electrical systems in Chapter 6 have been edited for a more concise presentation of the material.

- The former Chapter 11 has been split into two chapters to focus more concisely on PID control system design issues (the new Chapter 11) and compensator design (the new Chapter 12).

- The final chapter (Vibration Applications) now includes coverage of active vibration control systems and nonlinear vibration.

Retained/hallmark features:

- The first edition's extensive coverage of mechanical, electrical, fluid, and thermal systems is retained.

- Function discovery, parameter estimation, and system identification techniques are covered in several chapters.

- MATLAB is introduced in the first chapter and used throughout the book as an optional feature.

- Simulink is introduced in Chapter 5 and used as an optional feature in remaining chapters for doing systems simulation.