Report on Slip-line field theory

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1 Introduction

Generally, plasticity problem is much more difficult to solve comparing with linear elastic problems. Therefore, the Slip Line Field Theory, a graphical technique, is introduced to analyze the stresses of the material undergoing plastic deformation. In a deforming material, the directions of maximum shear stresses form the orthogonal curvatures which are called α slip-line and β slip-line. The stress state in the solid can always determined based on sets of these lines.

For this paper, all the contents are based on the Chapter 6.1 from *Applied Mechanics of Solid* by Professor Allan F. Bower.[1] I will summarize the contents, derive the conclusions he omitted and re-solve the examples.

2 Assumptions

There are several assumptions for the slip line field theory:

- **Plane strain deformation** ($u_3 = 0, \sigma_{13} = \sigma_{23} = 0$): The general directions of 3D deformation is still hard to solve. Therefore, for this theory, we assume the strain on e_3 direction is zero.

- **Quasi-static loading**: Quasi-static loading means the load is applied so slowly that the strain rate is very small. Furthermore, the inertia can be neglected.

- **No temperature changes and body forces**

- **The solid has perfect rigid-plastic response**: For this theory, the material is assumed to be perfect rigid-plastic which means there is no elastic behaviours happened during this process.

3 Derivation of the Slip Line Field Theory

3.1 Stress state

From assumption, we know it is plane strain deformation which means

$$\sigma_{13} = \sigma_{23} = 0$$
$$\sigma_{33} = \frac{1}{2}(\sigma_{11} + \sigma_{22})$$

therefore, the stress state tensor in e_1 and e_2 directions is:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \frac{1}{2}(\sigma_{11} + \sigma_{22}) \end{bmatrix}$$

For slip line field theory, the slip lines are always parallel to the direction of maximum shear stress as shown as Fig 1. ϕ is angle between e_1 and e_α. Now, the stress state tensor in e_α and e_β direction is:

$$[\sigma]_{\alpha,\beta} = \begin{bmatrix} \bar{\sigma} & k & 0 \\ k & \bar{\sigma} & 0 \\ 0 & 0 & \bar{\sigma} \end{bmatrix}$$

where $\bar{\sigma} = \sigma_h = \frac{1}{2}(\sigma_{11} + \sigma_{22})$ and k is the yield shear stress. From the geometry of Mohr's circle, we have:

$$\sigma_{11} = \bar{\sigma} - ksin2\phi$$

$$\sigma_{22} = \bar{\sigma} + ksin2\phi$$

$$\sigma_{12} = kcos2\phi$$

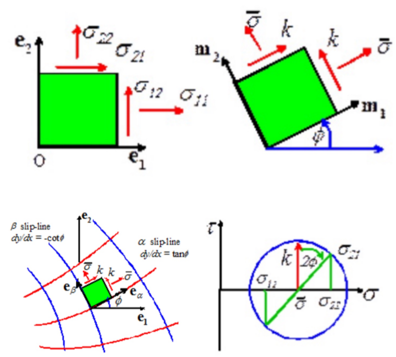

Figure 1: Coordinate system of Slip lines

3.2 Hencky equations

From 2-D stress equilibrium, we know:

$$\sigma_{11,1} + \sigma_{12,2} = 0$$

$$\sigma_{12,1} + \sigma_{22,2} = 0$$

From previous result, in e_1 and e_2 direction:

$$\sigma_{11} = \bar{\sigma} - ksin2\phi$$

$$\sigma_{22} = \bar{\sigma} + ksin2\phi$$

$$\sigma_{12} = kcos2\phi$$

therefore, the stress equilibrium in e_1 and e_2 directions are:

$$\frac{\partial\bar{\sigma}}{\partial x_1} - 2k(cos2\phi\frac{\partial\phi}{\partial x_1} + sin2\phi\frac{\partial\phi}{\partial x_2}) = 0$$
$$\frac{\partial\bar{\sigma}}{\partial x_2} - 2k(sin2\phi\frac{\partial\phi}{\partial x_1} - cos2\phi\frac{\partial\phi}{\partial x_2}) = 0$$

In order to get these equations. Let's assume:

$$\frac{\partial}{\partial s_\alpha}(\bar{\sigma} - 2k\phi) = 0$$
$$\frac{\partial}{\partial s_\beta}(\bar{\sigma} + 2k\phi) = 0$$

where s_α and s_β are coordinate system along α and β lines. therefore,

$$\begin{aligned}\frac{\partial}{\partial s_\alpha}(\bar{\sigma} - 2k\phi) &= \frac{\partial(\bar{\sigma} - 2k\phi)}{\partial x_1}\frac{x_1}{\partial s_\alpha} + \frac{\partial(\bar{\sigma} - 2k\phi)}{\partial x_2}\frac{x_2}{\partial s_\alpha} \\ &= cos\phi\frac{\partial(\bar{\sigma} - 2k\phi)}{\partial x_1} + sin\phi\frac{\partial(\bar{\sigma} - 2k\phi)}{\partial x_2} \\ &= cos\phi\bar{\sigma}_{,x_1} + sin\phi\bar{\sigma}_{,x_2} - 2k(cos\phi * \phi_{,x_1} + sin\phi * \phi_{,x_2})\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{\partial}{\partial s_\beta}(\bar{\sigma} + 2k\phi) &= \frac{\partial(\bar{\sigma} + 2k\phi)}{\partial x_1}\frac{x_1}{\partial s_\beta} + \frac{\partial(\bar{\sigma} + 2k\phi)}{\partial x_2}\frac{x_2}{\partial s_\beta} \\ &= sin\phi\frac{\partial(\bar{\sigma} + 2k\phi)}{\partial x_1} + cos\phi\frac{\partial(\bar{\sigma} + 2k\phi)}{\partial x_2} \\ &= -sin\phi\bar{\sigma}_{,x_1} + cos\phi\bar{\sigma}_{,x_2} + 2k(sin\phi * \phi_{,x_1} + cos\phi * \phi_{,x_2})\end{aligned} \tag{2}$$

By relating Eq(1) with Eq(2), we can get previous stress equilibrium relations in e_1 and e_2 directions.

$$Eq(1) * cos\phi - Eq(2) * sin\phi \Rightarrow \bar{\sigma}_{,x_1} - 2k(cos2\phi * \phi_{,x_1} + sin2\phi * \phi_{,x_2}) = 0$$
$$Eq(1) * sin\phi + Eq(2) * cos\phi \Rightarrow \bar{\sigma}_{,x_2} - 2k(sin2\phi * \phi_{,x_1} - cos2\phi * \phi_{,x_2}) = 0$$

Therefore, our assumption is confirmed and the hydrostatic stress and maximum shear stress along the slip lines has relations as following, which is called Hencky equations.

$$\bar{\sigma} - 2k\phi = constant \text{ } (\alpha \text{ slip line})$$

$$\bar{\sigma} + 2k\phi = constant \text{ } (\beta \text{ slip line})$$

3.3 Governing equations

For slip line field theory, we have several governing equations.

- Yield criterion

  From above result, we know the stress state tensor in e_1 and e_2 direction is:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \frac{1}{2}(\sigma_{11} + \sigma_{22}) \end{bmatrix}$$

  therefore,

$$[S] = \begin{bmatrix} \frac{1}{2}(\sigma_{11} - \sigma_{22}) & \sigma_{12} & 0 \\ \sigma_{12} & \frac{1}{2}(\sigma_{22} - \sigma_{11}) & 0 \\ 0 & 0 & 0) \end{bmatrix}$$

  we know for Von Mise yield criterion,

$$\sqrt{\tfrac{3}{2}S_{ij}S_{ij}} - Y = 0 \Rightarrow \tfrac{3}{2}S_{ij}S_{ij} = Y^2 = 3k^2$$
$$\Rightarrow \tfrac{3}{2} * 2 * \tfrac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \tfrac{3}{2} * 2\sigma_{12}^2 = 3k^2$$

  therefore, we have:

$$\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2 = k^2 \tag{3}$$

- Plastic flow rule

  Since for plastic flow, we know:

$$\dot{\epsilon_{ij}} = \dot{\lambda} S_{ij}$$

  from this relation, we know:

$$\dot{\epsilon_{11}} + \dot{\epsilon_{22}} = \dot{\lambda}(S_{11} + S_{12}) = 0$$

  therefore,

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{4}$$

  we also know that:

$$\dot{\lambda} = \tfrac{\dot{\epsilon}_{11} - \dot{\epsilon}_{22}}{S_{11} - S_{22}} = \tfrac{\dot{\epsilon}_{12}}{S_{12}}$$
$$\Rightarrow \tfrac{\dot{\epsilon}_{11} - \dot{\epsilon}_{22}}{\sigma_{11} - \sigma_{12}} = \tfrac{\dot{\epsilon}_{12}}{\sigma_{12}}$$

  therefore,

$$(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2})\sigma_{12} = \frac{1}{2}(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1})(\sigma_{11} - \sigma_{22}) \tag{5}$$

- Stress Equilibrium

  Since we know for stress equilibrium in solid is following if we neglected the body force and inertia:

$$\tfrac{\partial \sigma_{ij}}{\partial x_j} = 0$$

  therefore, we have last two governing equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \tag{6}$$

$$\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{21}}{\partial x_1} = 0 \tag{7}$$

3.4 Geiringer Equations

In order to solve the velocity field along e_1, e_2 or e_α, e_β directions, We are going to do the calculation in matrix form.

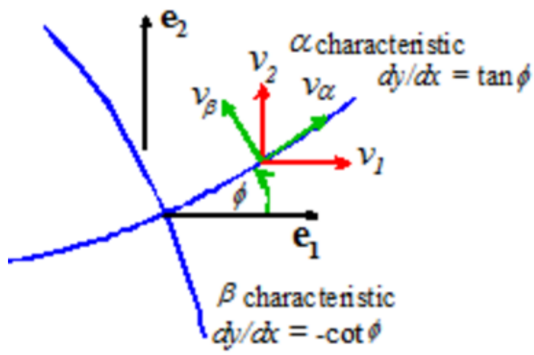

Figure 2: The velocity field

The governing equations (3)-(7) can be expressed in matrix form:

$$A_{ij}\frac{\partial q_j}{\partial x_1} + B_{ij}\frac{\partial q_j}{\partial x_2} = 0 \tag{8}$$

where,

$$\text{q} = \begin{bmatrix} \phi \\ v_1 \\ v_2 \\ \bar{\sigma} \end{bmatrix} \quad \text{A} = \begin{bmatrix} 0 & -2kcos2\phi & -2ksin2\phi & 0 \\ -2kcos2\phi & 0 & 0 & 1 \\ -2ksin2\phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$
B = \begin{bmatrix} 0 & -2ksin2\phi & 2kcos2\phi & 0 \\ -2ksin2\phi & 0 & 0 & 0 \\ 2kcos2\phi & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

The first step to solve this equation is to find eigenvalues μ and eigenvector r_i that satisfy

$$
\tau_i A_{ij} = \mu r_i B_{ij} \tag{9}
$$

the most straight way to find eigenvalues is to calculate

$$
\det(A\text{-}\mu B) = 0
$$

therefore,

$$
A\text{-}\mu B = \begin{bmatrix} 0 & -2k(cos2\phi - \mu sin2\phi) & -2k(sin2\phi + \mu cos2\phi) & 0 \\ -2k(cos2\phi - \mu sin2\phi) & 0 & 0 & 1 \\ -2k(sin2\phi + \mu cos2\phi) & 0 & 0 & -\mu \\ 0 & 1 & -\mu & 0 \end{bmatrix}
$$

$$\det(A\text{-}\mu B) = \begin{bmatrix} 0 & -2k(cos2\phi - \mu sin2\phi) & -2k(sin2\phi + \mu cos2\phi) \\ -2k(sin2\phi + \mu sin2\phi) & 0 & 0 \\ 0 & 1 & -\mu \end{bmatrix} +$$

$$\mu \begin{bmatrix} 0 & -2k(cos2\phi - \mu sin2\phi) & -2k(sin2\phi + \mu cos2\phi) \\ -2k(cos2\phi - \mu sin2\phi) & 0 & 0 \\ 0 & 1 & -\mu \end{bmatrix}$$

$$= 2\mathrm{k}(\sin 2\phi + \mu cos2\phi)[2k\mu(cos2\phi - \mu sin2\phi) + 2k(sin2\phi + \mu cos2\phi)]$$

$$+\mu 2k(cos2\phi - \mu sin2\phi)[2k\mu(cos2\phi - \mu sin2\phi) + 2k(sin2\phi + \mu cos2\phi)] = 0$$

we can divide by $2k\mu(cos2\phi - \mu sin2\phi) + 2k(sin2\phi + \mu cos2\phi)$ both sides, then we can get following:

$$2k(sin2\phi + \mu cos2\phi) + \mu 2k(cos2\phi - \mu sin2\phi) = 0$$

then, divide by $2kcos2\phi$ both sides, we have the final equations for eighenvalues μ:

$$tan2\phi(1 - \mu^2) + 2\mu = 0 \tag{10}$$

the solutions of Eq 10 are:

$$\mu_1 = -tan\phi$$

$$\mu_2 = cot\phi$$

Let's assume the eigenvector r is:

$$r = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \end{bmatrix}$$

For eigenvalue $\mu_1 = -tan\phi$, we have:

$$[r][A] = \mu[r][B]$$

then we have:

$$\text{-2kcos}2\phi r_2 - 2ksin2\phi r_3 = -tan\phi(-2ksin2\phi r_2 + 2kcos2\phi r_3)$$

$$\text{-2kcos}2\phi r_1 + r_4 = -tan\phi(-2ksin2\phi)r_1$$

$$\text{-2ksin}2\phi r_1 = -tan\phi(2kcos2\phi)r_1 + r_4$$

$$\text{r}_2 = -tan\phi r_3$$

the eigenvector of above equations are:

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 2k \end{bmatrix} or \begin{bmatrix} 0 & 1 & -cot\phi & 0 \end{bmatrix}$$

For eigenvalue $\mu_2 = cot\phi$, we have:

$$\text{-2kcos}2\phi r_2 - 2ksin2\phi r_3 = cot\phi(-2ksin2\phi r_2 + 2kcos2\phi r_3)$$

$$\text{-2kcos}2\phi r_1 + r_4 = cot\phi(-2ksin2\phi)r_1$$

$$\text{-2ksin}2\phi r_1 = cot\phi(2kcos2\phi)r_1 + r_4$$

$$\text{r}_2 = cot\phi r_3$$

the eigenvector of above equations are:

$$\tau = \begin{bmatrix} 1 & 0 & 0 & -2k \end{bmatrix} or \begin{bmatrix} 0 & 1 & tan\phi & 0 \end{bmatrix}$$

Relating Eq 8 and Eq 9, we can substitute A by B:

$$\text{r}_i B_{ij}(\mu\frac{\partial q_j}{\partial x_1} + \frac{\partial q_j}{\partial x_2}) = 0$$

$$\Rightarrow \sqrt{1+\mu^2}(\frac{\mu}{\sqrt{1+\mu^2}}\frac{\partial q_j}{\partial x_1} + \frac{1}{\sqrt{1+\mu^2}})\frac{\partial q_j}{\partial x_2} = 0$$

Let's assume:

$$\frac{\partial x_1}{\partial s} = \frac{\mu}{\sqrt{1+\mu^2}}$$
$$\frac{\partial x_2}{\partial s} = \frac{1}{\sqrt{1+\mu^2}}$$

then we have:

$$\mathrm{r}_i B_{ij}\left(\frac{\partial x_1}{\partial s}\frac{\partial q_j}{\partial x1} + \frac{\partial x_2}{\partial s}\frac{\partial q_j}{\partial x_2}\right) = 0$$
$$\Rightarrow r_i B_{ij}\frac{\partial q_j}{\partial s} = 0$$

which s is direction along the slip line, and

$$\frac{dx_2}{dx_1} = \frac{1}{\mu}$$

which means

$$\frac{dx_2}{dx_1} = tan\phi \text{ along the } \alpha \text{ slip line}$$
$$\frac{dx_2}{dx_1} = -cot\phi \text{ along the } \beta \text{ slip line}$$

Now, we have equation:

$$r_i B_{ij}\frac{\partial q_j}{\partial s} = 0 \tag{11}$$

where

$$\mathrm{q} = \begin{bmatrix} \phi \\ v_1 \\ v_2 \\ \bar{\sigma} \end{bmatrix} \mathrm{B} = \begin{bmatrix} 0 & -2ksin2\phi & 2kcos2\phi & 0 \\ -2ksin2\phi & 0 & 0 & 0 \\ 2kcos2\phi & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\text{when } \mu = cot\phi,\ \mathrm{r} = \begin{bmatrix} 1 & 0 & 0 & -2k \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & \tan\phi & 0 \end{bmatrix}$$
$$\text{when } \mu = -tan\phi,\ \mathrm{r} = \begin{bmatrix} 1 & 0 & 0 & 2k \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & \text{-}\cot\phi & 0 \end{bmatrix}$$

Next, we expand the matrix calculation:

when $\mu = cot\phi$ (α slip line),

$$\mathrm{r}_i B_{ij} = \begin{bmatrix} 0 & -2ksin2\phi & 2kcos2\phi - 2k & 0 \end{bmatrix}$$
$$\text{or } \mathrm{r}_i B_{ij} = \begin{bmatrix} -2ksin2\phi + 2kcos2\phi tan\phi & 0 & 0 & tan\phi \end{bmatrix}$$

therefore, we have:

$$-2ksin2\phi \tfrac{\partial v_1}{\partial s} + 2k(cos2\phi - 1)\tfrac{\partial v_2}{\partial s} = 0$$
$$\Rightarrow sin2\phi \tfrac{\partial v_1}{\partial s} + (1 - cos2\phi)\tfrac{\partial v_2}{\partial s} = 0$$
$$\Rightarrow 2sin\phi cos\phi \tfrac{\partial v_1}{\partial s} + 2sin^2\phi \tfrac{\partial v_2}{\partial s} = 0$$
$$\Rightarrow \tfrac{\partial v_1}{\partial s} + tan\phi \tfrac{\partial v_2}{\partial s} = 0$$

or

$$-2k(sin2\phi - cos2\phi tan\phi)\tfrac{\partial \phi}{\partial s} + tan\phi \tfrac{\partial \bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow -2k[(2sin\phi cos\phi - (cos^2\phi - sin^2\phi)tan\phi)]\tfrac{\partial \phi}{\partial s} + tan\phi \tfrac{\partial \bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow -2k(2cos^2 2\phi - cos^2\phi + sin^2\phi)\tfrac{\partial \phi}{\partial s} + \tfrac{\partial \bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow -2k\tfrac{\partial \phi}{\partial s} + \tfrac{\partial \bar{\sigma}}{\partial s} = 0$$

when $\mu = -tan\phi$ (β slip line),

$$\mathrm{r}_i B_{ij} = \begin{bmatrix} 0 & -2ksin2\phi & 2kcos2\phi - 2k & 0 \end{bmatrix}$$
$$\text{or } \mathrm{r}_i B_{ij} = \begin{bmatrix} -2ksin2\phi + 2kcos2\phi tan\phi & 0 & 0 & tan\phi \end{bmatrix}$$

therefore, we have:

$$-2ksin2\phi \tfrac{\partial v_1}{\partial s} + 2k(cos2\phi + 1)\tfrac{\partial v_2}{\partial s} = 0$$

$$\Rightarrow sin2\phi\frac{\partial v_1}{\partial s} - (cos2\phi + 1)\frac{\partial v_2}{\partial s} = 0$$
$$\Rightarrow 2sin\phi cos\phi\frac{\partial v_1}{\partial s} - 2cos^2\phi\frac{\partial v_2}{\partial s} = 0$$
$$\Rightarrow \frac{\partial v_1}{\partial s} - cot\phi\frac{\partial v_2}{\partial s} = 0$$

or

$$-2k(sin2\phi - cos2\phi cot\phi)\frac{\partial\phi}{\partial s} - cot\phi\frac{\partial\bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow 2k[(2sin\phi cos\phi - (cos^2\phi - sin^2\phi)cot\phi)]\frac{\partial\phi}{\partial s} + cot\phi\frac{\partial\bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow 2k(2cos^2\phi - cos^2\phi + sin^{\phi})\frac{\partial\phi}{\partial s} + \frac{\partial\bar{\sigma}}{\partial s} = 0$$
$$\Rightarrow 2k\frac{\partial\phi}{\partial s} + \frac{\partial\bar{\sigma}}{\partial s} = 0$$

Therefore, we have following results:

For α slip line:

$$\begin{aligned} \frac{\partial v_1}{\partial s} + tan\phi\frac{\partial v_2}{\partial s} &= 0 \\ -2k\frac{\partial\phi}{\partial s} + \frac{\partial\bar{\sigma}}{\partial s} &= 0 \end{aligned} \tag{12}$$

For β slip line:

$$\begin{aligned} \frac{\partial v_1}{\partial s} - cot\phi\frac{\partial v_2}{\partial s} &= 0 \\ 2k\frac{\partial\phi}{\partial s} + \frac{\partial\bar{\sigma}}{\partial s} &= 0 \end{aligned} \tag{13}$$

Since from the Fig 2, we know the basis transformation is:

$$v_\alpha = v_1 cos\phi + v_2 sin\phi$$
$$\mathrm{v}_\beta = -v_1 sin\phi + v_2 cos\phi$$

and

$$\mathrm{v}_1 = v_\alpha cos\phi - v_\beta sin\phi$$

$$\mathrm{v}_2 = v_\alpha sin\phi + v_\beta cos\phi$$

therefore we have:

$$\frac{dv_1}{ds} = \frac{dv_\alpha}{ds} cos\phi + v_\alpha \frac{dcos\phi}{ds} - \frac{dv_\beta}{ds} sin\phi - v_\beta \frac{dsin\phi}{ds}$$
$$\frac{dv_2}{ds} = \frac{dv_\alpha}{ds} sin\phi + v_\alpha \frac{dsin\phi}{ds} + \frac{dv_\beta}{ds} cos\phi + v_\beta \frac{dcos\phi}{ds}$$

therefore, substitute above equation into Eq 12:

$$\frac{dv_1}{ds} + tan\phi \frac{dv_2}{ds} = 0$$
$$\Rightarrow \frac{dv_\alpha}{ds}(cos\phi + tan\phi sin\phi) + v_\alpha(\frac{dcos\phi}{ds} + tan\phi \frac{dsin\phi}{ds}) + v_\beta(\frac{dcos\phi}{ds} tan\phi - \frac{dsin\phi}{ds}) = 0$$
$$\Rightarrow \frac{dv_\alpha}{ds}(cos\phi + tan\phi sin\phi) + v_\alpha(-sin\phi + tan\phi cos\phi)\frac{d\phi}{ds} + v_\beta(-sin\phi tan\phi - cos\phi)\frac{d\phi}{ds} = 0$$
$$\Rightarrow \frac{dv_\alpha}{ds} - v_\beta \frac{d\phi}{ds} = 0$$

and substitute above equation into Eq 13:

$$\frac{dv_1}{ds} - cot\phi \frac{dv_2}{ds} = 0$$
$$\Rightarrow v_\alpha(\frac{dcos\phi}{ds} - cot\phi \frac{dsin\phi}{ds}) - \frac{dv_\beta}{ds}(sin\phi + cot\phi cos\phi) - v_\beta(\frac{dsin\phi}{ds} + cot\phi \frac{dcos\phi}{ds}) = 0$$
$$\Rightarrow v_\alpha(-sin\phi - cot\phi cos\phi)\frac{d\phi}{ds} - \frac{dv_\beta}{ds}(sin\phi + cot\phi cos\phi) - v_\beta(cos\phi - cot\phi sin\phi)\frac{d\phi}{ds} = 0$$
$$\Rightarrow \frac{dv_\beta}{ds} + v_\alpha \frac{d\phi}{ds} = 0$$

therefore, we have:

$$\frac{dv_\alpha}{ds} = v_\beta \frac{d\phi}{ds}$$
$$\frac{dv_\beta}{ds} = -v_\alpha \frac{d\phi}{ds}$$

which is known as Geiringer equation.

4 Examples

4.1 Simple Compression

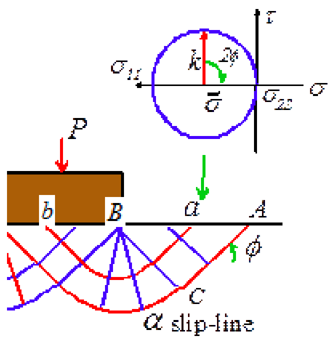

Figure 3: Simple Compression

First example is to find P, total force per unit length acting in the solid shown as Fig 3. At point a, we know the angle between e_α and e_1 is 45° which means

$$\phi_a = \frac{\pi}{4}$$

From the result of stress state in Section 3.1 we have

$$\sigma_{11} = \bar{\sigma} - ksin2\phi = \bar{\sigma} - k$$

$$\sigma_{22} = \bar{\sigma} + ksin2\phi = \bar{\sigma} + k$$

$$\sigma_{12} = kcos2\phi = 0$$

Since at point a, it's traction free, the normal vector is on e_2 direction. We have following boundary conditions:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus,

$$\sigma_{22} = \bar{\sigma} + k = 0$$

$$\Rightarrow \bar{\sigma} = -k$$

For point b, we know the angle between e_α and e_1 is $-45°$, which means:

$$\phi_b = -\frac{\pi}{4}$$

then, by Hencky equation, for α slip line:

$$\bar{\sigma}_a - 2k\phi_a = constant = \bar{\sigma}_b - 2k\phi_b$$

$$\Rightarrow -k - 2k\frac{\pi}{4} = \bar{\sigma}_b - 2k(-\frac{\pi}{4})$$

$$\Rightarrow \bar{\sigma}_b = -k - k\pi$$

therefore, the stress state of solid is:

$$\sigma_{11} = \bar{\sigma}_b - ksin(-\frac{\pi}{2}) = -k\pi$$

$$\sigma_{22} = \bar{\sigma}_b + ksin(-\frac{\pi}{2}) = -2k - k\pi$$

$$\sigma_{12} = kcos(-\frac{\pi}{2}) = 0$$

The force per unit length, P, we have:

$$P + \int_0^w \sigma_{22} dx_1 = 0$$

$$\Rightarrow P = wk(\pi + 2)$$

where w is the width of the solid.

4.2 Plan Strain Extrusion

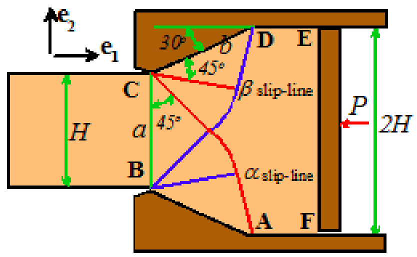

Figure 4: Plain Strain Extrusion

Second example is to find P of plain strain extrusion for tapered end. The friction is neglected. At point a, we know the angle between e_α and e_1 is $-45°$ which means:

$$\phi_a = -\tfrac{\pi}{4}$$

From the result of stress state in Section 3.1 we have

$$\sigma_{11} = \bar{\sigma} - ksin2\phi = \bar{\sigma} + k$$

$$\sigma_{22} = \bar{\sigma} + ksin2\phi = \bar{\sigma} - k$$

$$\sigma_{12} = kcos2\phi = 0$$

Since at point a, it's traction free, the normal vector is on $-e_1$ direction. We have following boundary conditions:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus,

$$\sigma_{11} = \bar{\sigma} + k = 0$$
$$\Rightarrow \bar{\sigma} = -k$$

For point b, from the geometry, we know the angle between e_α and e_1 is $-75°$, which means:

$$\phi_b = -\tfrac{5\pi}{12}$$

then, by Hencky equation, for α slip line:

$$\bar{\sigma}_a - 2k\phi_a = constant = \bar{\sigma}_b - 2k\phi_b$$
$$\Rightarrow -k + k\tfrac{\pi}{2} = \bar{\sigma}_b + k(\tfrac{5\pi}{6})$$
$$\Rightarrow \bar{\sigma}_b = -k - \tfrac{\pi}{3}k$$

Finally, the angle between slip line and CD is $45°$. Therefore, the stress state of solid is:

$$\sigma_{11} = \bar{\sigma}_b - ksin(\tfrac{\pi}{2}) = -2k - \tfrac{\pi}{3}k$$
$$\sigma_{22} = \bar{\sigma}_b + ksin(\tfrac{\pi}{2}) = -\tfrac{\pi}{3}k$$
$$\sigma_{12} = kcos(\tfrac{\pi}{2}) = 0$$

From the symmetry, the stress at AB in e_1 direction is same as that at CD. The height of AB and CD are all $\frac{H}{2}$. From force balance in e_1 direction we have:

$$\sum F_1 = -P + 2 * \int_0^{\frac{H}{2}} \sigma_{11} dx_2 = 0$$
$$\Rightarrow P = kH(\tfrac{\pi}{3} + 2)$$

4.3 Double-notched Plate in Tension

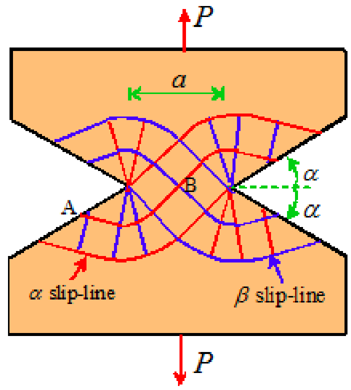

Figure 5: Double-notched Plate in Tension

Example 3 is to find force P per unit length acting on a double-notched plate as shown as Fig 5. At point A, from geometry we can tell:

$$\phi_A = \alpha - \frac{\pi}{4}$$

From the result of stress state in Section 3.1 we have

$$\sigma_{11} = \bar{\sigma} - ksin(2\alpha - \frac{\pi}{2}) = \bar{\sigma} + kcos2\alpha$$

$$\sigma_{22} = \bar{\sigma} + ksin(2\alpha - \frac{\pi}{2}) = \bar{\sigma} - kcos2\alpha$$

$$\sigma_{12} = kcos(2\alpha - \frac{\pi}{2}) = ksin2\alpha$$

Since at point A, it's traction free, the normal vector of this plane is:

$$\text{n} = \begin{bmatrix} -sin\alpha \\ cos\alpha \end{bmatrix}$$

Now, we have following boundary conditions:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -sin\alpha \\ cos\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus,

$$-\sigma_{11} sin\alpha + \sigma_{12} cos\alpha = 0$$

$$\Rightarrow -(\bar{\sigma} + kcos2\alpha) sin\alpha + ksin2\alpha cos\alpha = 0$$

$$\Rightarrow -\bar{\sigma} sin\alpha - k(cos^2\alpha - sin^2\alpha) sin\alpha + 2ksin\alpha cos^2\alpha = 0$$

$$\Rightarrow -\bar{\sigma} sin\alpha + ksin^3\alpha + ksin\alpha cos^2\alpha = 0$$

$$\Rightarrow -\bar{\sigma} + k(sin^2\alpha + cos^2\alpha) = 0$$

$$\Rightarrow \bar{\sigma}_A = k$$

For point B, from the geometry, we know the angle between e_α and e_1 is $45°$, which means:

$$\phi_b = \frac{pi}{4}$$

then, by Hencky equation, for α slip line:

$$\bar{\sigma}_a - 2k\phi_a = constant = \bar{\sigma}_b - 2k\phi_b$$

$$\Rightarrow k - 2k(\alpha - \tfrac{\pi}{4}) = \bar{\sigma}_B - 2k\tfrac{\pi}{4}$$

$$\Rightarrow \bar{\sigma}_B = k - 2k\alpha + k\pi = k(\pi - 2\alpha + 1)$$

Note: From Professor Bower's book, the hydrostastic stress at point B is $k(\pi - 2\alpha)$. I think it is just a typo because his final result is same as mine.

Back to calculations, we can find the stress state at point B is:

$$\sigma_{11} = \bar{\sigma}_b - ksin(2\phi_B) = k(\pi - 2\alpha + 1) - k = k(\pi - 2\alpha)$$

$$\sigma_{22} = \bar{\sigma}_b + ksin(2\phi_B) = k(\pi - 2\alpha + 1) + k = k(\pi - 2\alpha + 2)$$

$$\sigma_{12} = kcos(2\phi_B) = 0$$

From force balance in e_2 direction we have:

$$\sum F_2 = -P + \int_0^a \sigma_{22} dx_1 = 0$$

$$\Rightarrow P = ak(\pi - 2\alpha + 2)$$

5 References

References

[1] A.F.Bower, (2012), Applied Mechanics of Solids, Chapter 6.1, Retrieved from http://solidmechanics.org/Text/Chapter6$_1$/$Chapter6_1.php$